Research Article

An Inverse Problem for a Two-Dimensional Time-Fractional Sideways Heat Equation

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In this paper, we consider a two-dimensional (2D) time-fractional inverse diffusion problem which is severely ill-posed; i.e., the solution (if it exists) does not depend continuously on the data. A modified kernel method is presented for approximating the solution of this problem, and the convergence estimates are obtained based on both a priori choice and a posteriori choice of regularization parameters. The numerical examples illustrate the behavior of the proposed method.

1. Introduction

In the past decades, fractional differential equations have attracted wide attention. Various models using fractional partial differential equations have been successfully applied to describe a range of problems in mechanical engineering [1], viscoelasticity [2], electron transport [3], dissipation [4], heat conduction [5, 6], and high-frequency financial data [7].

The time-fractional diffusion equation is deduced by replacing the standard time derivative with a time-fractional derivative and can be used to describe the superdiffusion and subdiffusion phenomena [8–13]. The direct problems, i.e., initial value problems and initial boundary value problems, for the time-fractional diffusion equation have been studied extensively in recent years. For instance, a well-posedness analysis [14–18] and numerical methods and simulations [19–24] are typical examples of this kind of analysis.

However, in some practical situations, part of boundary data, or initial data, or diffusion coefficient, or source term, or the order of fractional derivative may not be given and we want to find them by additional measured data which will yield some inverse problems of the fractional diffusion equation. The early papers on inverse problems of time-fractional diffusion equation were provided by Murio in [25, 26] for solving the time-fractional diffusion equation by mollification methods. Cheng et al. [27] gave the uniqueness

in determining the parameter α and $p(x)$ by means of observation data at one end point. Liu and Yamamoto [28] considered a backward problem in time for a time-fractional partial diffusion equation in one-dimensional case. Zhang and Xu [29] studied an inverse source problem in the time-fractional diffusion equation and proved uniqueness for identifying a space-dependent source term by using analytic continuation and Laplace transform. Zheng and Wei [30] applied a new regularization method to solve an inverse problem for a time-fractional diffusion equation in a one-dimensional semi-infinite domain. Jin and Rundell [31] studied an inverse problem of recovering a spatially varying potential term in the 1D time-fractional diffusion equation from the flux measurements and proposed a quasi-Newton-type reconstruction algorithm. Recently, Wei and Wang [32] proposed a modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation. Wei et al. [33] identified a time-dependent source term in a multidimensional time-fractional diffusion equation from boundary Cauchy data.

As for time-fractional inverse diffusion problem for one-dimensional (1D) models, theoretical concepts and computational implementation have been discussed by many authors, and a number of solution methods have been proposed, i.e., spectral regularization method [34], iteration regularization method [35], optimal regularization method

[36], and a new dynamic method [37]. However, for the two-dimensional time-fractional inverse diffusion problem, few results are available. Xiong et al. [38] gave a conditional stability estimate for the inverse heat conduction problem in the 2D time-fractional heat equation and studied a dynamic spectral regularization method with numerical testification. However, error estimates were given in [38] by only choosing the regularization parameter by a priori choice rule.

In the present paper, we will consider the 2D time-fractional inverse diffusion problem by using a modified kernel method and a posteriori parameter choice rule is given. For a priori choice and a posteriori choice of the regularization parameter, we obtain the convergence estimates, respectively.

The rest of this paper is organized as follows. In Section 2, we describe the 2D time-fractional inverse diffusion problem and give an analysis on the ill-posedness of this problem. In Section 3, we propose a modified kernel method and prove the convergence estimates under a priori and a posteriori parameter choice rule, respectively. Section 4 is the numerical aspect of the proposed method. Finally, we give a brief conclusion in Section 5.

2. Description of the 2D Time-Fractional Inverse Diffusion Problem

We consider the following 2D time-fractional inverse diffusion problem:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - u_{xx} - u_{yy} = 0, \quad x > 0, y > 0, t > 0, \quad (1)$$

$$u(1, y, t) = g(y, t), \quad y \geq 0, t \geq 0, \quad (2)$$

with the corresponding measured data function $g^\delta(y, t)$ and initial and boundary conditions

$$u(x, y, 0) = 0, \quad x > 0, y \geq 0, \quad (3)$$

$$u(x, 0, t) = 0, \quad x > 0, t \geq 0, \quad (4)$$

$$u_y(x, 0, t) = 0, \quad x > 0, t \geq 0, \quad (5)$$

$$u(x, y, t)|_{x \rightarrow \infty} \text{ bounded}, \quad y > 0, t > 0, \quad (6)$$

where the time-fractional derivative $(\partial^\alpha u / \partial t^\alpha)$ is the Caputo fractional derivative of order α ($0 < \alpha \leq 1$) defined by (see [39])

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha}(x, y, t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, y, s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1, \\ \frac{\partial^\alpha u}{\partial t^\alpha}(x, y, t) &= \frac{\partial u(x, y, t)}{\partial t}, \quad \alpha = 1. \end{aligned} \quad (7)$$

Here, we wish to determine the temperature $u(x, y, t)$ for $0 \leq x < 1$ from the temperature measurements $g^\delta(y, t)$.

In order to apply the Fourier transform techniques, we extend the functions $u(x, \cdot, \cdot)$, $g(\cdot, \cdot)$, and $g^\delta(\cdot, \cdot)$ to the whole plane $-\infty < y < +\infty$ and $-\infty < t < +\infty$ by setting the functions to be zero for $y < 0$ or $t < 0$. We assume that these functions are in $L^2(\mathbb{R}^2)$ and use the corresponding L^2 -norm, defined as follows:

$$\|g\| = \left(\int_{\mathbb{R}^2} |g(y, t)|^2 dy dt \right)^{1/2}. \quad (8)$$

We also assume that the measured data function $g^\delta(y, t)$ satisfies

$$\|g^\delta - g\| \leq \delta, \quad (9)$$

where the constant $\delta > 0$ represents a bound on the measurement error.

Let

$$\widehat{g}(\xi, \eta) = \frac{1}{2\pi} \int_{\mathbb{R}^2} g(y, t) e^{-i(\xi y + \eta t)} dy dt, \quad \xi, \eta \in \mathbb{R}, \quad (10)$$

be the Fourier transform of a function $g(y, t)$. Taking the transform to equations (1)–(6) with respect to y and t , we can get the solution of equations (1)–(6) in the frequency domain (see [38]):

$$\widehat{u}(x, \xi, \eta) = e^{(1-x)\sqrt{(i\eta)^\alpha + \xi^2}} \widehat{g}(\xi, \eta), \quad (11)$$

where

$$(i\eta)^\alpha = |\eta|^\alpha \left(\cos\left(\frac{\alpha\pi}{2}\right) + i \operatorname{sign}(\eta) \sin\left(\frac{\alpha\pi}{2}\right) \right). \quad (12)$$

Denote γ as follows:

$$\gamma := \sqrt{(i\eta)^\alpha + \xi^2} = A + iB, \quad (13)$$

where

$$\begin{aligned} A &:= \sqrt{\frac{\sqrt{|\eta|^{2\alpha} \sin^2(\alpha\pi/2) + (\xi^2 + |\eta|^\alpha \cos(\alpha\pi/2))^2} + (\xi^2 + |\eta|^\alpha \cos(\alpha\pi/2))}{2}}, \\ B &:= \operatorname{sign}\left(\operatorname{sign}(\eta) |\eta|^\alpha \sin\left(\frac{\alpha\pi}{2}\right)\right) \\ &\quad \times \sqrt{\frac{\sqrt{|\eta|^{2\alpha} \sin^2(\alpha\pi/2) + (\xi^2 + |\eta|^\alpha \cos(\alpha\pi/2))^2} - (\xi^2 + |\eta|^\alpha \cos(\alpha\pi/2))}{2}}. \end{aligned} \quad (14)$$

For the above problem, since $|e^{(1-x)\sqrt{(i\eta)^a + \xi^2}}|$ is unbounded with respect to variables ξ and η for fixed $0 \leq x < 1$, the small error in the high-frequency components will be amplified. Therefore, the 2D time-fractional inverse diffusion problem is severely ill-posed. To solve the 2D time-fractional inverse diffusion problem, a natural way to stabilize the problem is eliminate the high frequencies or to replace the “kernel” $e^{(1-x)\sqrt{(i\eta)^a + \xi^2}}$ by a bounded approximation.

3. A Modified Method and Convergence Estimates

In this section, we will give a modified “kernel” method and obtain the convergence estimates. The regularization solution is given by

$$\hat{u}_\beta^\delta(x, \xi, \eta) = k_\beta(x, \xi, \eta) \hat{g}^\delta(\xi, \eta), \quad (15)$$

where

$$k_\beta(x, \xi, \eta) = \begin{cases} e^{(1-x)(A+iB)}, & e^{(1-x)A} \leq \beta(x), \\ \beta(x)e^{i(1-x)B}, & e^{(1-x)A} > \beta(x), \end{cases} \quad (16)$$

where $\beta(x) > 1$ is the regularization parameter.

To obtain the convergence estimates between the regularization solution and the exact solution, we need to assume a priori bound:

$$\|u(0, \cdot, \cdot)\|_p \leq E, \quad p > 0, \quad (17)$$

where $E > 0$ is a constant and $\|\cdot\|_p$ denotes the norm in Sobolov space $H^p(\mathbb{R}^2)$ defined by

$$\|u(x, \cdot, \cdot)\|_p := \left(\int_{\mathbb{R}^2} (1 + \xi^2 + \eta^2)^p |\hat{u}(0, \cdot, \cdot)|^2 d\xi d\eta \right)^{1/2}. \quad (18)$$

Remark 1. Obviously, when $p = 0$, we can know that $H^p(\mathbb{R}^2) = H^0(\mathbb{R}^2) = L^2(\mathbb{R}^2)$ and formula (17) is bounded in the $L^2(\mathbb{R}^2)$ -norm.

3.1. A Priori Parameter Choice. In the following, we give the convergence estimate for $\|u_\beta^\delta(x, \cdot, \cdot) - u(x, \cdot, \cdot)\|$ by using a priori choice rule for the regularization parameter.

Theorem 1. Suppose that $u_\beta^\delta(x, y, t)$ is the regularization solution with noisy data $g^\delta(y, t)$ and that $u(x, y, t)$ is the exact solution with the exact data $g(y, t)$. Let assumption (9) be satisfied and let $\|u(0, \cdot, \cdot)\| \leq E$. If we choose

$$\beta(x) = x \left(\frac{\delta}{E} \right)^{x-1}, \quad (19)$$

then for every $x \in (0, 1)$, we obtain the error estimate

$$\|u_\beta^\delta(x, \cdot, \cdot) - u(x, \cdot, \cdot)\| \leq \delta^x E^{1-x}. \quad (20)$$

Proof. Due to Parseval's identity and triangle inequality, we can obtain

$$\begin{aligned} & \|u_\beta^\delta(x, \cdot, \cdot) - u(x, \cdot, \cdot)\| \\ &= \|\hat{u}_\beta^\delta(x, \cdot, \cdot) - \hat{u}(x, \cdot, \cdot)\| \\ &= \|k_\beta(x, \xi, \eta) \hat{g}^\delta - e^{(1-x)(A+iB)} \hat{g}\| \\ &= \|k_\beta(x, \xi, \eta) \hat{g}^\delta - k_\beta(x, \xi, \eta) \hat{g} + k_\beta(x, \xi, \eta) \hat{g} - e^{(1-x)(A+iB)} \hat{g}\| \\ &\leq \|k_\beta(x, \xi, \eta) \hat{g}^\delta - k_\beta(x, \xi, \eta) \hat{g}\| + \|k_\beta(x, \xi, \eta) \hat{g} - e^{(1-x)(A+iB)} \hat{g}\| \\ &= I_1 + I_2. \end{aligned} \quad (21)$$

For I_1 , we have

$$I_1 = \|k_\beta(x, \xi, \eta) \hat{g}^\delta - k_\beta(x, \xi, \eta) \hat{g}\| \leq \delta \sup_{\xi, \eta \in \mathbb{R}} |k_\beta(x, \xi, \eta)| \leq \delta \beta(x). \quad (22)$$

From (11), we know

$$\hat{g}(\xi, \eta) = e^{-(A+iB)} \hat{u}(0, \xi, \eta). \quad (23)$$

We now estimate I_2 ; note that from (23),

$$\begin{aligned} I_2 &= \|k_\beta(x, \xi, \eta) \hat{g} - e^{(1-x)(A+iB)} \hat{g}\| \\ &= \left\| \frac{e^{(1-x)(A+iB)} - k_\beta(x, \xi, \eta)}{e^{(A+iB)}} \hat{u}(0, \xi, \eta) \right\| \\ &= \left\| \frac{e^{(1-x)(A+iB)} - \min\{e^{(1-x)A}, \beta(x)\} e^{(1-x)iB}}{e^{(A+iB)}} \hat{u}(0, \xi, \eta) \right\| \\ &= \left\| \frac{(e^{(1-x)A} - \min\{e^{(1-x)A}, \beta(x)\}) e^{(1-x)iB}}{e^{(A+iB)}} \hat{u}(0, \xi, \eta) \right\| \\ &\leq \sup_{\xi, \eta \in \mathbb{R}} \left| \frac{(e^{(1-x)A} - \min\{e^{(1-x)A}, \beta(x)\}) e^{(1-x)iB}}{e^{(A+iB)}} \right| E \\ &\leq E \sup_{\xi, \eta \in \mathbb{R}, e^{(1-x)A} > \beta(x)} \frac{e^{(1-x)A} - \beta(x)}{e^A}. \end{aligned} \quad (24)$$

Let

$$\varphi(A) = \frac{e^{(1-x)A} - \beta(x)}{e^A}. \quad (25)$$

Differentiating $\varphi(A)$ and setting the derivative equal to zero, we find that

$$A^* = \frac{1}{1-x} \ln\left(\frac{1}{x} \beta(x)\right), \quad (26)$$

is the maximum value point of the function $\varphi(A)$. Therefore, we know

$$I_2 \leq \frac{((1/x) - 1) \beta(x)}{((1/x) \beta(x))^{1/(1-x)}} E. \quad (27)$$

Combining (19), (21), (22), and (27), we have the Hölder-type error estimate (20).

Remark 2. The error estimate in Theorem 1 does not give any useful information on the continuous dependence of the solution at $x = 0$. To retain the continuous dependence of the solution at $x = 0$, one has to introduce a stronger a priori assumption.

Theorem 2. Suppose that $u_\beta^\delta(0, y, t)$ is the regularization solution with noisy data $g^\delta(y, t)$ and that $u(0, y, t)$ is the exact solution with the exact data $g(y, t)$ at $x = 0$. Let assumption (9) be satisfied and let $\|u(0, \cdot, \cdot)\|_p \leq E$ ($p > 0$). If we choose

$$\beta = \left(\frac{\delta}{E} \right)^{-(1/2)}, \quad (28)$$

then for $x = 0$, we obtain the error estimate

$$\|u_\beta^\delta(0, y, t) - u(0, y, t)\| \leq \delta^{1/2} E^{1/2} + E \left(\frac{1}{4} \ln \frac{E}{\delta} \right)^{-p}. \quad (29)$$

Proof. Due to Parseval's identity and the triangle inequality, we have

$$\begin{aligned} & \|u_\beta^\delta(0, y, t) - u(0, y, t)\| \\ &= \|\hat{u}_\beta^\delta(0, \cdot, \cdot) - \hat{u}(0, \cdot, \cdot)\| \\ &= \|k_\beta(0, \xi, \eta) \hat{g}^\delta - e^{(A+iB)} \hat{g}\| \\ &= \|k_\beta(0, \xi, \eta) \hat{g}^\delta - k_\beta(0, \xi, \eta) \hat{g} + k_\beta(0, \xi, \eta) \hat{g} - e^{(A+iB)} \hat{g}\| \\ &\leq \|k_\beta(0, \xi, \eta) \hat{g}^\delta - k_\beta(0, \xi, \eta) \hat{g}\| + \|k_\beta(0, \xi, \eta) \hat{g} - e^{(A+iB)} \hat{g}\| \\ &= I_3 + I_4, \end{aligned} \quad (30)$$

where

$$\begin{aligned} I_3 &= \|k_\beta(0, \xi, \eta) \hat{g}^\delta - k_\beta(0, \xi, \eta) \hat{g}\| \\ &= \|k_\beta(0, \xi, \eta) (g^\delta - g)\| \\ &\leq \delta \sup_{\xi, \eta \in \mathbb{R}} |k_\beta(0, \xi, \eta)| \\ &\leq \delta \beta, \\ I_4 &= \|k_\beta(0, \xi, \eta) \hat{g} - e^{(A+iB)} \hat{g}\| \\ &= \left\| \frac{e^{((A+iB))} - \min\{e^A, \beta\} e^{iB}}{e^{(A+iB)}} \hat{u}(0, \xi, \eta) \right\| \\ &= \left\| \frac{(e^A - \min\{e^A, \beta\}) e^{iB}}{(1 + \xi^2 + \eta^2)^{p/2} e^{(A+iB)}} (1 + \xi^2 + \eta^2)^{p/2} \hat{u}(0, \xi, \eta) \right\| \\ &\leq \sup_{\xi, \eta \in \mathbb{R}} \left| \frac{(e^A - \min\{e^A, \beta\}) e^{iB}}{(1 + \xi^2 + \eta^2)^{p/2} e^{(A+iB)}} \right| E \\ &\leq E \sup_{\xi, \eta \in \mathbb{R}, e^A > \beta} \frac{e^A - \beta}{(1 + \xi^2 + \eta^2)^{p/2} e^A}. \end{aligned} \quad (31)$$

Using the fact $|\gamma| = |\sqrt{(i\eta)^\alpha + \xi^2}| \leq |\eta|^{\alpha/2} + |\xi|$, we can get

$$A \leq \sqrt{A^2 + B^2} = |\gamma| \leq |\eta|^{\alpha/2} + |\xi|, \text{ i.e., } A \leq |\eta|^{\alpha/2} + |\xi|. \quad (33)$$

Note that also $\beta < e^A$, and we have $A > \ln \beta$; therefore, we can derive

$$\ln \beta \leq |\eta|^{\alpha/2} + |\xi|. \quad (34)$$

If $|\xi| > |\eta|^{\alpha/2} \geq 0$, we know that

$$\ln \beta \leq |\eta|^{\alpha/2} + |\xi| \leq 2|\xi|, \quad (35)$$

i.e., $|\xi| > (1/2) \ln \beta$. Therefore,

$$(1 + \xi^2 + \eta^2)^{-(p/2)} \leq (\xi^2 + \eta^2)^{-(p/2)} < |\xi|^{-p} < \left(\frac{1}{2} \ln \beta \right)^{-p}. \quad (36)$$

If $|\eta|^{\alpha/2} \geq |\xi| \geq 0$, we have

$$\ln \beta \leq |\eta|^{\alpha/2} + |\xi| \leq 2|\eta|^{\alpha/2}, \quad (37)$$

i.e., $|\eta| \geq ((1/2) \ln \beta)^{2/\alpha}$. Therefore,

$$(1 + \xi^2 + \eta^2)^{-(p/2)} \leq (\xi^2 + \eta^2)^{-(p/2)} \leq |\eta|^{-p} < \left(\frac{1}{2} \ln \beta \right)^{-((2p)/\alpha)}. \quad (38)$$

Combining (32), (36), and (38), we can obtain

$$\begin{aligned} I_4 &\leq E \sup_{\xi, \eta \in \mathbb{R}, e^A > \beta} \frac{1}{(1 + \xi^2 + \eta^2)^{p/2}} \leq E \max \\ &\cdot \left\{ \left(\frac{1}{2} \ln \beta \right)^{-p}, \left(\frac{1}{2} \ln \beta \right)^{-((2p)/\alpha)} \right\}. \end{aligned} \quad (39)$$

Combining (28), (30), (31), and (39), we can obtain

$$\begin{aligned} \|u_\beta^\delta(0, y, t) - u(0, y, t)\| &\leq I_3 + I_4 \\ &\leq \delta^{1/2} E^{1/2} + E \max \left\{ \left(\frac{1}{4} \ln \frac{E}{\delta} \right)^{-p}, \right. \\ &\quad \left. \left(\frac{1}{4} \ln \frac{E}{\delta} \right)^{-((2p)/\alpha)} \right\} \\ &= \delta^{1/2} E^{1/2} + E \left(\frac{1}{4} \ln \frac{E}{\delta} \right)^{-p}. \end{aligned} \quad (40)$$

□

3.2. A Posteriori Parameter Choice. In this section, we first give the following lemma.

Lemma 1 (see [40]). Let the function $f(\lambda): (0, a] \rightarrow \mathbb{R}$ be given by

$$f(\lambda) = \lambda^b \left[d \ln \frac{1}{\lambda} \right]^{-c}, \quad (41)$$

with a constant $c \in \mathbb{R}$ and positive constants $a < 1$, b , and d ; then, for the inverse function, we have

$$f^{-1}(\lambda) = \lambda^{1/b} \left[\frac{d}{b} \ln \frac{1}{\lambda} \right]^{c/b} (1 + o(1)), \quad \text{for } \lambda \rightarrow 0. \quad (42)$$

In this following, we give the convergence estimate for $\|u_\beta^\delta(x, \cdot, \cdot) - u(x, \cdot, \cdot)\|$ by using a posteriori choice rule for the regularization parameter, i.e., Morozov's discrepancy principle.

According to Morozov's discrepancy principle [41], we adopt the regularization parameter β as the solution of the following equation:

$$\|e^{-(A+iB)(1-x)} k_\beta(x, \xi, \eta) \hat{g}^\delta - \hat{g}^\delta\| = \tau \delta, \quad (43)$$

where $\tau > 1$ is a constant. Denote

$$T_\beta = e^{-(A+iB)(1-x)} k_\beta(x, \xi, \eta) - 1. \quad (44)$$

Here and in the following, β is a function of the variable x and sometimes the function $\beta(x)$ is also used.

Lemma 2. Let $\rho(\beta) := \|e^{-(A+iB)(1-x)} k_\beta(x, \xi, \eta) \hat{g}^\delta - \hat{g}^\delta\|$, and then, the following results hold:

- (a) $\rho(\beta)$ is a continuous function
- (b) $\lim_{\beta \rightarrow 1} \rho(\beta) = \|(e^{-A(1-x)} - 1) \hat{g}^\delta(\xi, \eta)\|$
- (c) $\lim_{\beta \rightarrow +\infty} \rho(\beta) = 0$
- (d) $\rho(\beta)$ is a strictly decreasing function for $\beta \in (1, +\infty)$

The proof is obvious, and we omit it here.

Remark 3. According to Lemma 2, we find that, if $0 < \tau \delta < (e^{-A(1-x)} - 1) \hat{g}^\delta(\xi, \eta)$, equation (43) has a unique solution.

Lemma 3. If $\beta(x)$ is the solution of equation (43), then the following inequality holds

$$\beta(x) \leq \left(\frac{(\tau - 1)\delta}{E} \right)^{x-1} \left(\frac{1}{2} \ln \frac{E}{(\tau - 1)\delta} \right)^{-p(1-x)} (1 + o(1)). \quad (45)$$

Proof. Due to the triangle inequality and (43), note that $|T_\beta| \leq 1$, and we obtain

$$\tau \delta = \|T_\beta \hat{g}^\delta\| \leq \|T_\beta(\hat{g}^\delta - \hat{g})\| + \|T_\beta \hat{g} - \hat{g}\| + \|T_\beta \hat{g}\|. \quad (46)$$

We now estimate the second term on the right hand side of (46):

$$\begin{aligned} T_\beta \hat{g} &= \left\| e^{-(A+iB)(1-x)} k_\beta(x, \xi, \eta) - 1 \right\| e^{-(A+iB)} \hat{u}(0, \cdot, \cdot) \\ &= \left\| \frac{1 - e^{-(A+iB)(1-x)} k_\beta(x, \xi, \eta)}{e^{A+iB} (1 + \xi^2 + \eta^2)^{p/2}} \hat{u}(0, \cdot, \cdot) \right\| \\ &\leq E \sup_{\xi, \eta \in \mathbb{R}, e^{A(1-x)} > \beta(x)} \left| \frac{1 - e^{-A(1-x)} \beta(x)}{e^A (1 + \xi^2 + \eta^2)^{p/2}} \right| \\ &= E \sup_{\xi, \eta \in \mathbb{R}, e^{A(1-x)} > \beta(x)} \left| \frac{e^{A(1-x)} - \beta(x)}{e^{A(2-x)} (1 + \xi^2 + \eta^2)^{p/2}} \right| \\ &= E \sup_{\xi, \eta \in \mathbb{R}, e^{A(1-x)} > \beta(x)} \frac{e^{A(1-x)} - \beta(x)}{e^{A(1-x)}} e^{-A} (1 + \xi^2 + \eta^2)^{-(p/2)} \\ &\leq E \sup_{\xi, \eta \in \mathbb{R}, e^{A(1-x)} > \beta(x)} e^{-A} (1 + \xi^2 + \eta^2)^{-(p/2)}. \end{aligned} \quad (47)$$

Similar to the proof of Theorem 2, we can know

$$\begin{aligned} \|T_\beta \hat{g}\| &\leq E \sup_{\xi, \eta \in \mathbb{R}, e^{A(1-x)} > \beta(x)} e^{-A} (1 + \xi^2 + \eta^2)^{-(p/2)} \\ &\leq E \beta(x)^{-(1/(1-x))} \max \left\{ \left(\frac{1}{2(1-x)} \ln \beta(x) \right)^{-p}, \right. \\ &\quad \left. \left(\frac{1}{2(1-x)} \ln \beta(x) \right)^{-(2p/\alpha)} \right\} \\ &= E \beta(x)^{-(1/(1-x))} \left(\frac{1}{2(1-x)} \ln \beta(x) \right)^{-p}. \end{aligned} \quad (48)$$

Combining (46) and (48), we obtain

$$(\tau - 1)\delta \leq E \beta(x)^{-(1/(1-x))} \left(\frac{1}{2(1-x)} \ln \beta(x) \right)^{-p}. \quad (49)$$

Let $(\tau - 1)\delta = E \beta_*(x)^{-(1/(1-x))} ((1/(2(1-x))) \ln \beta_*(x))^{-p}$. From Lemma 1, we have

$$\beta_*(x) = \left(\frac{(\tau - 1)\delta}{E} \right)^{x-1} \left(\frac{1}{2} \ln \frac{E}{(\tau - 1)\delta} \right)^{-p(1-x)} (1 + o(1)). \quad (50)$$

Let $G(\beta) := E\beta(x)^{-1/(1-x)}((1/(2(1-x)))\ln\beta(x))^{-p}$. For $\beta > 1$ and $0 \leq x < 1$, the function $G(\beta)$ is strictly decreasing monotonically with β , so we have

$$\beta(x) \leq \beta_*(x). \quad (51)$$

Lemma 4. If $\beta(x)$ is the solution of equation (43), then the following inequality holds:

$$\|T_\beta \hat{g}\| \leq (\tau + 1)\delta. \quad (52)$$

Proof. Using the triangle inequality and (43), we have

$$\begin{aligned} \|T_\beta \hat{g}\| &= \|T_\beta(\hat{g} - \hat{g}^\delta) + T_\beta \hat{g}^\delta\| \leq \|T_\beta(\hat{g} - \hat{g}^\delta)\| + \|T_\beta \hat{g}^\delta\| \\ &\leq (\tau + 1)\delta. \end{aligned} \quad (53)$$

Theorem 3. Assume that conditions (9) and (17) hold and taking the solution of equation (43) as the regularization parameter, then there holds the following error estimate:

$$\|u_\beta^\delta(x, \cdot, \cdot) - u(x, \cdot, \cdot)\| \leq \delta^x E^{1-x} \left(\frac{1}{2} \ln \frac{E}{(\tau - 1)\delta} \right)^{-p(1-x)} (C + o(1)), \quad (54)$$

where $C = (\tau - 1)^{x-1} (2\tau + 1)$, $0 \leq x < 1$.

Proof. Due to the Parseval formula and the triangle inequality, we obtain

$$\begin{aligned} \|u_\beta^\delta(x, \cdot, \cdot) - u(x, \cdot, \cdot)\| &= \|\hat{u}_\beta^\delta(x, \cdot, \cdot) - \hat{u}(x, \cdot, \cdot)\| \\ &= \|k_\beta \hat{g}^\delta - e^{(A+iB)(1-x)} \hat{g}\| \\ &\leq \|k_\beta(\hat{g}^\delta - \hat{g})\| + \|k_\beta - e^{(A+iB)(1-x)}\| \|\hat{g}\| \\ &\leq \beta(x)\delta + \|k_\beta e^{-(A+iB)(1-x)} - 1\| \|\hat{u}(x, \cdot, \cdot)\| \\ &= \beta(x)\delta + \|T_\beta \hat{u}(x, \cdot, \cdot)\|. \end{aligned} \quad (55)$$

From Lemma 3, we can know that

$$\beta(x)\delta \leq \left(\frac{E}{\tau - 1} \right)^{1-x} \delta^x \left(\frac{1}{2} \ln \frac{E}{(\tau - 1)\delta} \right)^{-p(1-x)} (1 + o(1)). \quad (56)$$

In the following, we estimate the second term of (55). Denote $\varphi = T_\beta \hat{u}(x, \cdot, \cdot)$ and take $\beta(x) = (E/((\tau - 1)\delta))^{1-x} ((1/2)\ln(E/((\tau - 1)\delta)))^{-p(1-x)}$; we have

$$\begin{aligned} \|\varphi\| &= \|\varphi - k_\beta e^{-(A+iB)(1-x)} \varphi + k_\beta e^{-(A+iB)(1-x)} \varphi\| \\ &\leq \|\varphi - k_\beta e^{-(A+iB)(1-x)} \varphi\| + \|k_\beta e^{-(A+iB)(1-x)} \varphi\| \\ &= M_1 + M_2. \end{aligned} \quad (57)$$

Note that $\hat{u}(x, \cdot, \cdot) = e^{(A+iB)(1-x)} \hat{g}(\cdot, \cdot)$, we get

$$\begin{aligned} M_1 &= \|T_\beta e^{(A+iB)(1-x)} \hat{g} (1 - k_\beta e^{-(A+iB)(1-x)})\| \\ &= \|T_\beta (e^{(A+iB)(1-x)} - k_\beta) \hat{g}\| \\ &\leq \|(e^{(A+iB)(1-x)} - k_\beta) \hat{g}\|. \end{aligned} \quad (58)$$

Denote $M := \|(e^{(A+iB)(1-x)} - k_\beta) \hat{g}\|$, we have

$$\begin{aligned} M &= \|(e^{(A+iB)(1-x)} - k_\beta) \hat{g}\| \leq E \sup_{\xi, \eta \in \mathbb{R}} \left| \frac{e^{(A+iB)(1-x)} - k_\beta}{e^{(A+iB)(1-x)} (1 + \xi^2 + \eta^2)^{p/2}} \right| \\ &\leq E \sup_{\xi, \eta \in \mathbb{R}} \left| \frac{e^{(A+iB)(1-x)} - \min\{e^{A(1-x)}, \beta(x)\} e^{iB(1-x)}}{e^{(A+iB)(1-x)} (1 + \xi^2 + \eta^2)^{p/2}} \right| \\ &\leq E \sup_{\xi, \eta \in \mathbb{R}, e^{A(1-x)} > \beta(x)} \frac{e^{A(1-x)} - \beta(x)}{e^{A(1-x)} (1 + \xi^2 + \eta^2)^{p/2}} \\ &= E \sup_{\xi, \eta \in \mathbb{R}, e^{A(1-x)} > \beta(x)} \frac{e^{A(1-x)} - \beta(x)}{e^{A(1-x)}} (1 + \xi^2 + \eta^2)^{-(p/2)} e^{-Ax} \\ &\leq E \sup_{\xi, \eta \in \mathbb{R}, e^{A(1-x)} > \beta(x)} (1 + \xi^2 + \eta^2)^{-(p/2)} e^{-Ax}. \end{aligned} \quad (59)$$

Similar to the proof of Theorem 2, we can know that

$$\begin{aligned}
M &\leq E \sup_{\xi, \eta \in \mathbb{R}, e^{A(1-x)} > \beta(x)} \left(1 + \xi^2 + \eta^2\right)^{-(p/2)} e^{-Ax} \\
&\leq E \beta(x)^{-(x/(1-x))} \max \left\{ \left(\frac{1}{2(1-x)} \ln \beta(x) \right)^{-p}, \left(\frac{1}{2(1-x)} \ln \beta(x) \right)^{-(2p/\alpha)} \right\} \\
&= E \beta(x)^{-(x/(1-x))} \left(\frac{1}{2(1-x)} \ln \beta(x) \right)^{-p} \\
&= E \left[\frac{1}{2(1-x)} \ln \left(\left(\frac{E}{(\tau-1)\delta} \right)^{1-x} \left(\frac{1}{2} \ln \frac{E}{(\tau-1)\delta} \right)^{-p(1-x)} \right) \right]^{-p} \\
&\quad \times \left[\left(\frac{E}{(\tau-1)\delta} \right)^{1-x} \left(\frac{1}{2} \ln \frac{E}{(\tau-1)\delta} \right)^{-p(1-x)} \right]^{-(x/(1-x))} \\
&= E \left[\frac{1}{2} \ln \frac{E}{(\tau-1)\delta} + \ln \left(\frac{1}{2} \ln \frac{E}{(\tau-1)\delta} \right)^{-(p/2)} \right]^{-p} \left(\frac{E}{(\tau-1)\delta} \right)^{-x} \left(\frac{1}{2} \ln \frac{E}{(\tau-1)\delta} \right)^{px} \\
&= E \left(\frac{1}{2} \ln \frac{E}{(\tau-1)\delta} \right)^{-p} \left[1 + \frac{\ln(1/2 \ln E / ((\tau-1)\delta)^{-(p/2)})}{1/2 \ln E / ((\tau-1)\delta)} \right]^{-p} \left(\frac{(\tau-1)\delta}{E} \right)^x \left(\frac{1}{2} \ln \frac{E}{(\tau-1)\delta} \right)^{px} \\
&= (\tau-1)^x \delta^x E^{1-x} \left(\frac{1}{2} \ln \frac{E}{(\tau-1)\delta} \right)^{-p(1-x)} (1 + o(1)).
\end{aligned} \tag{60}$$

For M_2 , from Lemma 4 and assuming that $|k_\beta| \leq \beta(x)$, we have

$$\begin{aligned}
M_2 &\leq \beta(x) \left\| e^{-(A+iB)(1-x)} \varphi \right\| \\
&= \beta(x) \left\| T_\beta \hat{g} \right\| \\
&\leq (\tau+1)(\tau-1)^{x-1} \delta^x E^{1-x} \left(\frac{1}{2} \ln \frac{E}{(\tau-1)\delta} \right)^{-p(1-x)}.
\end{aligned} \tag{61}$$

Combining (56)–(61) with (55), the proof of Theorem 3 is completed. $\square$

4. Numerical Aspect

In this section, we present two numerical examples intended to illustrate the behavior of the proposed method.

The numerical examples are constructed in the following way. First, we present the initial data $u(0, y, t) = f(y, t)$ of 2D time-fractional diffusion problem at $x = 0$ and computed the function $u(1, y, t) = g(y, t)$ by solving a direct problem, which is a well-posed problem. Then, we added a random distributed perturbation to the data function obtaining vector $g^\delta(y, t)$, i.e.,

$$g^\delta(y, t) = g(y, t) + \varepsilon \text{randn}(\text{size}(g(y, t))), \tag{62}$$

where

$$g(y, t) = (g(y_1, t_1), g(y_2, t_2), \dots, g(y_n, t_n))^T,$$

$$y_i = (i-1)\Delta y, \quad \Delta y = \frac{1}{n-1}, \quad i = 1, 2, \dots, n, \tag{63}$$

$$t_j = (j-1)\Delta t, \quad \Delta t = \frac{1}{n-1}, \quad j = 1, 2, \dots, n,$$

where δ indicates the error level of g , i.e.,

$$\delta := \sqrt{\frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n (g^\delta(y_i, t_j) - g(y_i, t_j))^2}. \tag{64}$$

In numerical implementations, we give the data $f(y, t)$ and sample at an equidistant grid in the domain $[0, 1] \times [0, 1]$ with 64×64 grid points. The function “randn(·)” generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$; “randn(size($g(y, t)$))” returns an array of random entries which has the same size as $g(y, t)$. Let RMS denotes the mean square for a sampled function $\varphi(\cdot, \cdot)$ which is defined by

$$\text{RMS}(\varphi) := \sqrt{\frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n (\varphi(y_i, t_j))^2}, \tag{65}$$

where n is the total number of test points. Similarly, we can define the mean square error (RMSE) between the computed data and the exact data.

Finally, we solved the 2D time-fractional inverse diffusion problem by the modified method.

Example 1. We consider a smooth function

FIGURE 1: The exact solution and its approximation solution at $\varepsilon = 0.001$ and $\alpha = 0.1$: (a) the exact solution at $x = 0.9$; (b) its approximation solution at $x = 0.9$.

$$u(0, y, t) = f(y, t) = 16 \sin(2\pi y) \sin(2\pi t). \quad (66)$$

Example 2. Let $\Omega = \{(y, t) \mid 0.1 \leq y \leq 0.3, 0.1 \leq t \leq 0.3\} \cup \{(y, t) \mid 0.6 \leq y \leq 0.8, 0.6 \leq t \leq 0.8\}$. Consider a nonsmooth function

$$f(y, t) = \begin{cases} 1, & \text{if } (y, t) \in \Omega, \\ 0, & \text{if else.} \end{cases} \quad (67)$$

Figures 1–3 are the comparison of the exact solution and the approximation solution for $\alpha = 0.1$ with noise level $\varepsilon = 0.001$ at different points $x = 0.9, 0.5$, and 0.1 under a priori regularization parameter choice rule with Example 1. Figures 4–6 are the comparison of the exact solution and the approximation solution for $\alpha = 0.1$ with noise level $\varepsilon = 0.0001$ at different points $x = 0.9, 0.5$, and 0.1 under a

posteriori regularization parameter choice rule with Example 1.

Table 1 shows the comparison of RMSE of Example 1 for different x with $\varepsilon = 0.0001$ and $\alpha = 0.1$. We find that the smaller the x is, the worse the computed approximation will be.

Table 2 shows the comparison of RMSE of Example 1 for different ε with $\alpha = 0.1$ at $x = 0.5$. The numerical error is decreasing as the level of noise becomes smaller.

Table 3 shows the comparison of RMSE of Example 1 for different α with $\varepsilon = 0.0001$ at $x = 0.5$. The numerical accuracy is stable to the fractional order α .

Figure 7 is the comparison of the exact solution and the approximation solution for $\alpha = 0.1$ with noise level $\varepsilon = 0.0001$ at $x = 0.4$ under both a priori and a posteriori regularization parameter choice rule with Example 2.

FIGURE 2: The exact solution and its approximation solution at $\varepsilon = 0.001$ and $\alpha = 0.1$: (a) the exact solution at $x = 0.5$; (b) its approximation solution at $x = 0.5$.

FIGURE 3: The exact solution and its approximation solution at $\varepsilon = 0.001$ and $\alpha = 0.1$: (a) the exact solution at $x = 0.1$; (b) its approximation solution at $x = 0.1$.

FIGURE 4: The exact solution and its approximation solution at $\varepsilon = 0.0001$ and $\alpha = 0.1$: (a) the exact solution at $x = 0.9$; (b) its approximation solution at $x = 0.9$.

FIGURE 5: The exact solution and its approximation solution at $\varepsilon = 0.0001$ and $\alpha = 0.1$: (a) the exact solution at $x = 0.5$; (b) its approximation solution at $x = 0.5$.

FIGURE 6: The exact solution and its approximation solution at $\varepsilon = 0.0001$ and $\alpha = 0.1$: (a) the exact solution at $x = 0.1$; (b) its approximation solution at $x = 0.1$.

TABLE 1: Error behavior of Example 1 for different x with $\alpha = 0.1$ and $\varepsilon = 0.0001$.

x	0.5	0.3	0.1
A priori choice rule: RMSE	4.59×10^{-2}	1.73×10^{-1}	6.37×10^{-1}
A posteriori choice rule: RMSE	4.55×10^{-2}	1.63×10^{-1}	5.88×10^{-1}

TABLE 2: Error behavior of Example 1 for different ε with $\alpha = 0.1$ at $x = 0.5$.

ε	0.01	0.001	0.0001
A priori choice rule: RMSE	9.22×10^{-2}	5.16×10^{-2}	4.59×10^{-2}
A posteriori choice rule: RMSE	8.94×10^{-2}	4.90×10^{-2}	4.55×10^{-2}

TABLE 3: Error behavior of Example 1 for different α with $\varepsilon = 0.0001$ at $x = 0.5$.

α	0.1	0.3	0.8
A priori choice rule: RMSE	4.59×10^{-2}	4.38×10^{-2}	3.89×10^{-2}
A posteriori choice rule: RMSE	4.55×10^{-2}	4.35×10^{-2}	3.88×10^{-2}

FIGURE 7: The exact solution and its approximation solution at $\varepsilon = 0.0001$ and $\alpha = 0.1$: (a) the exact solution at $x = 0.4$; (b) its approximation solution under a priori parameter choice rule at $x = 0.4$; (c) its approximation solution under a posteriori parameter choice rule at $x = 0.4$.

5. Conclusion

In this paper, we propose a modified method to solve the time-fractional inverse diffusion problem in the two-dimensional setting. For the choice of regularization parameter, we give not only a priori but also a posteriori rules. Moreover, under both a priori rule and a posteriori rule, we prove the error estimates from the viewpoint of theoretical analysis.

The numerical examples are presented to illustrate the validity and effectiveness of the proposed method. In general, $E > 0$ such that a priori condition (17) holds will not be known. But such a E has to be known if one wants to construct a priori parameter choice rule. We propose a posteriori parameter choice rule which is independent of E . So it is more implementable in practical application. Moreover, the numerical experiments show that a posteriori parameter choice rule also works well.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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