Research Article

H-Coverings of Path-Amalgamated Ladders and Fans

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Received 13 August 2020; Accepted 24 October 2020; Published 20 November 2020

Academic Editor: Francisco R. Villatoro

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Let $\mathbb{G}$ be a connected, simple graph with finite vertices v and edges e . A family $\{\mathbb{G}_1, \mathbb{G}_2, \dots, \mathbb{G}_p\} \subset \mathbb{G}$ of subgraphs such that for all $e \in E$, $e \in \mathbb{G}_l$, for some l , $l = 1, 2, \dots, p$ is an edge-covering of $\mathbb{G}$. If $\mathbb{G}_l \cong \mathbb{H}$, $\forall l$, then $\mathbb{G}$ has an $\mathbb{H}$ -covering. Graph $\mathbb{G}$ with $\mathbb{H}$ -covering is an (a_d, d) - $\mathbb{H}$ -antimagic if $\psi: V(\mathbb{G}) \cup E(\mathbb{G}) \rightarrow \{1, 2, \dots, v + e\}$ a bijection exists and the sum over all vertex-weights and edge-weights of $\mathbb{H}$ forms a set $\{a_d, a_d + d, \dots, a_d + (p - 1)d\}$. The labeling ψ is super for $\psi(V(\mathbb{G})) = \{1, 2, 3, \dots, v\}$ and graph $\mathbb{G}$ is $\mathbb{H}$ -supermagic for $d = 0$. This manuscript proves results about super $\mathbb{H}$ -antimagic labeling of path amalgamation of ladders and fans for several differences.

1. Introduction and Preliminaries

Let $\mathbb{G} = (V_{\mathbb{G}}, E_{\mathbb{G}})$ be a connected, finite, and simple graph [1, 2]. An edge-covering of $\mathbb{G}$ is a family $\{\mathbb{G}_1, \mathbb{G}_2, \dots, \mathbb{G}_p\} \subset \mathbb{G}$ such that for all $e \in E(\mathbb{G})$, $e \in E(\mathbb{G}_l)$, for some l , $l = 1, 2, \dots, p$. If $\mathbb{G}_l \cong \mathbb{H}$, $\forall l$, then $\mathbb{G}$ has a $\mathbb{H}$ -covering. Graph $\mathbb{G}$ with $\mathbb{H}$ -covering is an (a_d, d) - $\mathbb{H}$ -antimagic if $\exists$ a one-to-one correspondence $\psi: V_{\mathbb{G}} \cup E_{\mathbb{G}} \rightarrow \{1, 2, \dots, v + e\}$, $\exists$ for all $\mathbb{H} \subset \mathbb{G}$, $\mathbb{H} \cong \mathbb{G}$,

$$\begin{aligned} wt_{\psi}(\mathbb{H}) &= \left\{ \sum (\psi(v_{\mathbb{H}}) + \psi(E_{\mathbb{H}})) \right\} \\ &= \{a_d, a_d + d, \dots, a_d + (q - 1)d\}. \end{aligned} \quad (1)$$

For $\psi(V_{\mathbb{G}}) = \{1, 2, 3, \dots, |V(\mathbb{G})|\}$, the labeling ψ becomes super (a_d, d) - $\mathbb{H}$ -antimagic and it would be $\mathbb{H}$ -supermagic for $d = 0$. Gutiérrez and Lladó, in [3], defined the $\mathbb{H}$ -supermagic graphs for $K_{1,m}$, $K_{p,m}$, P_m , and C_m for some subgraph $\mathbb{H}$. Jeyanthi and Selvagopal [4] proved $\mathbb{H}$ -supermagic results for 2-connected graphs, k -polygonal snake, and one point union of m -disjoint paths. kC_m -path, $\mathbb{G} \cong K_{1,m} + K_1$, book, and

ladder-related graphs are $\mathbb{H}$ -supermagic proved in [5]. Inayah et al. [6] introduced $\mathbb{H}$ -antimagic graphs with weights forming an arithmetic progression. She also proved some bounds for a_d and d for general graphs and fans. Susanto [7] derived bound for cycle-antimagic labeling of disjoint union of cycles. Recent results on $\mathbb{H}$ -antimagic labeling of graphs can be seen in [8–14]. Also, in [12], Baca et al. discussed the tree-antimagicness of disconnected graphs. Recently, authors in [15] discussed the super $(a, 1)$ -tree-antimagicness of Sun graphs. The (super) $\mathbb{H}$ -antimagic labeling is also related to a (super) d -antimagic labeling of type $(1, 1, 0)$ of a plane graph [16]: a generalization of a face-magic labeling introduced by Lih [17]. Baca et al. proved d -antimagic labeling of type $(1, 1, 1)$ for toroidal fullerenes in [18], while, in [19], Baca et al. proved labeling for plane graphs containing Hamiltonian paths. For more details, we refer [20–29] and the references therein. In the present article, we have studied super $\mathbb{H}$ -antimagic labeling of path-amalgamation of ladder and fan for several differences, where $\mathbb{H}$ is isomorphic to cycles C_3 , C_4 , C_5 , and P_2 -amalgamation of two cycles C_3 and C_4 .

FIGURE 1: P_6 -Amalgamation of ladder L_6 and fan F_6 .

2. Path Amalgamation

In [30, 31], authors defined vertex amalgamation of m isomorphic copies of $\mathbb{G}$, denoted with $\text{Amal}(\mathbb{G}, \{u\}, m)$, for $u \in \mathbb{G}$. Maryati et al. proved $\text{Amal}(\mathbb{G}, \{v\}, m)$ is G -supermagic in [32]. G -supermagic labeling of (i) edge amalgamation of 2-connected simple graph $\mathbb{G}$ is proved by Jeyanthi and Selvagopal in [4] and of (ii) $\text{Amal}(G, P_n, m)$ by Salman and Maryati in [33]. In this paper, we extend this idea and use path graph P_m instead of a vertex to define a path amalgamation for two graphs ladders and fans. A ladder graph L_m is defined as the Cartesian product of P_m with P_2 , i.e., $L_m \cong P_m \square P_2$. A fan graph F_m is defined as the join of P_m with an isolated vertex x , i.e., $x F_m \cong P_m + x$.

Definition 1. Let L_m and F_m be ladder and fan graph for $m \geq 3$. A path amalgamation of ladder and fan denoted by $\text{Amal}(L_m, F_m, P_m)$ is obtained by identifying path P_m in ladder and fan.

The vertex set of $\text{Amal}(L_m, F_m, P_m)$ is $\{a_i, b_i, x: 1 \leq i \leq m\}$ and edge set is $(a_i b_i, x b_i: 1 \leq i \leq m) \cup \{a_i a_{i+1}, b_i b_{i+1}: 1 \leq i \leq m-1\}$.

Figure 1 depicts P_6 -amalgamation of ladder L_6 and fan F_6 .

Definition 2. Let C_3 and C_4 be two cycles on 3 and 4 vertices, respectively. By identifying one edge of both cycles C_3 and C_4 , we obtain P_2 -amalgamation denoted by $\text{Amal}(C_4, C_3, P_2)$.

Figure 2 depicts P_2 -amalgamation of cycles C_4 and C_3 .

In Section 3, we will study the existence of super (a_d, d) - $\mathbb{G}'$ -antimagic labeling of path amalgamation of

FIGURE 2: P_2 -Amalgamation of C_3 and C_4 .

ladder and fan, where $\mathbb{G}'$ is C_3, C_4 , and $\text{Amal}(C_3, C_4, P_2)$ for several differences.

3. Super $\mathbb{H}$ -Antimagic Labeling of Path Amalgamation

From Figures 1 and 2, it is clear that path amalgamation $\text{Amal}(L_m, F_m, P_m)$ has $\mathbb{G}'$ -coverings by subgraphs $\mathbb{G}_i$, $i = 1, 2, 3, 4$, where $\mathbb{G}_1 = C_3$, $\mathbb{G}_2 = C_4$, $\mathbb{G}_3 = C_5$, and $\mathbb{G}_4 = \text{Amal}(C_3, C_4, P_2)$.

The set of vertices and edges in subgraphs $\mathbb{G}_i$, $i = 1, 2, 3, 4$, $j = 1, 2, \dots, m-1$ are

$$\begin{aligned} V(\mathbb{G}_1^{(j)}) &= \{x, b_j, b_{j+1}\}, \\ V(\mathbb{G}_2^{(j)}) &= \{a_j, a_{j+1}, b_j, b_{j+1}\}, \\ V(\mathbb{G}_{3,4}^{(j)}) &= \{x, a_j, a_{j+1}, b_j, b_{j+1}\}, \\ E(\mathbb{G}_1^{(j)}) &= \{x b_j, x b_{j+1}, b_j b_{j+1}\}, \\ E(\mathbb{G}_2^{(j)}) &= \{a_j a_{j+1}, b_j b_{j+1}, a_j b_j, a_{j+1} b_{j+1}\}, \\ E(\mathbb{G}_3^{(j)}) &= \frac{E(\mathbb{G}_1) \cup E(\mathbb{G}_2)}{\{b_j b_{j+1}\}}, \\ E(\mathbb{G}_4^{(j)}) &= E(\mathbb{G}_1) \cup E(\mathbb{G}_2). \end{aligned} \quad (2)$$

Let ψ be the total labeling of $\text{Amal}(L_m, F_m, P_m)$, then $wt_\psi(\mathbb{G}_i^{(j)})$, $i = 1, 2, 3, 4$, $j = 1, 2, \dots, m$ would be

$$wt_\psi((\mathbb{G}_1)^j) = \psi(x) + \psi(b_j) + \psi(b_{j+1}) + \psi(x b_j) + \psi(x b_{j+1}) + \psi(b_j b_{j+1}), \quad (3)$$

$$wt_\psi((\mathbb{G}_2)^j) = \psi(a_j) + \psi(a_{j+1}) + \psi(b_j) + \psi(b_{j+1}) + \psi(a_j b_j) + \psi(a_j a_{j+1}) + \psi(b_j b_{j+1}) + \psi(a_{j+1} b_{j+1}), \quad (4)$$

$$wt_\psi((\mathbb{G}_3)^j) = wt_\psi((\mathbb{G}_1)^j) + wt_\psi((\mathbb{G}_2)^j) - \psi(b_j) - \psi(b_{j+1}) - 2\psi(b_j b_{j+1}), \quad (5)$$

$$wt_\psi((\mathbb{G}_4)^j) = wt_\psi((\mathbb{G}_3)^j) + \psi(b_j b_{j+1}). \quad (6)$$

Theorem 1. Let $\text{Amal}(L_m, F_m, P_m)$ be amalgamation of ladder L_m and fan F_m and $m \geq 3$ be a positive integer. Then, $\text{Amal}(L_m, F_m, P_m)$ posses a super (a_d, d) - $\mathbb{G}_i$ -antimagic labeling for $i = 1, 2$ and $d \in \{0, 1, 2, \dots, 8\}$.

Proof. Consider the total labeling ψ as follows:

$$\begin{aligned}
 \psi_d(x) &= 2m + 1, \\
 \psi_d(a_i) &= \begin{cases} \left\lfloor \frac{m}{2} \right\rfloor + \frac{i}{2} & i = \text{even}, d = 0, 1, 2, 3, \\ \frac{i+1}{2} & i = \text{odd}, d = 0, 1, 2, 3, \\ i & i = 1, 2, \dots, m, d = 4, \\ 2i - 1 & i = 1, 2, \dots, m, d = 5, 6, 7, 8, \end{cases} \\
 \psi_d(b_i) &= \begin{cases} m + \left\lfloor \frac{m}{2} \right\rfloor + \frac{i}{2} & i = \text{even}, d = 1, 2, 4, \\ m + \frac{i+1}{2} & i = \text{odd}, d = 1, 2, 4, \\ m + i & i = 1, 2, \dots, m, d = 0, 3, \\ 2i & i = 1, 2, \dots, m, d = 5, 6, 7, 8, \end{cases} \\
 \psi_d(a_i a_{i+1}) &= 4m - ii = 1, 2, \dots, m - 1, d = 0, 1, \dots, 8, \\
 \psi_d(xb_i) &= \begin{cases} 5m - 1 + \left\lfloor \frac{m}{2} \right\rfloor + \frac{i}{2} & i = \text{even}, d = 1, 6, 7, \\ 5m - 1 + \frac{i+1}{2} & i = \text{odd}, d = 1, 6, 7, \\ 6m - \left\lfloor \frac{m}{2} \right\rfloor - \frac{i}{2} & i = \text{even}, d = 0, \\ 6m - \frac{i+1}{2} & i = \text{odd}, d = 0, \\ 5m - 1 + i & i = 1, 2, \dots, m, d = 2, 3, 4, 5, 8, \end{cases} \\
 \psi_d(b_i b_{i+1}) &= \begin{cases} 3m + 1 - i & i = 1, 2, \dots, m - 1, d = 0, 1, 2, 3, 5, \\ 2m + 1 + i & i = 1, 2, \dots, m - 1, d = 4, 6, \\ 2(m + i) & i = 1, 2, \dots, m - 1, d = 7, 8, \end{cases} \\
 \psi_d(a_i b_{i+1}) &= \begin{cases} 4m - 1 + i & i = 1, 2, \dots, m, d = 2, 3, \\ 4m - 1 + \left\lfloor \frac{m}{2} \right\rfloor + \frac{i}{2} & i = \text{even}, d = 1, 4, \\ 4m - 1 + \frac{i+1}{2} & i = \text{odd}, d = 1, 4, \\ 5m - \left\lfloor \frac{m}{2} \right\rfloor - \frac{i}{2} & i = \text{even}, d = 0, 5, 8, \\ 5m - \frac{i+1}{2} & i = \text{odd}, d = 0, 5, 8, \\ 5m - i & i = 1, 2, \dots, m, d = 6, 7. \end{cases} \quad (7)
 \end{aligned}$$

Since, $\psi_d(V(\text{Amal}(L_m, F_m, P_m))) = \{1, 2, \dots, 2m + 1\}$, ψ_d is a super labeling for $d = \{0, 1, \dots, 8\}$ and $\psi_d(E(\text{Amal}(L_m, F_m, P_m))) = \{2m + 2, 2m + 3, \dots, 6m - 1\}$. Therefore, ψ_d is a total labeling.

Using (3) and (4) and $j = 1, 2, \dots, m - 1$,

$$\begin{aligned}
 wt_{\psi_0}((\mathbb{G}_i)^j) &= \begin{cases} 19m + 2 - \left\lfloor \frac{m}{2} \right\rfloor & i = 1, \\ 19m + 2 & i = 2, \end{cases} \\
 wt_{\psi_1}((\mathbb{G}_i)^j) &= \begin{cases} 17m + 2 + 2\left\lfloor \frac{m}{2} \right\rfloor + i & i = 1, \\ 17m + 2 + 3\left\lfloor \frac{m}{2} \right\rfloor + i & i = 2, \end{cases} \\
 wt_{\psi_2}((\mathbb{G}_i)^j) &= \begin{cases} 17m + \left\lfloor \frac{m}{2} \right\rfloor + 2 + 2i & i = 1, \\ 17m + 2\left\lfloor \frac{m}{2} \right\rfloor + 2 + 2i & i = 2, \end{cases} \\
 wt_{\psi_3}((\mathbb{G}_i)^j) &= \begin{cases} 17m + 2 + 3i & i = 1, \\ 17m + \left\lfloor \frac{m}{2} \right\rfloor + 2 + 3i & i = 2, \end{cases} \\
 wt_{\psi_4}((\mathbb{G}_i)^j) &= \begin{cases} 16m + \left\lfloor \frac{m}{2} \right\rfloor + 2 + 4i & i = 1, \\ 16m + 2\left\lfloor \frac{m}{2} \right\rfloor + 2 + 4i & i = 2, \end{cases} \quad (8) \\
 wt_{\psi_5}((\mathbb{G}_i)^j) &= \begin{cases} 15m + 3 + 5i & i = 1, \\ 17m - \left\lfloor \frac{m}{2} \right\rfloor + 2 + 5i & i = 2, \end{cases} \\
 wt_{\psi_6}((\mathbb{G}_i)^j) &= \begin{cases} 14m + \left\lfloor \frac{m}{2} \right\rfloor + 3 + 6i & i = 1, \\ 16m + 2 + 6i & i = 2, \end{cases} \\
 wt_{\psi_7}((\mathbb{G}_i)^j) &= \begin{cases} 14m + \left\lfloor \frac{m}{2} \right\rfloor + 2 + 7i & i = 1, \\ 16m + 1 + 7i & i = 2, \end{cases} \\
 wt_{\psi_8}((\mathbb{G}_i)^j) &= \begin{cases} 14m + 2 + 8i & i = 1, \\ 16m - \left\lfloor \frac{m}{2} \right\rfloor + 1 + 8i & i = 2. \end{cases}
 \end{aligned}$$

All the above equations show an arithmetic progression with differences $d \in \{0, 1, 2, \dots, 8\}$, respectively. This completes the proof. $\square$

Theorem 2. Let $Amal(L_m, F_m, P_m)$ be amalgamation of ladder L_m and fan F_m and $m \geq 3$ be a positive integer. Then, $Amal(L_m, F_m, P_m)$ possesses a super (a', d') - $\mathbb{G}_3$ -antimagic labeling for $d' \in \{0, 3, 5, 6, 8\}$ and (a'', d'') - $\mathbb{G}_4$ -antimagic labeling for $d'' \in \{0, 1, 2, \dots, 8\}$.

Proof. The subgraphs $\mathbb{G}_i$, $i = 3, 4$ contain the subgraphs $\mathbb{G}_i$, $i = 1, 2$; therefore, differences d' depends upon differences d obtained in Theorem 1. We will use the labeling ψ_d from Theorem 1. Let v denote a vertex element in $\mathbb{G}_i$, $i = 3, 4$ and ω denote an edge element in $\mathbb{G}_i$, $i = 3, 4$; then, the total labeling $\psi'_{d'}$ is defined as follows:

$$\begin{aligned}\psi'_{d'}(v) &= \psi''_{d''}(v) = \psi_d(v), \\ \psi'_{d'}(\omega) &= \psi''_{d''}(\omega) = \psi_d(\omega).\end{aligned}\quad (9)$$

Since ψ_d is a super total labeling, therefore $\psi'_{d'}$ is super total. For differences $d' \in \{0, 3, 5, 6, 8\}$, the total labeling $\psi''_{d''}$ of $(\mathbb{G}_3)^j$ is obtained as follows:

$$\begin{aligned}\psi'_0((\mathbb{G}_3)^j) &= \psi_0((\mathbb{G}_3)^j), \\ \psi'_3((\mathbb{G}_3)^j) &= \psi_1((\mathbb{G}_3)^j), \\ \psi'_5((\mathbb{G}_3)^j) &= \psi_2((\mathbb{G}_3)^j), \\ \psi'_6((\mathbb{G}_3)^j) &= \psi_3((\mathbb{G}_3)^j), \\ \psi'_8((\mathbb{G}_3)^j) &= \psi_5((\mathbb{G}_3)^j).\end{aligned}\quad (10)$$

For differences $d'' \in \{1, 2, 5, 6, 7, 8, 10\}$, the total labeling $\psi''_{d''}$ of $(\mathbb{G}_4)^j$ is obtained as follows:

$$\begin{aligned}\psi''_1((\mathbb{G}_4)^j) &= \psi_0((\mathbb{G}_4)^j), \\ \psi''_2((\mathbb{G}_4)^j) &= \psi_1((\mathbb{G}_4)^j), \\ \psi''_5((\mathbb{G}_4)^j) &= \psi_3((\mathbb{G}_4)^j), \\ \psi''_6((\mathbb{G}_4)^j) &= \psi_4((\mathbb{G}_4)^j), \\ \psi''_7((\mathbb{G}_4)^j) &= \psi_5((\mathbb{G}_4)^j), \\ \psi''_8((\mathbb{G}_4)^j) &= \psi_7((\mathbb{G}_4)^j), \\ \psi''_{10}((\mathbb{G}_4)^j) &= \psi_8((\mathbb{G}_4)^j).\end{aligned}\quad (11)$$

Using (5) and (6) and $j = 1, 2, \dots, m-1$,

$$\begin{aligned}wt_{\psi'_{d'}}((\mathbb{G}_3)^j) &= \begin{cases} 30m + 1 - \left\lceil \frac{m}{2} \right\rceil & d' = 0, \\ 26m + 4 \left\lceil \frac{m}{2} \right\rceil + 1 + 3i & d' = 3, \\ 26m + 2 \left\lceil \frac{m}{2} \right\rceil + 1 + 5i & d' = 5, \\ 26m + 2 \left\lceil \frac{m}{2} \right\rceil + 1 + 6i & d' = 6, \\ 26m - \left\lceil \frac{m}{2} \right\rceil + 1 + 8i & d' = 8, \end{cases} \\ wt_{\psi''_{d''}}((\mathbb{G}_4)^j) &= \begin{cases} 33m + 2 - \left\lceil \frac{m}{2} \right\rceil - i & d' = 1, \\ 29m + 4 \left\lceil \frac{m}{2} \right\rceil + 3 + 2i & d' = 2, \\ 29m + \left\lceil \frac{m}{2} \right\rceil + 2 + 5i & d' = 5, \\ 28m + 2 \left\lceil \frac{m}{2} \right\rceil + 2 + 6i & d' = 6, \\ 29m - \left\lceil \frac{m}{2} \right\rceil + 2 + 7i & d' = 7, \\ 28m + \left\lceil \frac{m}{2} \right\rceil + 1 + 8i & d' = 8, \\ 28m - \left\lceil \frac{m}{2} \right\rceil + 1 + 10i & d' = 10. \end{cases}\end{aligned}\quad (12)$$

All the above equations show an arithmetic progression with required differences d' and d'' . This completes the proof. $\square$

4. $\mathbb{H}$ -Antimagic Coverings for Disjoint Union

Theorem 3. Let path amalgamation of ladder and fan $Amal(L_m, F_m, P_m)$, $m \geq 3$ admit $\mathbb{G}_i$ -supermagic labeling for $i = 1, 2, 3, 4$. Then, the disjoint union of arbitrary number of copies of $Amal(L_m, F_m, P_m)$, i.e., $\tau Amal(L_m, F_m, P_m)$ also admits a $\mathbb{G}_i$ -supermagic labeling for $i = 1, 2, 3$ and super $(a, 1)$ - $\mathbb{G}_4$ -antimagic for $\tau \geq 1$ a positive integer.

Proof. Let $\tau \geq 1$ be a positive integer. In our proof, u_k , $k = 1, 2, \dots, \tau$, denotes a vertex or an edge in the k th copy of the path amalgamation of ladder and fan, denoted by $\mathbb{G}: = Amal(L_m, F_m, P_m)(k)$, corresponding to u in $Amal(L_m, F_m, P_m)$, i.e., $u \in V(Amal(L_m, F_m, P_m)) \cup E(Amal(L_m, F_m, P_m))$. In the same way, let $\mathbb{G}_i^{(r)}(k)$, $i = 1, 2, 3, 4$, $k = 1, 2, \dots, \tau$, $r = 1, 2, \dots, m-1$, be the subgraph in the k th copy of $Amal(L_m, F_m, P_m)$ corresponding to the subgraph $\mathbb{G}_i^{(r)}$, $i = 1, 2, 3, 4$, $r = 1, 2, \dots, m-1$ in $Amal(L_m, F_m, P_m)$. The total labeling h' of $\tau Amal(L_m, F_m, P_m)$ is defined as follows:

$$\phi(u_k) = \begin{cases} \tau(\phi(u) - 1) + k & u \in V(Amal(L_m, F_m, P_m)), \\ \tau\phi(u) + 1 - k & u \in E(Amal(L_m, F_m, P_m)). \end{cases} \quad (13)$$

First, we will prove that vertices of $\mathbb{G}: = Amal(L_m, F_m, P_m)(k)$ use integers from 1 up to $p\tau$ under the labeling ϕ , where p is number of vertices in graph $Amal(L_m, F_m, P_m)$, i.e.,

$$\phi(V(\mathbb{G})) = \begin{cases} \{1, \tau + 1, 2\tau + 1, \dots, (p-1)\tau + 1\}, & k = 1, \\ \{2, \tau + 2, 2\tau + 2, \dots, (p-1)\tau + 2\}, & k = 2, \\ \dots & \dots \\ \dots & \dots \\ \dots & \dots \\ \{j, \tau + j, 2\tau + j, \dots, (p-1)\tau + j\}, & k = j, \\ \dots & \dots \\ \dots & \dots \\ \dots & \dots \\ \{p\tau, 2\tau, 3\tau, \dots, p\tau\}, & k = \tau. \end{cases} \quad (14)$$

Secondly, for edges of $\cup_{k=1}^{\tau} Amal(L_m, F_m, P_m)(k)$ with $|E(\mathbb{G})| = q$, under the labeling ϕ ,

$$\phi\{E(\mathbb{G})\} = \begin{cases} \{(p+1)\tau, (p+2)\tau, (p+3)\tau, \dots, (p+q)\tau\} & k = 1, \\ \{(p+1)\tau - 1, (p+2)\tau - 1, (p+3)\tau - 1, \dots, (p+q)\tau - 1\} & k = 2, \\ \dots & \dots \\ \dots & \dots \\ \dots & \dots \\ \{(p+1)\tau + 1 - j, (p+2)\tau + 1 - j, (p+3)\tau + 1 - j, \dots, (p+q)\tau + 1 - j\} & k = j, \\ \dots & \dots \\ \dots & \dots \\ \dots & \dots \\ \{p\tau + 1, (p+1)\tau + 1, (p+2)\tau + 1, \dots, (p+q-1)\tau + 1\} & k = \tau. \end{cases} \quad (15)$$

From equations (14) and (15), it is clear the labeling ϕ is a total labeling, since ϕ is a bijection between the integers $\{1, 2, \dots, (p+q)\tau\}$ and the vertices and edges of $\cup_{k=1}^{\tau} Amal(L_m, F_m, P_m)(k)$.

Under the total labeling ϕ , the $\mathbb{G}_i^{(r)}(k)$ -weights, $i = 1, 2, 3, 4$, $1 \leq k \leq \tau$, $1 \leq r \leq m-1$ would be

$$\begin{aligned} wt_{\phi}(\mathbb{G}_i^{(r)}(k)) &= \sum_{v \in V(\mathbb{G}_i^{(r)}(k))} \phi(v) + \sum_{e \in E(\mathbb{G}_i^{(r)}(k))} \phi(e) \\ &= \sum_{v \in V(\mathbb{G}_i^{(r)}(k))} (\tau(\phi(v) - 1) + k) + \sum_{e \in E(\mathbb{G}_i^{(r)}(k))} (\tau\phi(e) + 1 - k) \\ &= \tau \sum_{v \in V(\mathbb{G}_i^{(r)}(k))} \phi(v) - \tau|V(\mathbb{G}_i^{(r)}(k))| + k|V(\mathbb{G}_i^{(r)}(k))| \\ &\quad + \tau \sum_{e \in E(\mathbb{G}_i^{(r)}(k))} \phi(e) + |E(\mathbb{G}_i^{(r)}(k))| - k|E(\mathbb{G}_i^{(r)}(k))|, \end{aligned} \quad (16)$$

where r is the number of $\mathbb{G}_i(k)$'s $i = 1, 2, 3, 4$, $r = 1, 2, \dots, m-1$ in $Amal(L_m, F_m, P_m)(k)$:

$$\begin{aligned} wt_\phi(\mathbb{G}_i^{(r)}(k)) &= \tau \left(\sum_{v \in V(\mathbb{G}_i^{(r)}(k))} \phi(v) + \sum_{e \in E(\mathbb{G}_i^{(r)}(k))} \phi(e) \right) - \tau |V(\mathbb{G}_i^{(r)}(k))| \\ &\quad + |E(\mathbb{G}_i^{(r)}(k))| k |V(\mathbb{G}_i^{(r)}(k))| - k |E(\mathbb{G}_i^{(r)}(k))| \\ &= \tau wt_\phi(\mathbb{G}_i^{(r)}(k)) - \tau |V(\mathbb{G}_i^{(r)}(k))| + |E(\mathbb{G}_i^{(r)}(k))| \\ &\quad + k |V(\mathbb{G}_i^{(r)}(k))| - k |E(\mathbb{G}_i^{(r)}(k))|. \end{aligned} \quad (17)$$

As every $\mathbb{G}_i^{(r)}(k)$, $i = 1, 2, 3, 4$, $k = 1, 2, \dots, \tau$, $r = 1, 2, \dots, m-1$, is isomorphic to the cycle $\mathbb{G}_i$, $i = 1, 2, 3, 4$, it holds

$$\begin{aligned} |V(\mathbb{G}_1^{(r)}(k))| &= |V(\mathbb{G}_1^{(r)})| = 3, & |V(\mathbb{G}_2^{(r)}(k))| &= |V(\mathbb{G}_2^{(r)})| = 4, \\ |V(\mathbb{G}_3^{(r)}(k))| &= |V(\mathbb{G}_3^{(r)})| = 5, & |V(\mathbb{G}_4^{(r)}(k))| &= |V(\mathbb{G}_4^{(r)})| = 5, \\ |E(\mathbb{G}_1^{(r)}(k))| &= |E(\mathbb{G}_1^{(r)})| = 3, & |E(\mathbb{G}_2^{(r)}(k))| &= |E(\mathbb{G}_2^{(r)})| = 4, \\ |E(\mathbb{G}_3^{(r)}(k))| &= |E(\mathbb{G}_3^{(r)})| = 5, & |E(\mathbb{G}_4^{(r)}(k))| &= |E(\mathbb{G}_4^{(r)})| = 6. \end{aligned} \quad (18)$$

Thus, for the $\mathbb{G}_i$ -weights for $i = 1, 2, 3, 4$, we obtain

$$\begin{aligned} wt_\phi(\mathbb{G}_1^{(r)}(k)) &= \tau wt_\phi(\mathbb{G}_1^{(r)}(k)) + 3(1-\tau) = \frac{\tau}{2} (29(m-1) + 43) + 3(1-\tau) = \frac{\tau}{2} (29(m-1) + 35) + 3, \\ wt_\phi(\mathbb{G}_2^{(r)}(k)) &= \tau wt_\phi(\mathbb{G}_2^{(r)}(k)) + 4(1-\tau) = \frac{\tau}{2} (29(m-1) + 43) + 4(1-\tau) = \frac{\tau}{2} (29(m-1) + 35) + 3, \\ wt_\phi(\mathbb{G}_3^{(r)}(k)) &= \tau wt_\phi(\mathbb{G}_3^{(r)}(k)) + 5(1-\tau) = \frac{\tau}{2} (29(m-1) + 43) + 5(1-\tau) = \frac{\tau}{2} (29(m-1) + 35) + 5, \\ wt_\phi(\mathbb{G}_4^{(r)}(k)) &= \tau wt_\phi(\mathbb{G}_4^{(r)}(k)) + 6(1-\tau) = \frac{\tau}{2} (29(m-1) + 43) + 6(1-\tau) - k = \frac{\tau}{2} (29(m-1) + 35) + 6 - k. \end{aligned} \quad (19)$$

It is easy to see that the set of all $\mathbb{G}_i^{(r)}(k)$ -weights for $i = 1, 2, 3$ consists of the same integers and $\mathbb{G}_4$ -weight consists of consecutive integers in $\cup_{k=1}^r Amal(L_m, F_m, P_m)(k)$. Thus, the graph $\cup_{k=1}^r Amal(L_m, F_m, P_m)$ is a $\mathbb{G}_i$, $i = 1, 2, 3$ -supermagic and super $(a, 1)$ - $\mathbb{G}_4$ -antimagic. This completes the proof. $\square$

Data Availability

No such data were used in preparing the present manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

The research was supported by the National Natural Science Foundation of China (Grant nos. 11971142, 11871202, 61673169, 11701176, 11626101, and 11601485).

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