

Research Article

Effect of Nonlinear Baseline Length Constraint on Global Navigation Satellite System Compass: A Theoretical Analysis

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GNSS (global navigation satellite system) compass is a low-cost, high-precision, and temporally stable north-finding technique. While the nonlinear baseline length constraint is widely known to be important in ambiguity resolution of GNSS compass, its direct effect on yaw angle estimation is theoretically analyzed in this work. Four different methods are considered with different ways in which the length constraint is made use of as follows: one without considering the constraints, one with simple scaling, one with indirect statistical scaling, and one with direct statistical scaling. It is found that simple scaling does not have any effect on yaw estimation; indirect and direct statistical scalings are equivalent to each other with both being able to increase the precision. The analysis and the conclusion developed in this work can go in parallel for the case of the tilt angle estimation.

1. Introduction

Control of a vehicle often relies on measuring necessary parameters of the vehicle [1, 2]. The vehicle's attitude, especially the heading or yaw angle as one of the attitude components, is a vital one of these parameters in many control applications [3]. GNSS compass is a cost-effective method to provide heading information in real time. It is one of the high-precision short-baseline applications [4–6]. By high precision, it is meant that carrier-phase measurements are used in addition to code pseudo ranges. GNSS compass is also a special case of GNSS attitude determination (AD), which can further be viewed as a special case of GNSS antenna array applications [7–9]. In compass, only the yaw or heading of the complete three attitude elements is of interest. GNSS compass (also GNSS AD) is of low cost and is temporally stable and hence finds wide applications. Each piece of carrier-phase measurement can be viewed as high-precision version of pseudo range only when its integer ambiguity has been correctly fixed [10–13]. From this regard, ambiguity resolution (AR) is inevitable in GNSS compass as in other high-precision

applications [4, 14–16]. Though single-epoch AR is the most challengeable [17], we should do AR sequentially to fully explore the temporally constant property of the ambiguities [18], as long as cycle slips are absent or repaired in real time [19]. In this work, we are concerned with yaw angle estimation with carrier-phase measurements whose ambiguities have already been fixed at previous epochs. It is widely known that the length constraints of the baselines connecting antennas fixed on the vehicle's body should be given full consideration in the AR of GNSS compass or AD. The nonlinear length constraint enhances the GNSS compass AR model in terms of not only precision but also reliability [20]. However, in AD with vector measurements, the lengths of the vector measurements do not have an effect in general [3, 21]. The AD with vector measurement can be pointwise or sequential, namely, Wahba's problem [22–26] or the attitude filtering problem [27–32]. This is the reason why the vector measurements can be normalized for the sake of better numerical stability.

Does the nonlinear baseline length constraint have an effect on GNSS compass, as long as the ambiguity-fixed measurements are used? Or, how can the constraint have an

effect? If there is an effect, will this effect be positive, namely, with precision increased? In this work, these questions are studied in theory thoroughly and clear answers are given. Four different methods are analyzed with different ways in which the length constraint is considered. The first is called No-Constraint in which the length constraint is completely ignored. The second is called Simple-Scaling in which the baseline vector estimate is simply scaled to be with the known length. The third is called Indirect-Statistical-Scaling in which the baseline vector estimate is scaled statistically by taking the covariance of the estimate into consideration. The fourth is called Direct-Statistical-Scaling in which the baseline vector is estimated by solving a constrained least-squares problem. After briefly introducing the measurement model in Section 2, the four methods are analyzed in Section 3. The effects of the length constraints on the compassing, together with the relationships among them, are the focus of the analysis. This work is concluded in Section 4. Some derivations and proofs are presented as appendices.

2. GNSS Compass Measurement Model

It is shown in Figure 1 that a pair of antennas is rigidly mounted to a vehicle whose yaw or heading is to be determined. The baseline vector linking the antenna-pair is denoted as \mathbf{x} . Without loss of generality, the baseline vector is in the vehicle body's right-front plane. Then, the baseline vector defines the yaw angle and the tilt angle. We further assume that the baseline vector points to the front. Then, the tilt angle is exactly the pitch angle. The coordinate vector in the local ENU reference frame of the baseline is also denoted as \mathbf{x} , and let $\mathbf{x} = [a \ b \ c]^T$. This will not introduce any confusion, since only the coordinates in the reference frame are involved in this work. So, in the following by the baseline vector, we mean its coordinate vector in the reference frame. The yaw can be computed from the coordinates as follows:

$$\varphi = \arctan\left(\frac{a}{b}\right). \quad (1)$$

For nondedicated receivers [33], double difference carrier-phase measurements can be used to simplify the measurement model by eliminating both satellite and receiver clock errors [17, 34, 35]. Considering only the measurements with corresponding ambiguities fixed, the measurement model is as follows:

$$\mathbf{y} = \mathbf{B}\mathbf{x} + \boldsymbol{\varepsilon}, \quad (2)$$

with \mathbf{y} , \mathbf{B} , and $\mathbf{Q} = \text{cov}[\boldsymbol{\varepsilon}]$ known. Note that the elements of the measurement vector \mathbf{y} may not necessarily be the original double difference carrier phases, but rather some linear combinations of them. The corresponding combined ambiguities, e.g., the wide-lane ones, are relatively easy to be fixed; we assume they have already been fixed at certain previous epochs. Further assume the vehicle's body is rigid and hence we have the following length constraint:

$$\mathbf{x}^T \mathbf{x} = l^2, \quad (3)$$

with l known and time invariant. Note that this is the nonlinear and so-called hard constraint [36, 37]. The models

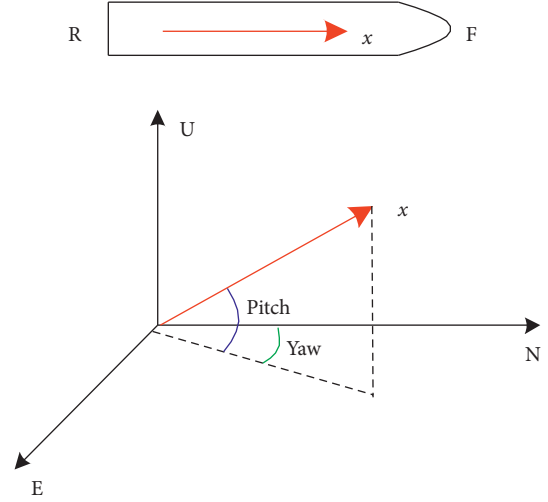


FIGURE 1: The baseline vector (red line) in the vehicle's body frame (top) and in the local reference frame (bottom). (\mathbf{x}) = baseline vector, R = rear, F = front, E = east, N = north, and U = upward.

(1)-(3) represent all the available information relevant to the compassing. Different GNSS compass methods are results of different ways in which the information is used. An important issue concerning these methods is whether the constraint in [3] is used and how it is used.

3. Effect of Baseline Length Constraint in Different Methods

In the following, the four different methods are presented, the effect of the baseline length constraint on the final solution of yaw angle is analyzed, and potential relations among different methods are revealed.

3.1. No-Constraint Method. The first method is called No-Constraint which ignores the constraint completely. It first estimates the baseline vector with least-squares:

$$\begin{aligned} \hat{\mathbf{x}}_1 &= [\hat{a}_1 \ \hat{b}_1 \ \hat{c}_1]^T, \\ &= (\mathbf{B}^T \mathbf{Q}^{-1} \mathbf{B})^{-1} \mathbf{B}^T \mathbf{Q}^{-1} \mathbf{y}, \end{aligned} \quad (4)$$

with the covariance of this estimate being $\mathbf{P}_1 = (\mathbf{B}^T \mathbf{Q}^{-1} \mathbf{B})^{-1}$. Then, the yaw angle with this estimate is calculated according to [1]. For easy reference, the formula is displayed as follows:

$$\hat{\varphi}_1 = \arctan\left(\frac{\mathbf{e}_1^T \hat{\mathbf{x}}_1}{\mathbf{e}_2^T \hat{\mathbf{x}}_1}\right). \quad (5)$$

with $\mathbf{e}_1 = [1 \ 0 \ 0]^T$ and $\mathbf{e}_2 = [0 \ 1 \ 0]^T$. The variance of this estimate can be worked according to the error propagation law:

$$\sigma_1^2 = \mathbf{g}^T \mathbf{P}_1 \mathbf{g}. \quad (6)$$

with $\mathbf{g}^T = ((\hat{b}_1 \mathbf{e}_1^T - \hat{a}_1 \mathbf{e}_2^T) / (\hat{a}_1^2 + \hat{b}_1^2))$. A derivation of [6] can be found in Appendix A. It is needless to say that the length constraints play no role in this method.

3.2. Simple-Scaling Method. The second method is carried out in steps as follows. First, it estimates the baseline vector as in [4]. Second, it scales the baseline vector as

$$\hat{\mathbf{x}}_2 = \frac{l\hat{\mathbf{x}}_1}{\sqrt{\hat{\mathbf{x}}_1^T \hat{\mathbf{x}}_1}} = k\hat{\mathbf{x}}_1. \quad (7)$$

This step is exactly the reason why it is called simple-scaling. Third, it calculates the yaw angle according to [1]: $\hat{\varphi}_2 = \arctan(\mathbf{e}_1^T \hat{\mathbf{x}}_2 / \mathbf{e}_2^T \hat{\mathbf{x}}_2) = \arctan(k\mathbf{e}_1^T \hat{\mathbf{x}}_1 / k\mathbf{e}_2^T \hat{\mathbf{x}}_1) = \hat{\varphi}_1$. So, as long as the yaw is to be estimated, the first and the second methods are the same, and it is readily known that their variances are also the same. The baseline length constraint does not have any effect on GNSS compass in this method, either. This may be reminiscent of the case of AD with vector measurements, namely, Wahba's problem, in which the specific length of the vector may be irrelevant to the AD.

As a final note, as long as the baseline vector, rather than the yaw, is of interest, the precision of the solution after simple scaling can be increased or decreased, or remain unchanged, depending on the length of the vector before scaling. For more information on this topic, the interested readers are referred to Appendix B.

3.3. Indirect-Statistical-Scaling Method. The third method is also a stepwise one as the second method. The difference from the second method lies in the second step. In this step, a constrained least-squares estimation is done with the estimate from [4] being treated as pseudo measurement with covariance \mathbf{P}_1 . The nonlinear constraint equation in [3] is linearized, say around the estimate in [4], as follows. Let $d = l^2 + \hat{\mathbf{x}}_1^T \hat{\mathbf{x}}_1$ and $c = 2\hat{\mathbf{x}}_1$; then, we have the linearized constraint as follows:

$$d = \mathbf{c}^T \mathbf{x}. \quad (8)$$

The resulting baseline vector estimate is defined as follows:

$$\hat{\mathbf{x}}_3 = \arg \min_x \eta_3 = \arg \min_x [(\mathbf{x} - \hat{\mathbf{x}}_1)^T \mathbf{P}_1^{-1} (\mathbf{x} - \hat{\mathbf{x}}_1) + 2\lambda(d - \mathbf{c}^T \mathbf{x})], \quad (9)$$

where λ denotes the Lagrange multiplier and η is called the Lagrangian. It turns out that the estimate defined in [9] is as follows:

$$\hat{\mathbf{x}}_3 = \mathbf{H}\hat{\mathbf{X}}_1 + d\mathbf{h}. \quad (10)$$

with $\mathbf{h} = (1/\mathbf{c}^T \mathbf{P}_1 \mathbf{c})\mathbf{P}_1 \mathbf{c}$ and $\mathbf{H} = (\mathbf{I}_3 - \mathbf{h}\mathbf{c}^T)$. A derivation of [10] can be found in Appendix C. The covariance is readily known as $\mathbf{P}_3 = \mathbf{H}\mathbf{P}_1\mathbf{H}^T$. We call [10] an Indirect-Statistical-Scaling for brevity. It is indirect because we first estimate the baseline vector without considering the constraint and then modify the estimate by considering the constraint. It is statistical because the statistical information, namely, the covariance \mathbf{P}_1 , is used in this scaling. The yaw is calculated using this estimate according to [1] and denoted as $\hat{\varphi}_3$.

It is clear that $\hat{\varphi}_3 \neq \hat{\varphi}_1$ in general. The question is can one of the two be always more accurate than the other? The answer is positive as proved in the following. Similar to [6],

TABLE 1: A summary of the five GNSS compass methods.

Method	Description	Effect
No-Constraint	[4] + [1]	None
Simple-Scaling	[4] + [7] + [1]	None
Indirect-Statistical-Scaling	[4] + [10] + [1]	Positive
Direct-Statistical-Scaling	[12] + [1]	Positive

we know that $\sigma_3^2 = \mathbf{g}^T \mathbf{P}_3 \mathbf{g}$. The following can be readily proved:

$$\Delta = \mathbf{P}_1 - \mathbf{P}_3 = (\mathbf{c}^T \mathbf{P}_1 \mathbf{c}) \mathbf{h} \mathbf{h}^T. \quad (11)$$

A derivation can be found in Appendix D. It is readily known that Δ is of rank one; furthermore, besides the zero double eigenvalue, the only nonzero eigenvalue is $(\mathbf{c}^T \mathbf{P}_1 \mathbf{c}) \mathbf{h}^T \mathbf{h} \geq 0$, because $\mathbf{c}^T \mathbf{P}_1 \mathbf{c} \geq 0$ and $\mathbf{h}^T \mathbf{h} \geq 0$. So the matrix Δ is positive semidefinite. So $\sigma_1^2 - \sigma_3^2 = \mathbf{g}^T \mathbf{P}_1 \mathbf{g} - \mathbf{g}^T \mathbf{P}_3 \mathbf{g} = \mathbf{g}^T (\mathbf{P}_1 - \mathbf{P}_3) \mathbf{g} = \mathbf{g}^T \Delta \mathbf{g} \geq 0$. It means that the precision of $\hat{\varphi}_3$ cannot be lower than $\hat{\varphi}_1$. Only when either of the following two conditions are fulfilled, the two are of equal precision: (1) $\Delta = 0$; (2) \mathbf{g} is one of the eigenvectors of Δ . In practice, the probability of either of the two conditions holding is zero. To summarize, with the indirect statistical scaling in the third method, the baseline length constraint has a positive effect in GNSS compassing, namely, that the precision can be improved by considering the length constraint.

3.4. Direct-Statistical-Scaling Method. The fourth method is a two-step one. It first estimates the baseline vector with constrained least-squares to consider the constraint in [3]. Then, it calculates the yaw using this estimate according to [1]. In the first step, the constrained least-squares solution to the baseline vector is defined as follows:

$$\hat{\mathbf{x}}_4 = \arg \min_x \eta_4 = \arg \min_x [(\mathbf{y} - \mathbf{B}\mathbf{x})^T \mathbf{Q}^{-1} (\mathbf{y} - \mathbf{B}\mathbf{x}) + 2\lambda(d - \mathbf{c}^T \mathbf{x})]. \quad (12)$$

It can be proved, as in Appendix E, that the baseline vector estimate defined in [12] is the same as the one in [10], and hence, the yaw, denoted as $\hat{\varphi}_4$, is the same as $\hat{\varphi}_3$. Their variances are also the same. So, the baseline length constraint has a positive effect on the GNSS compass in this method, namely, that by considering the length constraint, the precision of the yaw estimate can be improved.

3.5. Summary. The above analysis is summarized in Table 1. In a nutshell, (1) with the Simple-Scaling, the baseline length constraint does not have any effect on the GNSS compass; (2) the Indirect-Statistical-Scaling and the Direct-Statistical-Scaling can equivalently produce a positive effect of the baseline length constraint on the GNSS compass.

4. Conclusion

As long as the antennas are mounted rigidly to the vehicle's rigid body, the baseline length remains unchanged, independent of the vehicle's dynamics. This is a hard and

nonlinear constraint. In a GNSS compass, the yaw angle or heading of the vehicle can be determined with carrier-phase measurements whose ambiguities have been fixed at previous epochs. The question is answered in this work that whether the baseline length constraint has an effect on the yaw angle determined. In a nutshell, the answer is as follows: it depends on the specific method of considering the constraints. If we simply scale the estimated baseline vector to make its length be the true one, namely, to make the constraint fulfilled, the constraint does not have any effect on the yaw angle estimation. However, if the constraint is used through a statistical scaling, it can have a positive effect, namely, that the precision of the yaw estimation can be improved. The statistical scaling can be done indirectly or directly. In the indirect statistical scaling, the constraint is used after the baseline vector is estimated, whereas in the direct statistical scaling, the constraint is used in the baseline vector estimation. They are called statistical because the statistical information, namely, the covariance, is used in both of them. The two statistical scaling methods are equivalent to each other, namely, producing the same yaw estimate with the same variance of this estimation.

As a final note, the analysis and the conclusion developed in this work go in parallel for the case of tilt angle. Depending on the configuration of the baseline vector, this tilt angle can be pitch or roll angle.

Appendix

A. Derivation of [6]

According to the train rule of the derivative, the Jacobian (derivative) of $\hat{\varphi}_1$ with respect to $\hat{\mathbf{x}}_1$ is as follows:

$$\begin{aligned} \frac{\partial \hat{\varphi}_1}{\partial \hat{\mathbf{x}}_1^T} &= \frac{\partial \arctan(\mathbf{e}_1^T \hat{\mathbf{x}}_1 / \mathbf{e}_2^T \hat{\mathbf{x}}_1)}{\partial \hat{\mathbf{x}}_1^T}, \\ &= \frac{\partial \arctan(\mathbf{e}_1^T \hat{\mathbf{x}}_1 / \mathbf{e}_2^T \hat{\mathbf{x}}_1)}{\partial (\mathbf{e}_1^T \hat{\mathbf{x}}_1 / \mathbf{e}_2^T \hat{\mathbf{x}}_1)} \left[\frac{\partial (\mathbf{e}_1^T \hat{\mathbf{x}}_1 / \mathbf{e}_2^T \hat{\mathbf{x}}_1)}{\partial (\mathbf{e}_1^T \hat{\mathbf{x}}_1)} \frac{\partial (\mathbf{e}_1^T \hat{\mathbf{x}}_1)}{\partial \hat{\mathbf{x}}_1^T} \right. \\ &\quad \left. + \frac{\partial (\mathbf{e}_1^T \hat{\mathbf{x}}_1 / \mathbf{e}_2^T \hat{\mathbf{x}}_1)}{\partial (\mathbf{e}_2^T \hat{\mathbf{x}}_1)} \frac{\partial (\mathbf{e}_2^T \hat{\mathbf{x}}_1)}{\partial \hat{\mathbf{x}}_1^T} \right], \end{aligned} \quad (\text{A.1})$$

$$= \frac{1}{1 + (\mathbf{e}_1^T \hat{\mathbf{x}}_1 / \mathbf{e}_2^T \hat{\mathbf{x}}_1)^2} \left[\frac{1}{\mathbf{e}_2^T \hat{\mathbf{x}}_1} \mathbf{e}_1^T + \frac{-\mathbf{e}_1^T \hat{\mathbf{x}}_1}{(\mathbf{e}_2^T \hat{\mathbf{x}}_1)^2} \mathbf{e}_2^T \right].$$

Substituting $\begin{cases} \hat{a}_1 = \mathbf{e}_1^T \hat{\mathbf{x}}_1 \\ \hat{b}_1 = \mathbf{e}_2^T \hat{\mathbf{x}}_1 \end{cases}$ into the above, we have

$$\frac{\partial \hat{\varphi}_1}{\partial \hat{\mathbf{x}}_1^T} = \frac{1}{1 + (\hat{a}_1 / \hat{b}_1)^2} \left[\frac{1}{\hat{b}_1} \mathbf{e}_1^T + \frac{-\hat{a}_1}{(\hat{b}_1)^2} \mathbf{e}_2^T \right] = \frac{\hat{b}_1 \mathbf{e}_1^T - \hat{a}_1 \mathbf{e}_2^T}{\hat{a}_1^2 + \hat{b}_1^2}. \quad (\text{A.2})$$

which is exactly \mathbf{g}^T used in [6]. Note that [6] is obtained simply through error or covariance propagation.

B. Effect of Simple-Scaling on the Precision of Baseline Vector Estimation

According to the simple scaling formula, namely, $\hat{\mathbf{x}}_2 = k \hat{\mathbf{x}}_1$ with $k = l / \sqrt{\hat{\mathbf{x}}_1^T \hat{\mathbf{x}}_1}$, we have the following covariances for $\hat{\mathbf{x}}_2$:

$$\mathbf{P}_2 = k^2 \mathbf{P}_1. \quad (\text{B.1})$$

So, readily we have the following:

$$\begin{cases} \mathbf{P}_2 > \mathbf{P}_1, & \text{when } k > 1, \\ \mathbf{P}_2 = \mathbf{P}_1, & \text{when } k = 1, \\ \mathbf{P}_2 < \mathbf{P}_1, & \text{when } k < 1. \end{cases} \quad (\text{B.2})$$

When we say $\mathbf{P}_2 > \mathbf{P}_1$, we say that $\mathbf{P}_2 - \mathbf{P}_1$ is positive definite and the ‘‘smaller than’’ case goes similarly. From [16], we know that the simple scaling does not necessarily increase the precision of the baseline vector estimation. To be more specific, when the length of the unscaled baseline vector estimate is overly estimated, the simple scaling can even decrease the precision.

C. Derivation of [10]

In order for the Lagrangian to be minimum, its first-order derivative with respect to \mathbf{x} should be zero [38–40], namely,

$$\frac{\partial \eta_3}{\partial \mathbf{x}} = 0 \Leftrightarrow \mathbf{P}_1^{-1} \mathbf{x} = \mathbf{P}_1^{-1} \hat{\mathbf{x}}_1 + \lambda \mathbf{c}. \quad (\text{C.1})$$

The above is equivalent to the following:

$$\mathbf{x} = \hat{\mathbf{x}}_1 + \lambda \mathbf{P}_1 \mathbf{c}. \quad (\text{C.2})$$

Substitute this expression into the length constraint; we can compute the Lagrange multiplier, as follows:

$$\lambda = \frac{d - \mathbf{c}^T \hat{\mathbf{x}}_1}{\mathbf{c}^T \mathbf{P}_1 \mathbf{c}}. \quad (\text{C.3})$$

Substituting (C.3) into (C.2), we have the following estimate:

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_1 + \mathbf{P}_1 \mathbf{c} \frac{d - \mathbf{c}^T \hat{\mathbf{x}}_1}{\mathbf{c}^T \mathbf{P}_1 \mathbf{c}} = \left(\mathbf{I}_3 - \frac{1}{\mathbf{c}^T \mathbf{P}_1 \mathbf{c}} \mathbf{P}_1 \mathbf{c} \mathbf{c}^T \right) \hat{\mathbf{x}}_1 + d \frac{1}{\mathbf{c}^T \mathbf{P}_1 \mathbf{c}} \mathbf{P}_1 \mathbf{c}. \quad (\text{C.4})$$

The rightmost expression is exactly that in [10].

D. Derivation of [11]

First, the expression of \mathbf{P}_3 is expanded as follows:

$$\begin{aligned} \mathbf{P}_3 &= \mathbf{H} \mathbf{P}_1 \mathbf{H}^T, \\ &= \mathbf{P}_1 - \frac{2}{\mathbf{c}^T \mathbf{P}_1 \mathbf{c}} \mathbf{P}_1 \mathbf{c} \mathbf{c}^T \mathbf{P}_1 + \frac{1}{(\mathbf{c}^T \mathbf{P}_1 \mathbf{c})^2} \mathbf{P}_1 \mathbf{c} \mathbf{c}^T \mathbf{P}_1 \mathbf{c} \mathbf{c}^T \mathbf{P}_1, \\ &= \mathbf{P}_1 - \frac{2}{\mathbf{c}^T \mathbf{P}_1 \mathbf{c}} \mathbf{P}_1 \mathbf{c} \mathbf{c}^T \mathbf{P}_1 + \frac{1}{\mathbf{c}^T \mathbf{P}_1 \mathbf{c}} \mathbf{P}_1 \mathbf{c} \mathbf{c}^T \mathbf{P}_1, \\ &= \mathbf{P}_1 - \frac{1}{\mathbf{c}^T \mathbf{P}_1 \mathbf{c}} \mathbf{P}_1 \mathbf{c} \mathbf{c}^T \mathbf{P}_1. \end{aligned} \quad (\text{D.1})$$

So, readily we have

$$\begin{aligned}
\Delta &= \mathbf{P}_1 - \mathbf{P}_3, \\
&= \frac{1}{\mathbf{c}^T \mathbf{P}_1 \mathbf{c}} \mathbf{P}_1 \mathbf{c} \mathbf{c}^T \mathbf{P}_1, \\
&= (\mathbf{c}^T \mathbf{P}_1 \mathbf{c}) \mathbf{h} \mathbf{h}^T.
\end{aligned} \tag{D.2}$$

Furthermore, we have

$$(\mathbf{c}^T \mathbf{P}_1 \mathbf{c}) \mathbf{h} \mathbf{h}^T = \frac{1}{\mathbf{c}^T \mathbf{P}_1 \mathbf{c}} \mathbf{P}_1 \mathbf{c} \mathbf{c}^T \mathbf{P}_1. \tag{D.3}$$

With (D.2) and (D.3), we finally have [11].

E. Equivalence between [12] and [10]

Instead of directly working out the solution of [12], we will equivalently prove that the minimizer of η_3 can also minimize η_4 . We first rearrange η_3 and η_4 as follows:

$$\begin{aligned}
\eta_3 &= \mathbf{x}^T \mathbf{P}_1^{-1} \mathbf{x} - 2\mathbf{x}^T \mathbf{P}_1^{-1} \hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_1^T \mathbf{P}_1^{-1} \hat{\mathbf{x}}_1 + 2\lambda(d - \mathbf{c}^T \mathbf{x}), \\
&\propto \mathbf{x}^T \mathbf{P}_1^{-1} \mathbf{x} - 2\mathbf{x}^T \mathbf{P}_1^{-1} \hat{\mathbf{x}}_1 + 2\lambda(d - \mathbf{c}^T \mathbf{x}), \\
&= \mathbf{x}^T \mathbf{B}^T \mathbf{Q}^{-1} \mathbf{B} \mathbf{x} - 2\mathbf{x}^T \mathbf{B}^T \mathbf{Q}^{-1} \mathbf{B} (\mathbf{B}^T \mathbf{Q}^{-1} \mathbf{B})^{-1} \mathbf{B}^T \mathbf{Q}^{-1} \mathbf{y} \\
&\quad + 2\lambda(d - \mathbf{c}^T \mathbf{x}), \\
&= \mathbf{x}^T \mathbf{B}^T \mathbf{Q}^{-1} \mathbf{B} \mathbf{x} - 2\mathbf{x}^T \mathbf{B}^T \mathbf{Q}^{-1} \mathbf{y} + 2\lambda(d - \mathbf{c}^T \mathbf{x}), \\
&= \xi,
\end{aligned} \tag{E.1}$$

$$\begin{aligned}
\eta_4 &= \mathbf{x}^T \mathbf{B}^T \mathbf{Q}^{-1} \mathbf{B} \mathbf{x} - 2\mathbf{x}^T \mathbf{B}^T \mathbf{Q}^{-1} \mathbf{y} + \mathbf{y}^T \mathbf{Q}^{-1} \mathbf{y} + 2\lambda(d - \mathbf{c}^T \mathbf{x}), \\
&\propto \mathbf{x}^T \mathbf{B}^T \mathbf{Q}^{-1} \mathbf{B} \mathbf{x} - 2\mathbf{x}^T \mathbf{B}^T \mathbf{Q}^{-1} \mathbf{y} + 2\lambda(d - \mathbf{c}^T \mathbf{x}), \\
&= \xi.
\end{aligned} \tag{E.2}$$

In both of the above, uninteresting additive terms independent of \mathbf{x} and λ are omitted. So, the two Lagrangians are the same, after omitting different uninteresting additive constants which do not depend on the unknowns including the Lagrangian multipliers. So, the minimizer of one of the two will also minimize the other. This means that the two solutions are the same to each other.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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