Research Article

A Note on the Appell Hypergeometric Matrix Function F_2

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In this article, we introduce some of the mathematical properties of the second Appell hypergeometric matrix function $F_2(A, B_1, B_2, C_1, C_2; z, w)$ including integral representations, transformation formulas, and series formulas.

1. Introduction

Appell defined and studied in [1–3] four kinds of double series of two variables z, w as generalizations of the hypergeometric series:

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n (1)_n} z^n, \quad (1)$$

where z is a main variable in the unit disk $\{z \in \mathbb{C}: |z| < 1\}$, α, β, γ are complex parameters with $\gamma \neq 0, -1, -2, -3, \dots$, and $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)$ ($n \in \mathbb{N}$) and $(\alpha)_0 = 1$. Here and throughout, let $\mathbb{C}$ and $\mathbb{N}$ denote the sets of complex numbers and positive integers, respectively, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Appell hypergeometric functions F_s , $s = \{1, 2, 3, 4\}$ play an important role in mathematical physics in which broad practical applications can be found (see, e.g. [1, 3–7]). In particular, the Appell hypergeometric series F_2 arises frequently in various physical and chemical applications ([8–11]). The exact solutions of number of problems in quantum mechanics have been given [6, 7, 9, 12] in terms of Appell's function F_2 . For readers, they can find some results of the classical second Appell hypergeometric function F_2 in [13–17].

On the other hand, many authors [18–25] generalized the hypergeometric series $F(\alpha, \beta, \gamma; z)$ by extending parameters α, β , and γ to square matrices A, B , and C in the

complex space $\mathbb{C}^{d \times d}$. Recently, the extension of the classical Appell hypergeometric functions F_s , $s = \{1, 2, 3, 4\}$, to the Appell hypergeometric matrix functions has been a subject of intensive studies [26–30]. The purpose of the present work is to study the second Appell hypergeometric matrix function $F_2(A, B_1, B_2, C_1, C_2; z, w)$ on the domain $\{(z, w) \in \mathbb{C}^2: |z| + |w| < 1\}$, with square matrix valued parameters A, B_1, B_2, C_1 , and C_2 in $\mathbb{C}^{d \times d}$. We investigate some of the mathematical properties of this matrix function and introduce new integral representations, transformation formulas, and summation formulas.

2. Some Known Definitions and Results

We begin with a brief review of some definitions and notations. A matrix E is a positive stable matrix in $\mathbb{C}^{d \times d}$ if $\operatorname{Re}(\lambda) > 0$ for all $\lambda \in \sigma(E)$, where $\sigma(E)$ is the set of all eigenvalues of E . I and $\mathbf{0}$ stand for the identity matrix and the null matrix in $\mathbb{C}^{d \times d}$, respectively.

If $\Phi(z)$ and $\Psi(z)$ are holomorphic functions of the complex variable z , which are defined in an open set Ω of the complex plane and E is a matrix in $\mathbb{C}^{d \times d}$ such that $\sigma(E) \subset \Omega$; then, from the properties of the matrix functional calculus [28], it follows that

$$\Phi(E)\Psi(E) = \Psi(E)\Phi(E). \quad (2)$$

Hence, if F in $\mathbb{C}^{d \times d}$ is a matrix for which $\sigma(F) \subset \Omega$ and also if $EF = FE$, then

$$\Phi(E)\Psi(F) = \Psi(F)\Phi(E). \quad (3)$$

By application of the matrix functional calculus, for E in $\mathbb{C}^{d \times d}$, then from [23, 31], the Pochhammer symbol or shifted factorial defined by

$$(E)_n = \begin{cases} E(E+I) \dots (E+(n-1)I) = \Gamma^{-1}(E)\Gamma(E+nI), & n \in \mathbb{N}, \\ I, & n = 0, \end{cases} \quad (4)$$

with the condition

$$E + nI \text{ is invertible for all integers } n \in \mathbb{N}_0. \quad (5)$$

From (5), it is easy to find that

$$(E)_{n-k} = (-1)^k (E)_n [(I - E - nI)_k]^{-1}; \quad 0 \leq k \leq n, \quad (6)$$

$$(E)_{m+n} = (E)_n (E + nI)_m \text{ or } (E)_{m+n} = (E)_m (E + mI)_n. \quad (7)$$

From [28], one obtains

$$\frac{(-1)^k}{(n-k)!} I = \frac{(-n)_k}{n!} I = \frac{(-nI)_k}{n!}; \quad 0 \leq k \leq n. \quad (8)$$

Definition 1 (see [31]). If E is a matrix in $\mathbb{C}^{d \times d}$, such that $\operatorname{Re}(z) > 0$ for all eigenvalues z of E , then $\Gamma(E)$ is well defined as

$$\begin{aligned} \Gamma(E) &= \int_0^\infty \tau^{E-I} e^{-\tau} d\tau, \\ \tau^{E-I} &= \exp((E-I)\ln\tau). \end{aligned} \quad (9)$$

Definition 2 (see [31]). If E and F are positive stable matrices in $\mathbb{C}^{d \times d}$ and $EF = FE$, then the Beta matrix function is well defined by

$$\mathcal{B}(E, F) = \int_0^1 \tau^{E-I} (1-\tau)^{F-I} d\tau = \Gamma^{-1}(E+F)\Gamma(E)\Gamma(F). \quad (10)$$

Definition 3 (see [23]). Suppose that N_1 , N_2 , and N_3 are matrices in $\mathbb{C}^{d \times d}$, such that N_3 satisfies condition (5). Then,

the hypergeometric matrix function ${}_2F_1(N_1, N_2; N_3; z)$ is given by

$${}_2F_1(N_1, N_2; N_3; z) = \sum_{n \geq 0} \frac{(N_1)_n (N_2)_n [(N_3)_n]^{-1}}{n!} z^n. \quad (11)$$

Definition 4. If E is the positive stable matrix in $\mathbb{C}^{d \times d}$, then the Laguerre-type matrix polynomial is defined by [28]

$$\begin{aligned} L_n^E(z) &= \sum_{k=0}^n \frac{(-1)^k (z)^k}{k! (n-k)!} (E+I)_n (E+I)_k^{-1} \\ &= \frac{(E+I)_n}{n!} {}_1F_1(-nI; E+I; z), \quad n \in \mathbb{N}_0, \end{aligned} \quad (12)$$

where ${}_1F_1$ is the confluent hypergeometric matrix function (cf. [25]).

Definition 5 (see [28, 32, 33]). Let E and F be positive stable matrices in $\mathbb{C}^{d \times d}$, then the Jacobi matrix polynomial $\mathbb{P}_n^{(E,F)}(z)$ is defined by

$$\begin{aligned} \mathbb{P}_n^{(E,F)}(z) &= \sum_{k=0}^n \binom{n}{k} \Gamma^{-1}(E+F+(n+1)I) \Gamma^{-1}(F+(k+1)I) \\ &\quad \Gamma(F+(n+1)I) \Gamma(E+F+(n+k+1)I) \frac{(-1)^{n+k} (1+z)^k}{2^k n!}, \\ &= \frac{(E+I)_n}{n!} {}_2F_1\left(-nI, E+F+(n+1)I; E+I; \frac{1-z}{2}\right). \end{aligned} \quad (13)$$

Using (6) and (11), we can write the second kind of two complex variables Appell hypergeometric matrix function in the following definition (see [26, 28]).

Definition 6. Let A , B_1 , B_2 , C_1 , and C_2 be commutative matrices in $\mathbb{C}^{d \times d}$ with $C_1 + kI$ and $C_2 + kI$ being invertible for all integers $k \in \mathbb{N}_0$. Then, the second Appell hypergeometric matrix function $F_2(A, B_1, B_2, C_1, C_2; z, w)$ is defined in the following form:

$$\begin{aligned} F_2(A, B_1, B_2, C_1, C_2; z, w) &= \sum_{s_1, s_2=0}^{\infty} (A)_{s_1+s_2} (B_1)_{s_1} (B_2)_{s_2} [(C_1)_{s_1}]^{-1} [(C_2)_{s_2}]^{-1} \frac{z^{s_1} w^{s_2}}{s_1! s_2!} \\ &= \sum_{s_1=0}^{\infty} (A)_{s_1} (B_1)_{s_1} [(C_1)_{s_1}]^{-1} {}_2F_1(A+s_1I, B_2; C_2; w) \frac{z^{s_1}}{s_1!}, \quad (|z| + |w| < 1). \end{aligned} \quad (14)$$

3. Main Results

In this section, we investigate some of the main properties of the second Appell hypergeometric matrix function $F_2(A, B_1, B_2, C_1, C_2; z, w)$ such as integral representations, transformation formulas, and summation formulas

3.1. Integral Representations

Theorem 1. Let A , C_1 , and C_2 be positive stable matrices in $\mathbb{C}^{d \times d}$. Then, for $|z| + |w| < 1$, then the function $F_2(A, B_1, B_2, C_1, C_2; z, w)$ defined in (14) can be represented in the following integer forms:

$$F_2(A, B_1, B_2, C_1, C_2; z, w) = \Gamma^{-1}(A) \times \int_0^\infty u^{A-I} e^{-u} {}_1F_1(B_1; C_1; zu) {}_1F_1(B_2; C_2; wu) du, \quad (15)$$

$$F_2(A, -mI, -nI, C_1 + I, C_2 + I; z, w) = m!n! [(C_1 + I)_m]^{-1} [(C_2 + I)_n]^{-1} \Gamma^{-1}(A) \times \int_0^\infty u^{A-I} e^{-u} L_m^{C_1}(zu) L_n^{C_2}(wu) du. \quad (16)$$

Proof. Replacing the Pochhammer symbol $(A)_{m+n}$ in definition (14) by its integral representation which is obtained from (5) and (9), we get the desired result (15).

Using integral formula (15) and the relation given in (12), we have

$$F_2(A, -mI, -nI, C_1 + I, C_2 + I; z, w) = \Gamma^{-1}(A) \int_0^\infty u^{A-I} e^{-u} {}_1F_1(-mI; C_1 + I; zu) {}_1F_1(-nI; C_2 + I; wu) du \\ = m!n! [(C_1 + I)_m]^{-1} [(C_2 + I)_n]^{-1} \Gamma^{-1}(A) \times \int_0^\infty u^{A-I} e^{-u} L_m^{C_1}(zu) L_n^{C_2}(wu) du, \quad (17)$$

which completes proof relation (16). $\square$

Theorem 2. For the matrix function $F_2(A, B_1, B_2, C_1, C_2; z, w)$, we have the following transformations:

3.2. Transformation Formulas

$$F_2(A, B_1, B_2, C_1, C_2; z, w) = (1-z)^{-A} F_2\left(A, C_1 - B_1, B_2, C_1, C_2; \frac{-z}{1-z}, \frac{w}{1-w}\right), \quad (18)$$

$$F_2(A, B_1, B_2, C_1, C_2; z, w) = (1-w)^{-A} F_2\left(A, B_1, C_2 - B_2, C_1, C_2; \frac{z}{1-w}, \frac{w}{w-1}\right), \quad (19)$$

$$F_2(A, B_1, B_2, C_1, C_2; z, w) = (1-z-w)^{-A} F_2\left(A, C_1 - B_1, C_2 - B_2, C_1, C_2; \frac{-z}{1-z-w}, \frac{-w}{1-z-w}\right), \quad (20)$$

where A, B_1, B_2, C_1 , and C_2 are commutative matrices in $\mathbb{C}^{d \times d}$ with $C_1 + kI$ and $C_2 + kI$ being invertible for all integer $k \in \mathbb{N}_0$, and $B_1, B_2, C_1, C_2, C_1 - B_1$, and $C_2 - B_2$ are positively stable.

Proof. We will prove only (18) since the others can be proved similarly. Using matrix Kummer's first formula (cf. [8]),

$${}_1F_1(B; C; z) = e^z {}_1F_1(C - B; C; -z), \quad (21)$$

in (15), we have

$$F_2(A, B_1, B_2, C_1, C_2; z, w) = \Gamma^{-1}(A) \times \int_0^\infty u^{A-I} e^{-(1-z)u} {}_1F_1(C_1 - B_1; C_1; zu) {}_1F_1(B_2; C_2; wu) du. \quad (22)$$

Substituting $t = (1-z)u$ into (22), we obtain formula (18).

Now, connections with the Gauss hypergeometric matrix function is considered by the following theorem: $\square$

Theorem 3. Let $F_2(A, B, B', C_1, C_2; z, w)$ be given in (14). The following formulas hold true:

$$F_2(A, B_1, B_2, C_1, C_2; 0, w) = {}_2F_1(A, B_2; C_2; w), \quad (23)$$

$$F_2(A, B_1, B_2, C_1, C_2; z, 0) = {}_2F_1(A, B_1; C_1; z), \quad (24)$$

$$F_2(A, \mathbf{0}, B_2, C_1, C_2; z, w) = {}_2F_1(A, B_2; C_2; z), \quad (25)$$

$$F_2(A, B_1, \mathbf{0}, C_1, C_2; z, w) = {}_2F_1(A, B_1; C_1; z), \quad (26)$$

$$F_2(A, \mathbf{0}, B_2, C_1, C_2; z, 0) = (1-z)^{-A} {}_2F_1\left(A, B_2; C_2; \frac{z}{1-z}\right), \quad (27)$$

$$F_2(A, B_1, \mathbf{0}, C_1, C_2; 0, w) = (1-w)^{-A} {}_2F_1\left(A, B_1; C_1; \frac{w}{1-w}\right), \quad (28)$$

where ${}_2F_1$ is the Gauss hypergeometric matrix function defined in (11).

Proof. The proof of (23)–(26) is a direct consequence of definition (27). The relation (27) is obtained setting $C_1 = B_1$ in (18) and then using (25). Similarly, the relation (28) is derived setting $C_2 = B_2$ in (19) and then using (26). $\square$

3.3. Some Summation Formulas. We now present the summation formulas behavior of the second Appell hypergeometric matrix function $F_2(A, B_1, B_2, C_1, C_2; z, w)$ by the following results.

Theorem 4. *The following finite summation formula holds true:*

$$\begin{aligned} & \sum_{n=0}^{\mu} \frac{(C+I)_n}{n!} F_2(A, -nI, -nI, C+I, C+I; z, w) \\ &= \frac{(A-I)^{-1}(C+I)_{\mu+1}}{(z-w)\mu!} \{F_2(A-I, -\mu I, -(\mu+1)I, C \\ &+ I, C+I; z, w) + z \rightleftharpoons w\}, \end{aligned} \quad (29)$$

where A and C are positively stable in $\mathbb{C}^{d \times d}$ and $z \rightleftharpoons w$ indicates the presence of a second term that originates from the first by interchanging z and w .

Proof. Using (16), we find that

$$\begin{aligned} & \sum_{n=0}^{\mu} \frac{(C+I)_n}{n!} F_2(A, -nI, -nI, C+I, C+I; z, w) \\ &= \Gamma^{-1}(A) \sum_{n=0}^{\mu} n! [(C+I)_n]^{-1} \int_0^{\infty} u^{A-I} e^{-u} L_{\mu}^C(zu) L_n^C(wu) du. \end{aligned} \quad (30)$$

By interchanging the order of summation and integration and applying the following formula [28]:

$$\begin{aligned} & \sum_{n=0}^{\mu} n! [(C+I)_n]^{-1} L_n^C(z) L_n^C(w) \\ &= (\mu+1)! [(C+I)_{\mu}]^{-1} (z-w)^{-1} \{L_{\mu}^C(z) L_{\mu+1}^C(w) \\ &- L_{\mu+1}^C(z) L_{\mu}^C(w)\}, \end{aligned} \quad (31)$$

and then taking into consideration (16), we obtain formula (29).

To extend this theorem, we propose to obtain some more formulas centering around the Appell's matrix function F_2 ; it follows that $\square$

Theorem 5. *Suppose that A and B are positively stable in $\mathbb{C}^{d \times d}$ such that B satisfies spectral condition (5), with $|t| < 1$, $|zt/((1-w)(1-t))| < 1$ and $|w/((1-w)(1-t))| < 1$. The following generating matrix function holds true:*

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(B+I)_{\mu+n}}{n!} F_2(A, -nI, -(n+\mu)I; B+I, B+I; z, w) t^n \\ &= (B+I)_{\mu} (1-w)^{-A} (1-t)^{-(B+(1+\mu)I)} \\ &\quad \times F_4\left(A, B+(1+\mu)I; B+I, B+I; \frac{-zt}{(1-w)(1-t)}; \right. \\ &\quad \left. \frac{-w}{(1-w)(1-t)}\right). \end{aligned} \quad (32)$$

where F_4 is the four Appell's matrix function defined in [27–29].

Proof. To prove (32), we require formula (19) and the relations (12); thus, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(B+I)_{\mu+n}}{n!} F_2(A, -nI, -(n+\mu)I; B+I, B+I; z, w) t^n \\ &= (1-w)^{-A} \times \sum_{n=0}^{\infty} \frac{(B+I)_{\mu+n}}{n!} F_2\left(A, -nI; B+(\mu+n+1)I; \right. \\ &\quad \left. B+I, B+I; \frac{z}{(1-w)}; \frac{w}{(w-1)}\right) \\ &= (1-w)^{-A} \sum_{s,r=0}^{\infty} \sum_{n=0}^{\infty} (A)_{s+r} [(B+I)_r]^{-1} [(B+I)_s]^{-1} \\ &\quad \cdot (B+I)_{\mu+n+r} \times \frac{t^n}{(n-r)!s!r!} \left(\frac{-z}{(1-w)}\right)^r \left(\frac{-w}{(1-w)}\right)^s \\ &= (1-w)^{-A} \sum_{s,r=0}^{\infty} \frac{1}{s!r!} (A)_{s+r} [(B+I)_r]^{-1} [(B+I)_s]^{-1} \\ &\quad \cdot (B+I)_{\mu+s+r} \\ &\quad \times \left(\frac{-zt}{(1-w)}\right)^r \left(\frac{-w}{(1-w)}\right)^s \sum_{n=0}^{\infty} \frac{(B+(1+\mu+s+r)I)_n}{n!} \\ &= (B+I)_{\mu} (1-w)^{-A} (1-t)^{-(B+(1+\mu)I)} \\ &\quad \times F_4\left(A, B+(1+\mu)I; B+I, B+I; \frac{-zt}{(1-w)(1-t)}; \right. \\ &\quad \left. \frac{-w}{(1-w)(1-t)}\right). \end{aligned} \quad (33)$$

This completes the proof of Theorem 5.

Putting $\mu = 0$ and then using the following formula,

$$F_4(\mathbf{A}, \mathbf{B}; \mathbf{B}, \mathbf{B}; z, w) = (1 - z - w)^{-\mathbf{A}} {}_2F_1\left(\frac{\mathbf{A}}{2}, \frac{1}{2}(\mathbf{A} + I); \mathbf{B}; \frac{4zw}{(1 - z - w)^2}\right). \quad (34)$$

Thus, (32) reduces to

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\mathbf{B} + I)^{\mu+n}}{n!} F_2(\mathbf{A}, -nI, -(n + \mu)I; \mathbf{B} + I, \mathbf{B} + I; z, w) t^n &= (1 - w)^{-\mathbf{A}} \\ &\times \sum_{n=0}^{\infty} \frac{(\mathbf{B} + I)^{\mu+n}}{n!} F_2\left(\mathbf{A}, -nI; \mathbf{B} + (\mu + n + 1)I; \mathbf{B} + I, \mathbf{B} + I; \frac{z}{(1 - w)}; \frac{w}{(w - 1)}\right) \\ &= (1 - t)^{\mathbf{A} - (\mathbf{B} + I)} [1 - (1 - z - w)t]^{-\mathbf{A}} {}_2F_1\left(\frac{\mathbf{A}}{2}, \frac{1}{2}(\mathbf{A} + I); \mathbf{B} + I; \frac{4zwt}{[1 - (1 - z - w)t]^2}\right). \end{aligned} \quad (35)$$

Now, we shall see that (32) and (35) enable us to obtain some new formulas. By (29), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbf{A}^{-1} \frac{(\mathbf{B} + I)_{n+1}}{(z - w)n!} [F_2(\mathbf{A} - I, -nI, -(n + 1)I; \mathbf{B} + I, \mathbf{B} + I; z, w) \\ + z \rightleftharpoons w] t^n \\ = \sum_{n=0}^{\infty} \sum_{r=0}^n \frac{(\mathbf{B} + I)_k}{k!} F_2(\mathbf{A}, -kI, -kI; \mathbf{B} + I, \mathbf{B} + I; z, w) t^n. \end{aligned} \quad (36)$$

Using (35), we arrive at

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\mathbf{B} + I)_{n+1}}{n!} [F_2(\mathbf{A} - I, -nI, -(n + 1)I; \mathbf{B} + I, \mathbf{B} + I; z, w) \\ + z \rightleftharpoons w] t^n \\ = F_2(\mathbf{A}, -kI, -kI; \mathbf{B} + I, \mathbf{B} + I; z, w) t^n \\ = \mathbf{A}(z - w)(1 - t)^{\mathbf{A} - (\mathbf{B} + 2I)} [1 - (1 - z - w)t]^{-\mathbf{A}} \\ \times {}_2F_1\left(\frac{\mathbf{A}}{2}, \frac{1}{2}(\mathbf{A} + I); \mathbf{B} + I; \frac{4zwt}{[1 - (1 - z - w)t]^2}\right). \end{aligned} \quad (37)$$

With the help of a generating function for Jacobi matrix polynomials (see [28, 32, 33]),

$$\begin{aligned} \sum_{n=0}^{\infty} (\mathbf{B} + \mathbf{C} + I)_n [(\mathbf{B} + I)_n]^{-1} \mathbb{P}_n^{(\mathbf{B}, \mathbf{C})}(z) t^n \\ = (1 - t)^{-(\mathbf{B} + \mathbf{C} + I)} {}_2F_1\left(\frac{1}{2}(\mathbf{B} + \mathbf{C} + I), \frac{1}{2}(\mathbf{B} + \mathbf{C} + 2I); \mathbf{B} \right. \\ \left. + I; \frac{2t(z - 1)}{(1 - t)^2}\right). \end{aligned} \quad (38)$$

We rewrite (35) as

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\mathbf{B} + I)_{n+1}}{n!} [F_2(\mathbf{B} + \mathbf{C}, -nI, -(n + 1)I; \mathbf{B} + I, \mathbf{B} + I; z, w) \\ + z \rightleftharpoons w] t^n \\ = (\mathbf{B} + \mathbf{C} + I)(z - w)(1 - t)^{\mathbf{C} - I} \sum_{r=0}^{\infty} (\mathbf{B} + \mathbf{C} + I)_r [(\mathbf{B} + I)_r]^{-1} \\ \times \mathbb{P}_r^{(\mathbf{B}, \mathbf{C})}\left(\frac{(1 - w)(1 - z) + zw}{(1 - z - w)}\right) (1 - z - w)^r t^r, \end{aligned} \quad (39)$$

which yields

$$\begin{aligned} \mathbb{P}_n^{(\mathbf{B}, \mathbf{C})}\left(\frac{(1 - w)(1 - z) + zw}{(1 - z - w)}\right) &= (\mathbf{B} + I)_n (\mathbf{B} + \mathbf{C} + I) \\ &\cdot [(\mathbf{B} + \mathbf{C} + I)_n]^{-1} \\ &\times (z - w)^{-1} (1 - z - w)^{-n} \sum_{r=0}^n \frac{(\mathbf{C} - I)_r}{(n - r)!} \frac{(\mathbf{B} + I)_{r+1}}{r!} \\ &\times [F_2(\mathbf{B} + \mathbf{C}, -rI, -(r + 1)I; \mathbf{B} + I, \mathbf{B} + I; z, w) + z \rightleftharpoons w]. \end{aligned} \quad (40)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly to writing of this article. All the authors read and approved the final manuscript.

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