Research Article

Pricing Catastrophe Equity Put Options in a Mixed Fractional Brownian Motion Environment

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This paper considers the pricing of the CatEPut option (catastrophe equity put option) in a mixed fractional model in which the stock price is governed by a mixed fractional Brownian motion (mfBM model), which manifests long-range correlation and fluctuations from the financial market. Using the conditional expectation and the change of measure technique, we obtain an analytical pricing formula for the CatEPut option when the short interest rate is a deterministic and time-dependent function. Furthermore, we also derive analytical pricing formulas for the catastrophe put option and the influence of the Hurst index when the short interest rate follows an extended Vasicek model governed by another mixed fractional Brownian motion so that the environment captures the long-range dependence of the short interest rate. Based on the numerical experiments, we analyze quantitatively the impacts of different parameters from the mfBM model on the option price and hedging parameters. Numerical results show that the mfBM model is more close to the realistic market environment, and the CatEPut option price is evaluated accurately.

1. Introduction

As European put options, catastrophe equity put option (hereafter referred to as the CatEPut option) provides the insurance company the right to sell a specified amount of its stock to investors at a predetermined price at the expiration date of the option in case the total accumulated losses of the insured surpass a given trigger level of losses during the life time of the option. The CatEPut option is a catastrophe risk management tool and is firstly issued by RLI Corporation in 1996 to cover the potential losses caused by catastrophes such as typhoons, hurricanes, storms, waves, and earthquakes. Conceptually, CatEPut options and European put options are similar in that the two contractual financial instruments allow the holder special rights to sell securities. However, a CatEPut option differs from a European put option in two ways. One difference is that the CatEPut option is a sort of double trigger option. When the total accumulated losses surpass a given trigger level of losses during the lifetime of option, the CatEPut option is only exercisable. Other difference is that CatEPut options usually

have five years or more to maturity. Consequently, the pricing mechanism for the CatEPut option is more complex.

There are few studies on pricing the CatEPut option in recent years. Cox et al. [1] first applied arbitrage-free approach to obtain a closed-form solution for the price of this option when the underlying asset price process satisfies the famous Black-Scholes model with the aggregate catastrophe losses of an insurance company following a Poisson process. Besides, they assumed that only the event of a catastrophe affects the underlying asset price, while the size of the catastrophe is irrelevant. Jaimungal and Wang [2] extended the work of Cox et al. [1] to analyze the pricing and hedging of the CatEPut option in a stochastic interest rate framework suggested by Vasicek [3] and assumed that the aggregate catastrophe losses followed a compound Poisson process and the underlying asset price process depended on the total loss level rather than on only the total number of losses. Furthermore, the underlying asset price process is modeled through a jump-diffusion process which is correlated to the loss process. Chang and Hung [4] supposed that the underlying asset price process was driven by a Lévy process

with finite activity, and log jump sizes were independent identically distributed random variables with negative exponential density function and extended works of Cox et al. [1] and Jaimungal and Wang [2]. Lin et al. [5] considered different effects of catastrophe events and adopted a double Poisson process with log-normal intensity to model the number of catastrophe losses and derived the general pricing formula for contingent capital. Lin and Wang [6] applied a mixture of Erlangs as the distribution of the amounts for catastrophe losses and provided a pricing formula for the perpetual American CatEPut options using a penalty function approach. In addition, the pricing models for the CatEPut options with default risk have recently been studied (see Jiang et al. [7]; Jin and Zhong [8]; Koo and Kim [9] and Xu and Wang [10] for examples).

Recently, some researchers have adopted the stochastic volatility model or jump-diffusion model to describe the underlying asset price process. Kim et al. [11] extended the work of Lin and Wang [6] to the stochastic volatility model suggested by Heston [12] and investigated the value of the CatEPut options. Wang [13] presented CatEPut options with target variance which represents the insurance company's expectation of future realized variance under Heston's [12] model with stochastic interest rate and log-normal intensity of Poisson process. Wang et al. [14] considered pricing and hedging for the CatEPut options under a Markov-modulated jump-diffusion process with a Markov regime switching compensator. Kim et al. [15] provided a CatEPut option pricing model in which stock prices follow an additional jump term being moderately correlated with catastrophe losses and obtained a pricing formula for the perpetual American CatEPut options. Yu [16] and Yu et al. [17] studied pricing of the CatEPut options with double compound Poisson processes in which all jump terms are assumed to be independent mutually. The latest research studies on the valuing death benefits and population growth model were given by Yu et al. [18, 19] and Zhang et al. [20].

The aforementioned works have made significant contributions to the study of pricing catastrophe derivatives. However, none of the above works takes account of self-similarity and long-range dependence. A large number of empirical studies have shown that the distributions of the logarithm returns of the financial assets generally exhibit features of self-similarity and long-range dependence with heavy tails and volatility clustering in the real world (e.g., see Lo [21], Willinger et al. [22], Cont [23], Kang and Yoon [24], Kang et al. [25], and Huang et al. [26]). To incorporate this issue, the fractional Brownian motion (fBM) is introduced, which can capture the long-range dependence of the asset returns and produce burstiness in its sample path. Unfortunately, the fBM is neither a Markov nor a semimartingale and may, in some cases, still lead to implementable naive arbitrage opportunities (e.g., see Kuznetsov [27]; Bjork and Hult [28]). To get around this problem while taking into account the features of self-similarity and long-range dependence, the mixed fractional Brownian motion (mfBM) is suggested, which is a linear combination of a Brownian motion (BM) and an fBM with Hurst index $H \in (1/2, 1)$, to depict the underlying dynamics of the financial assets, and

hence, it is arbitrage-free (see Cheridito [29, 30]). Recently, there has been considerable interest in applications of the mfBM environment on the pricing of various derivatives. One can refer to Deng and Xi [31]; Prakasa Rao [32]; Sun [33]; Xiao et al. [34, 35]; Zhang et al. [36]; He and Chen [37]; and Mehrdoust et al. [38] and the references therein. On the contrary, long-range dependence in bond market is a stylized fact that has been empirically studied by McCarthy et al. [39], Gil-Alana [40], Tabaka and Cajueiro [41], Cajueiro and Tabak [42], and Cajueiro and Tabak [43]. In this case, it is important to infer that there exists the long-range dependence in stochastic interest rates. Motivated by these results mentioned above, to make pricing models of financial derivatives more realistic, it is natural to replace the BM with the mfBM in stochastic interest rate models (see Xiao et al. [35]; Zhou et al. [44]).

In this paper, to price the CatEPut options under the stochastic interest rate framework and to capture the long-range dependence of both the underlying asset price and the short interest rate, we consider the pricing problem of this option in the mixed fractional Brownian motion environment with the mixed fractional Vasicek interest rate model. The contribution of this paper is two-fold. First, using the fractional Girsanov transform approach and the fractional Ito formula, we proposed the pricing models for the CatEPut options with constant or stochastic interest rates driven by the mixed fractional Vasicek model. Second, to assess the performance of our model, we apply the proposed model to compare the pricing results with those obtaining from traditional models, such as the famous Black-Scholes model and the fBM model.

The outline of this paper is as follows. Section 2 introduces the model for the mfBM, the underlying asset, and the catastrophe losses. Section 3 derives the explicit pricing formulas for the CatEPut options with constant or stochastic interest rates driven by the fractional Vasicek model. In Section 4, we provide some numerical examples of option prices to compare the pricing results with those obtaining from the famous Black-Scholes and fBM models and to observe effects of model parameters. Section 5 presents the dynamical hedging strategies for the CatEPut options. Section 6 summarizes this paper and discusses possible directions for future research.

2. The Model

In the section, we briefly review some background concerning the mixed fractional Brownian motion and describe a mixed fractional Brownian motion with the jump environment.

Let $(\Omega, \mathcal{F}, Q)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. All the random processes in this paper are defined on this given probability space. A mfBM M_t^H of parameters α, β , and H is defined by a linear combination of a standard Brownian motion W_t and an independent standard fractional Brownian motion W_t^H with Hurst index H ($0 < H < 1$), that is, $M_t^H = \alpha W_t + \beta W_t^H$, where α and β are some real constants, not both are zero.

According to the definition above, it is clear that the mfBM M_t^H nests some popular processes in the literature. The mfBM degenerates to the fBM with $\alpha = 0$ and the BM with $\beta = 0$. The process M_t^H is a centered Gaussian process with $M_0^H = 0$ and with mean 0 for all $t \geq 0$ and covariation function $\text{Cov}(M_t^H, M_s^H) = \alpha^2 \min\{t, s\} + \beta^2/2 (|t|^{2H} + |s|^{2H} - |t-s|^{2H})$, for all $s, t \geq 0$. The increments of M_t^H are positively correlated if $H \in ((1/2), 1)$, uncorrelated if $H = 1/2$, and negatively correlated if $H \in (0, (1/2))$. It is remarkable that the mfBM M_t^H is the standard BM if $H = 1/2$, the increments of M_t^H are long-range dependent if and only if $H \in ((1/2), 1)$, and a self-similar process.

In order to ensure the market environment being arbitrage-free, we assume in this paper that the Hurst index $H \in ((3/4), 1)$, which leads to the mfBM being equivalent to a Wiener process (see Cheridito [29]). Now, consider a frictionless and arbitrage-free financial market where information arrives both continuously and discontinuously, and there exists a underlying stock S_t and a money market account B_t , which are traded continuously over a finite time interval $[0, T]$. Let r_t be the instantaneous interest rate process and L_t represent the total loss process of the insured over the time interval $[0, t]$, which is assumed to follow a compound Poisson process. Under the probability measure Q , the processes of S_t, B_t , and L_t can be expressed as follows:

$$\begin{cases} S_t = S_0 \exp\{-\mu L_t + \lambda \kappa t + X_t\}, \\ L_t = \sum_{i=1}^{N_t} Y_i, \\ B_t = B_0 e^{\int_0^t r_u du}, \quad B_0 = 1, \\ dX_t = \left[r_t - \frac{1}{2} \sigma_s^2 (\alpha^2 + 2\beta^2 H t^{2H-1}) \right] dt + \sigma_s dM_t^H, \end{cases} \quad (1)$$

where μ is a positive constant factor which denotes the percentage drop in the share value per unit loss, N_t is a Poisson process with constant intensity λ , $\{Y_i\}_{i=1}^\infty$ denote the size of i -th catastrophe loss and form a sequence of independent identically distributed random variables with probability density function $f_Y(y)$, and σ_s is the instantaneous volatility of the stock and is assumed to be constant. Additionally, all sources of randomness including the mfBM M_t^H , the Poisson process N_t , and the loss size sequence $(\{Y_i\}_{i=1}^\infty)$ are assumed to be mutually independent. We also define the filtration $\mathcal{F}_t$ generated by $\mathcal{F}_t = \mathcal{F}_t^S$, where $\mathcal{F}_t^S = \sigma(S_u, u \leq t)$ is the natural filtration generated by S_t .

Since the share price of the underlying stock specified in (1) follows the compound Poisson process, the financial market is incomplete, and there is no unique equivalent martingale measure. We assume in this paper that the probability measure Q is a risk-neutral probability measure provided by the agents, and the jump risk presents the nonsystematic and diversifiable risk (see, e.g., Cox et al. [1]; Jaimungal and Wang [2]). Hence, in order to ensure the discounted stock price is a Q -martingale, the parameter κ

satisfies $E(e^{-\mu L_t + \lambda \kappa t}) = 1$; here, $E(\cdot)$ denotes the quasi-expectation under measure Q .

Remark 1. $E(e^{-\mu L_t + \lambda \kappa t}) = 1 \Rightarrow \kappa = \int_0^{+\infty} (1 - e^{-\mu y}) f_Y(y) dy$. From (1), it is easy to know that

$$S_T = S_t \exp \left\{ \int_t^T r_u du - \mu (L_T - L_t) + \lambda \kappa (T - t) - \frac{1}{2} \sigma_s^2 [\alpha^2 (T - t) + \beta^2 (T^{2H} - t^{2H})] + \sigma_s (M_T^H - M_t^H) \right\}. \quad (2)$$

This implies that the stock price S_T is log-normally distributed with $\ln S_T \sim N(m, \nu^2)$ under r_t and L_t known on the time interval $[t, T]$, where $m = \ln S_t + \int_t^T r_u du - \mu (L_T - L_t) + \lambda \kappa (T - t) - (1/2) \sigma_s^2 [\alpha^2 (T - t) + \beta^2 (T^{2H} - t^{2H})]$, $\nu^2 = \sigma_s^2 (\alpha^2 (T - t) + \beta^2 (T^{2H} - t^{2H}))$, and $N(m, \nu^2)$ denotes the Gaussian distribution with mean m and variance ν^2 .

3. Pricing the CatEPut Option

In this section, we shall first calculate the price of the CatEPut option when the short interest rate is a time-dependent and deterministic function. Second, we also derive a closed-form analytical expression for the price of the CatEPut option with the short interest rate following extended Vasicek's model in which the Brownian motion is replaced by a mfBM to capture the long-range dependence of the short-term interest rate.

The catastrophe put option is similar to a European put option, but it is termed a "double trigger" option, and its payoff at maturity T is given by

$$\text{payoff} = (K - S_T)^+ 1_{(L_T - L_{t_0} > L)}, \quad (3)$$

where $x^+ = \max\{x, 0\}$, K is the strike price, $L_T - L_{t_0}$ is the total accumulated losses of the insured over time period $(t_0, T]$, L is the trigger level of losses, and $1_{(\cdot)}$ denotes the indicator function. Let $P_{\text{cat}}(t, t_0)$ denote the CatEPut option price at time t , which was signed at time $t_0 < t$ and matures at time $T > t$. In terms of the derivative pricing theory, we know that

$$P_{\text{cat}}(t, t_0) = E \left[e^{-\int_t^T r_u du} (K - S_T)^+ 1_{(L_T - L_{t_0} > L)} \middle| \mathcal{F}_t \right]. \quad (4)$$

Proposition 1. *Let the dynamics processes S_t and L_t be given by (1). When the short interest rate r_t is a time-dependent and deterministic function, then the analytical pricing formula of the CatEPut option is*

$$P_{\text{cat}}(t, t_0) = \sum_{n=1}^{\infty} \frac{[\lambda (T - t)]^n e^{-\lambda (T - t)}}{n!} \int_{\tilde{L}}^{+\infty} \Psi(y) f_Y^{(n)}(y) dy, \quad (5)$$

where $\tilde{L} = \max\{0, L + L_{t_0} - L_t\}$,

$$\begin{aligned}\Psi(y) &= Ke^{-\int_t^T r_u du} \Phi(-d_2(y)) - S_t e^{-\mu y + \lambda \kappa(T-t)} \Phi(-d_1(y)), \\ d_1(y) &= \frac{\ln(S_t/K) - \mu y + \lambda \kappa(T-t) + \int_t^T r_u du + 1/2 \sigma_s^2 [\alpha^2(T-t) + \beta^2(T^{2H} - t^{2H})]}{\sigma_s \sqrt{\alpha^2(T-t) + \beta^2(T^{2H} - t^{2H})}}, \\ d_2(y) &= d_1(y) - \sigma_s \sqrt{\alpha^2(T-t) + \beta^2(T^{2H} - t^{2H})},\end{aligned}\tag{6}$$

and $f_Y^{(n)}(y)$ is the n -fold convolution of the loss probability density function $f_Y(y)$ and $\Phi(\cdot)$ denotes the cumulated standard normal distribution function.

Proof. See Appendix A.

Since the life time of the CatEPut option can be 5 years or more, we generalize the price formula obtained above to the stochastic interest rate framework. Jaimungal and Wang [2], Jiang et al. [7], Yu [16], Wang [13], Koo and Kim [9], and Xu and Wang [10] extend the results in Cox et al. [1] by incorporating stochastic interest rate, which is assumed to satisfy Vasicek's model in which the driving process is supposed to be a Brownian motion. Note that a substantial amount of empirical evidence suggests that the process of the short interest rate may exhibit long-memory and self-similarity behaviors. Similar to Xiao et al. [35] and Zhou et al. [44], we assume that the short interest rate is controlled by the following process under the risk-neutral pricing measure Q :

$$dr_t = a(b - r_t)dt + \sigma_r dZ_t^H, \tag{7}$$

where a denotes the mean-reversion rate, b represents the long-run mean of the interest rate, σ_r is the instantaneous volatility of the interest rate, and $Z_t^H = \alpha B_t + \beta B_t^H$ is another mfBM with parameters α, β , and H defined on the probability space $(\Omega, \mathcal{F}, Q)$ and is independent of the catastrophe loss process L_t . Moreover, the mfBM Z_t^H is correlated to the mfBM M_t^H with correlation coefficient $\rho \in [-1, 1]$, which is given by $dM_t^H dZ_t^H = \rho(\alpha^2 dt + \beta^2 dt^{2H})$. Additionally, we assume that a, b, σ_r, ρ and the initial value r_0 of r_t are real constants, standard Brownian motions, W_t and B_t , are independent of the fractional Brownian motions, W_t^H, B_t^H , and the filtration $\mathcal{F}_t$ generated by $\mathcal{F}_t = \mathcal{F}_t^S \vee \mathcal{F}_t^r$, where $\mathcal{F}_t^r = \sigma(r_u, u \leq t)$ is the natural filtration generated by r_t .

Now, let us consider a zero-coupon bond that pays 1 unit of cash at maturity T . According to the risk-neutral pricing formula, the value of this bond at time $t \in [0, T]$, denoted by $P(t, T)$, is

$$P(t, T) = E \left[e^{-\int_t^T r_u du} \middle| \mathcal{F}_t^r \right]. \tag{8}$$

Using the fractional Ito formula (see Hu and Oksendal [45]), we have

$$\frac{dP(t, T)}{P(t, T)} = r_t dt + \sigma_r \frac{\partial P / \partial r}{P(t, T)} dZ_t^H. \tag{9}$$

Moreover, $P(t, T)$ satisfies the following P.D.E.:

$$\begin{cases} \frac{\partial P}{\partial t} + \frac{1}{2} \sigma_r^2 (\alpha^2 + 2\beta^2 H t^{2H-1}) \frac{\partial^2 P}{\partial r^2} + a(b - r) \frac{\partial P}{\partial r} - rP = 0, \\ P(T, T) = 1. \end{cases} \tag{10}$$

Similar to Xiao et al. [35], we obtain the following. $\square$

Corollary 1. When the short interest rate follows (7), the price of the zero-coupon bond is given by

$$P(t, T) = \exp\{A(t, T) - B(t, T)r_t\}, \tag{11}$$

where $B(t, T) = 1 - e^{-a(T-t)}/a$ and $A(t, T) = (b - (\alpha^2 \sigma_r^2 / 2a^2)) [B(t, T) - (T - t)] - (\alpha^2 \sigma_r^2 / 4a) B^2(t, T) + H \beta^2 \sigma_r^2 \int_t^T u^{2H-1} B^2(u, T) du$.

Now, we introduce the Radon-Nikodym derivative as follows:

$$\begin{aligned} \frac{dQ^T}{dQ} \bigg|_{\mathcal{F}_T} &= \frac{e^{-\int_t^T r_u du}}{P(t, T)} \\ &= \exp \left\{ -\frac{1}{2} \int_t^T \sigma_r^2 B^2(u, T) (\alpha^2 + 2H \beta^2 u^{2H-1}) du \right. \\ &\quad \left. - \int_t^T \sigma_r B(u, T) dZ_u^H \right\}. \end{aligned} \tag{12}$$

Using the fractional Girsanov theorem (see Hu and Oksendal [45]), then $\hat{M}_t^H = \alpha \hat{W}_t + \beta \hat{W}_t^H$, $\hat{Z}_t^H = \alpha \hat{Z}_t + \beta \hat{Z}_t^H$ defined by

$$\begin{aligned} \hat{M}_t^H &= M_t^H + \int_0^t \rho \sigma_r B(u, T) (\alpha^2 + 2H \beta^2 u^{2H-1}) du, \\ \hat{Z}_t^H &= Z_t^H + \int_0^t \sigma_r B(u, T) (\alpha^2 + 2H \beta^2 u^{2H-1}) du, \end{aligned} \tag{13}$$

are Q^T -mfBMs with the correlation coefficient ρ , and under Q^T ,

$$\begin{aligned} \ln S_T = & \ln \frac{S_t}{P(t, T)} - \mu(L_T - L_t) + \lambda\kappa(T - t) - \frac{1}{2}\sigma^2(u) \\ & + \int_t^T \sigma_s d\widehat{M}_t^H + \int_t^T \sigma_r B(u, T) d\widehat{Z}_t^H, \end{aligned} \quad (14)$$

$$\int_t^T \sigma_s d\widehat{M}_t^H + \int_t^T \sigma_r B(u, T) d\widehat{Z}_t^H \sim N(0, \theta^2(t, T)), \quad (15)$$

where $\theta^2(t, T) = \int_t^T \widetilde{B}(u, T)(\alpha^2 + 2H\beta^2 u^{2H-1})du$ and $\widetilde{B}(u, T) = \sigma_s^2 + 2\rho\sigma_s\sigma_r B(u, T) + \sigma_r^2 B^2(u, T)$. Therefore, (4) can be rewritten as

$$P_{\text{cat}}(t, t_0) = P(t, T)E^T \left\{ 1_{(L_T - L_{t_0} > L)} (K - S_T)^+ \middle| \mathcal{F}_t \right\}, \quad (16)$$

where $E^T[\mathcal{F}_t]$ denotes the quasi-conditional expectation under the T -forward measure Q^T .

Proposition 2. Let the dynamics processes S_t , L_t , and r_t be given by (1) and (7); then, the analytical pricing formula of the CatEPut option is

$$P_{\text{cat}}(t, t_0) = \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \int_{\widetilde{L}}^{+\infty} \widetilde{\Psi}(y) f_Y^{(n)}(y) dy, \quad (17)$$

where $\widetilde{L} = \max\{0, L + L_{t_0} - L_t\}$,

$$\widetilde{\Psi}(y) = KP(t, T)\Phi(-\widetilde{d}_2(y)) - S_t e^{-\mu y + \lambda\kappa(T-t)} \Phi(-\widetilde{d}_1(y)),$$

$$\widetilde{d}_1(y) = \frac{\ln(S_t/KP(t, T)) - \mu y + \lambda\kappa(T-t) + 1/2\theta^2(t, T)}{\theta(t, T)},$$

$$\widetilde{d}_2(y) = \widetilde{d}_1(y) - \theta(t, T). \quad (18)$$

Proof. See Appendix A.

Remark 2. When $(\alpha, \beta) = (1, 0)$, then the result of Proposition 2 reduces to that of Jaimungal and Wang [2]. When $(\alpha, \beta) = (0, 1)$, then the result of Proposition 2 is provided in the fBM environment. Therefore, the JW and fBM models are special cases of the mfBM model in this paper.

Remark 3. The integrations in the pricing model of equation (17), such as $\theta^2(t, T) = \int_t^T \widetilde{B}(u, T)(\alpha^2 + 2H\beta^2 u^{2H-1})du$, can be calculated using the Riemann integral.

Cox et al. [1] considered that the loss size conditional on a loss is fixed at l and the trigger level is an integer multiple of the loss size, i.e., $L = cl$. The parameter c , here labeled the trigger ratio level, represents the ratio of the trigger level to

the total expected loss. The probability density function $f_Y(y)$ of loss size Y_i in this fixed loss case is a Dirac density $f_Y(y) = \delta(y - l)$, and then the following corollary is given.

Corollary 2. If the loss size Y_i follows a Dirac density $f_Y(y) = \delta(y - l)$, the price function in (17) can be rewritten as follows:

$$\begin{aligned} P_{\text{cat}}(t, t_0) = & \sum_{n=\widetilde{N}+1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} [KP(t, T)\Phi(-\widehat{d}_2(n)) \\ & - S_t e^{-\mu nl + \lambda\kappa(T-t)} \Phi(-\widehat{d}_1(n))], \end{aligned} \quad (19)$$

where $\widetilde{N} = \max\{0, c + N_{t_0} - N_t\}$ and

$$\widehat{d}_1(n) = \frac{\ln(S_t/KP(t, T)) - \mu nl + \lambda\kappa(T-t) + (1/2)\theta^2(t, T)}{\theta(t, T)},$$

$$\widehat{d}_2(n) = \widehat{d}_1(n) - \theta(t, T). \quad (20)$$

It is clear that pricing model (17) depends on Hurst parameter H . Hence, we present the influence of this parameter based on price formulas (17) in the following proposition.

Proposition 3. The influence of the Hurst parameter H is given by

$$\begin{aligned} \frac{\partial P_{\text{cat}}(t, t_0)}{\partial H} = & KP(t, T)\beta^2 \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \int_{\widetilde{L}}^{+\infty} \int_t^T \\ & \cdot \left[\sigma_r^2 \Phi(-\widetilde{d}_2(y)) B^2(u, T) + \frac{\widetilde{B}(u, T) e^{-(\widetilde{d}_2(y)/2)}}{\sqrt{2\pi} \theta(t, T)} \right. \\ & \cdot \left. \left[u + H(2H-1) \right] u^{2H-2} f_Y^{(n)}(y) du dy \right]. \end{aligned} \quad (21)$$

Moreover, we have $(\partial P_{\text{cat}}(t, t_0)/\partial H) > 0$ for fixed $t \in [0, T]$.

Proof. We get from (17) that

$$\frac{\partial P_{\text{cat}}(t, t_0)}{\partial H} = \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \int_{\widetilde{L}}^{+\infty} \frac{\partial \widetilde{\Psi}(y)}{\partial H} f_Y^{(n)}(y) dy. \quad (22)$$

Since

$$\begin{aligned}
\frac{\partial \tilde{\Psi}(y)}{\partial H} &= K \frac{\partial P(t, T)}{\partial H} \Phi(-\tilde{d}_2(y)) - KP(t, T) \Phi'(-\tilde{d}_2(y)) \frac{\partial \tilde{d}_2(y)}{\partial H} + S_t e^{-\mu y + \lambda \kappa(T-t)} \Phi'(-\tilde{d}_1(y)) \frac{\partial \tilde{d}_1(y)}{\partial H}, \\
\frac{\partial P(t, T)}{\partial H} &= P(t, T) \frac{\partial A(t, T)}{\partial H}, \\
&= P(t, T) \beta^2 \sigma_r^2 \int_t^T [u + H(2H - 1)] B^2(u, T) u^{2H-2} du, \\
\frac{\partial \tilde{d}_2(y)}{\partial H} &= \frac{\partial \tilde{d}_1(y)}{\partial H} - \frac{\partial \theta(t, T)}{\partial H}.
\end{aligned} \tag{23}$$

On the contrary, we have

$$\begin{aligned}
\frac{\partial \tilde{d}_1(y)}{\partial H} &= \frac{[-(1/P(t, T))(\partial P(t, T)/\partial H) + \theta(t, T)(\partial \theta(t, T)/\partial H)] - \tilde{d}_1(y)(\partial \theta(t, T)/\partial H)}{\theta(t, T)}, \\
\frac{\partial \theta(t, T)}{\partial H} &= \beta^2 \int_t^T \tilde{B}(u, T) \left[\frac{u + tHn(2H-)}{\theta(t, T)} \right] u^{2H-2} du
\end{aligned} \tag{24}$$

With tedious calculation, we obtain that

$$\begin{aligned}
\frac{\partial \tilde{\Psi}(y)}{\partial H} &= KP(t, T) \beta^2 \int_t^T \left[\sigma_r^2 \Phi(-\tilde{d}_2(y)) B^2(u, T) \right. \\
&\quad \left. + \frac{\tilde{B}(u, T) e^{-\tilde{d}_2(y)/2}}{\sqrt{2\pi} \theta(t, T)} [u + H(2H - 1)] u^{2H-2} du \right] \\
&\quad + \frac{\beta^2}{\sqrt{2\pi}} \left[S_t e^{-\mu y + \lambda \kappa(T-t)} e^{-\tilde{d}_1(y)/2} - KP(t, T) e^{-\tilde{d}_1(y)/2} \right] \\
&\quad \cdot \int_t^T \left[-\sigma_r^2 B^2(u, T) + \left(1 - \frac{\tilde{d}_1(y)}{\theta(t, T)} \right) \tilde{B}(u, T) \right] \\
&\quad \cdot [u + H(2H - 1)] u^{2H-2} du.
\end{aligned} \tag{25}$$

Note that $e^{-\tilde{d}_2(y)/2} = e^{-\tilde{d}_1(y)/2 + \tilde{d}_1(y)\theta(t, T) - (\theta^2(t, T)/2)} = (S_t/KP(t, T)) e^{-\mu y + \lambda \kappa(T-t)} e^{-\tilde{d}_1(y)/2}$; then, (25) can be rewritten as

$$\begin{aligned}
\frac{\partial \tilde{\Psi}(y)}{\partial H} &= KP(t, T) \beta^2 \int_t^T \left[\sigma_r^2 \Phi(-\tilde{d}_2(y)) B^2(u, T) \right. \\
&\quad \left. + \frac{\tilde{B}(u, T) e^{-\tilde{d}_2(y)/2}}{\sqrt{2\pi} \theta(t, T)} [u + H(2H - 1)] u^{2H-2} du \right].
\end{aligned} \tag{26}$$

Substituting (26) into (22) leads to finish the proof.

Corollary 4. *The risk-neutral probability that the CatEPut option ends in the money under the dynamics processes S_t , L_t , and r_t given by (1) and (7) is*

$$\begin{aligned}
Q(S_T < K, L_T - L_{t_0} > L \mid \mathcal{F}_{t_0}) &= \sum_{n=1}^{\infty} \frac{[\lambda(T - t_0)]^n e^{-\lambda(T-t_0)}}{n!} \\
&\quad \int_L^{+\infty} \Phi(-h(y)) f_Y^{(n)}(y) dy,
\end{aligned} \tag{27}$$

where $h(y) = (\ln(S_{t_0}/KP(t_0, T)) - \mu y + \lambda \kappa(T - t_0) - 1/2\theta^2(t_0, T))/(\theta(t_0, T))$.

Proof. Similar to the proof of $\hat{I}_1$ in (B.2), it is obvious to obtain (27).

Although it appears that the pricing formula in (17) involves infinite sum, in practice, it is not necessary. In the following, we will prove that numerically, only the first few terms in the series are needed when the jump intensity rate λ is small. Similar to the argument of Jaimungal and Wang [2], we let

$$P_{\text{cat}}(t, t_0; M) = \sum_{n=1}^M \frac{[\lambda(T - t)]^n e^{-\lambda(T-t)}}{n!} \int_{\tilde{L}}^{+\infty} \tilde{\Psi}(y) f_Y^{(n)}(y) dy. \tag{28}$$

Then, by (17), we have

$$\begin{aligned}
|P_{\text{cat}}(t, t_0) - P_{\text{cat}}(t, t_0; M)| &= \sum_{n=M+1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \int_{\tilde{L}}^{+\infty} \tilde{\Psi}(y) f_Y^{(n)}(y) dy \\
&\leq \sum_{n=M+1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \int_{\tilde{L}}^{+\infty} KP(t, T) \Phi(-\tilde{d}_2(y)) f_Y^{(n)}(y) dy \\
&\leq \sum_{n=M+1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} KP(t, T) \\
&\leq KP(t, T) \left\{ \frac{[\lambda(T-t)]^M}{M!} (1 - e^{-\lambda(T-t)}) \right\}.
\end{aligned} \tag{29}$$

Consequently, the relative error in the price can be made very small by choosing M appropriately. For example, if $\lambda = 0.5$ and choosing $M = 20$, an accuracy of 0.4% can be obtained.

4. Numerical Results

In this section, we analyze numerically the proposed model by considering the stochastic interest rate environment. First, we clarify the effectiveness of the proposed model by making comparison with the results of Jaimungal and Wang [2] and the fBM model. Second, we present a sensitivity analysis for the CatEPut option prices under alternative parameter values. In particular, we examine the effects of two parameters of the claim size on the option price and the risk-neutral probability.

In the next numerical experiments, we consider the CatEPut option with maturity $T = 5$ year and show some parameter values in Table 1 if there is no special instruction. The loss size in this paper follows the gamma distribution with mean μ_l and standard deviation σ_l , that is, $f_Y(y) = ((\mu_l/\sigma_l^2)^{(\mu_l^2/\sigma_l^2)})/(\Gamma(\mu_l^2/\sigma_l^2)) y^{(\mu_l^2/\sigma_l^2)-1} e^{-(\mu_l/\sigma_l^2)y}$, $y > 0$, in which case, the n -fold convolution of the losses is given by

$$f_Y^{(n)}(y) = \frac{(\mu_l/\sigma_l^2)^{(n\mu_l^2/\sigma_l^2)}}{\Gamma(n\mu_l^2/\sigma_l^2)} y^{(n\mu_l^2/\sigma_l^2)-1} e^{-(\mu_l/\sigma_l^2)y}, \quad y > 0 \tag{30}$$

where $\Gamma(\cdot)$ is the gamma function. The trigger level of losses L is defined by $c\lambda(T-t)\mu_l$ which denotes a multiple c of the expected losses. According to Remark 1, the parameter κ in this case is $\kappa = 1 - (\mu_l/(\mu_l + \mu\sigma_l^2))^{(\mu_l^2/\sigma_l^2)}$. The parameter c denotes the real-valued trigger ratio level. Moreover, we also assume $L_t - L_{t_0} = 0$ and $(\alpha, \beta) = (1, 1)$ for the mfBM.

Figure 1 displays all values of the CatEPut option against different values of S_0 with the trigger ratio level of $c = 1$ and $c = 2$ for three different models, including the JW (Jaimungal and Wang [2]), fBM, and mfBM models in the stochastic interest rate framework.

As can be seen from Figure 1, the curvature of all option price curves with the trigger ratio level $c = 1$ and $c = 2$ for the three models is significantly decreased as the stock price

increases. This shows that the value of S_0 has a negative effect on the value of the CatEPut option. In addition, the option prices for the JW model are lower than the corresponding prices for the fBM and mfBM models. We point that the difference is caused by the existence of the long-range dependence. It implies that the proposed model can not only exhibit the long-range dependence but also produce closer results to the JW model in which BM has been used traditionally as the driving force for modeling the stock price.

Table 2 reports the option prices for the in-the-money ($S_0 = 60$), at-the-money ($S_0 = 80$), and out-of-the-money ($S_0 = 100$) CatEPut option under the constant/stochastic interest rate framework. As shown in Table 1, we know that the option price under stochastic interest rate is lower than the corresponding price under constant interest rate. This is reasonable because the introduction of stochastic interest rate reduces the variance of the return on the underlying stock compared with the situation with constant interest rate. No matter what the strike is, the price of the CatEPut option increases as the Hurst index H and the stock's volatility σ_s increase. One possible reason for this phenomenon is that the Hurst index represents the long-range dependency property. So, the bigger the Hurst index is, i.e., the stronger the long-memory property is, the higher the option price is. However, the option price decreases as the trigger ratio level c increases, which is obvious. In addition, the CatEPut option's price decreases from in-the-money to out-of-the-money.

Next, we examine how the option price is affected by the model parameter values of the spot interest rate process for a trigger ratio level $c = 1$. We focus on investigating the effects of two parameters σ_r and correlation coefficient ρ on the option price. The results are plotted in Figure 2.

Figure 2(a) indicates that the effect on option price is remarkable if the volatility of the spot interest rate is higher. This may be due to two facts: the uncertainty effect and the long-range dependence in the real world. The larger volatilities induce more uncertainty, and the long-range dependence produces burstiness in the sample path of asset returns, and these push the option prices upward. In Figure 2(b), we compare the option price with different

TABLE 1: The parameter values used in numerical examples.

Parameter	Value
K	80
r_0	0.02
σ_r	0.05
μ_l	25
λ	0.5
a	0.2
ρ	-0.1
σ_l	20
σ_s	0.2
b	0.3
μ	0.01
H	0.8

correlation coefficients ρ , which illustrates that the option prices decrease when the correlation coefficients ρ increase. By the light of nature, this is correct. The reason is if the asset price is strongly negatively correlated to the spot interest rates, the higher the asset price is, the lower the spot interest rates are and the higher the option price is. However, the changes in ρ do not affect the option prices as significantly as changes in the volatility.

In Figure 3, we illustrate how the option prices are affected by the mean μ_l and standard deviation σ_l of the claim size for two different trigger ratio levels of $c = 1$ and $c = 2$, respectively. For the trigger ratio level of $c = 2$, the option price decreases as the mean of the claim sizes increases, while for the trigger ratio level of $c = 1$, the option price increases as the mean of the claim sizes increases. This result is explained by exploring the risk-neutral probability in Figure 4(a) that the option is exercised. In Figure 4(a), the probability of exercising the option is an increasing function of the mean for smaller trigger ratio levels, while it is generally a constant function of the mean for larger trigger ratio levels. In addition, the probability of exercising the option for smaller trigger ratio levels is greater than that for larger ones. In Figure 3, we also find that, for the trigger ratio level of $c = 2$, the option price increases as the standard deviation of the claim sizes increases, while for the trigger ratio level of $c = 1$, the option price increases when the standard deviation of the claim sizes becomes smaller. However, after a certain value of σ_l and then onward, the option price decreases as σ_l increases. This reason is that, for reasonably small σ_l and the trigger ratio level of c , the option price is dominated by the risk-neutral probability, which is monotonically decreasing with σ_l , as shown in Figure 4(b).

5. Dynamically Hedging the CatEPut Option

A reinsurance company provides a CatEPut option to an insurance company, and they need to hedge risk for this contract to avoid taking on very large losses. In this section, we present how to hedge against stock price risk and interest rate risk. In a perfect market, delta-gamma hedging techniques are used to measure the sensitivity of the option's price to underlying asset price movements at the first

FIGURE 1: CatEPut option value for two trigger ratio levels $c = 1$ and $c = 2$ under the three specifications, JW, fBM, and mfBM models. (a) $c = 1$. (b) $c = 2$.

and second order. Jaimungal and Wang [2] further used the delta-gamma-rho hedging strategy to measure the price risk of the stock price and interest rate. Here, we also use this technique to measure the price risk and compute various Greek values. The Greek letters are described as follows.

Proposition 4. Under the dynamics processes S_t , L_t , and r_t modeled by (1) and (7), delta, gamma, and rho of the CatEPut option given in (17) are calculated as the following expressions:

TABLE 2: Option prices under constant/stochastic interest rate.

S_0	Constant interest rate			Stochastic interest rate		
	$c = 1$	$c = 2$	$c = 3$	$c = 1$	$c = 2$	$c = 3$
$H = 0.8$						
60	17.7834	6.19015	1.45349	8.92275	3.37690	0.849746
80	14.9291	5.56540	1.37039	7.14182	2.90722	0.776963
100	12.6302	5.01053	1.29149	5.82774	2.52269	0.711669
$H = 0.9$						
60	18.3864	6.25808	1.45731	9.72182	3.53258	0.870091
80	15.7647	5.67737	1.37779	8.03146	3.09770	0.802079
100	13.6421	5.16633	1.30325	6.75448	2.73960	0.741522
$H = 0.98$						
60	18.9851	6.33379	1.46235	10.4918	3.68761	0.891346
80	16.5656	5.79439	1.38677	8.88816	3.2833	0.827782
100	14.5964	5.3226	1.31667	7.6544	2.94864	0.771475
$\sigma_s = 0.2$						
60	17.7834	6.19015	1.45349	8.92275	3.37690	0.849746
80	14.9291	5.56540	1.37039	7.14182	2.90722	0.776963
100	12.6302	5.01053	1.29149	5.82774	2.52269	0.711669
$\sigma_s = 0.25$						
60	18.9757	6.33639	1.46282	10.0168	3.55346	0.864323
80	16.5352	5.79539	1.38735	8.43109	3.14982	0.800556
100	14.5444	5.32101	1.31725	7.21853	2.81694	0.74415
$\sigma_s = 0.5$						
60	24.6637	7.27578	1.56092	14.8864	4.45694	0.969254
80	23.5855	7.03926	1.52517	14.1160	4.27946	0.941041
100	22.6716	6.83390	1.49338	13.4775	4.12875	0.916482

— $\sigma_r = 0.05$
 - - $\sigma_r = 0.1$
 - - $\sigma_r = 0.15$

(a)

— $\rho = -0.1$
 - - $\rho = 0.0$
 - - $\rho = 0.1$

(b)

FIGURE 2: Option price for several parameter values of the interest rate. (a) Volatility levels. (b) Correlation coefficients.

FIGURE 3: The effect of σ_l on option price with different μ_l . (a) $c = 1$. (b) $c = 2$.

FIGURE 4: The effects of μ_l and σ_l on risk-neutral probability of ending in-the-money for several trigger ratio levels of c . (a) Mean of claims μ_l . (b) Std of claims σ_l .

TABLE 3: Values of delta, gamma, and rho under JW and mfBM models.

S_0	JW model			mfBM model		
	60	80	100	60	80	100
$c = 1$						
$\Delta_S(t, t_0)$	-0.12908	-0.07965	-0.05676	-0.10417	-0.07585	-0.04934
$\Gamma_S(t, t_0)$	0.00303	0.00194	0.00116	0.00173	0.00115	0.00078
$\rho(t, t_0)$	-72.3660	-54.2121	-39.8832	-80.4577	-70.0462	-61.0020
$c = 2$						
$\Delta_S(t, t_0)$	-0.03472	-0.02853	-0.02229	-0.02596	-0.02119	-0.01741
$\Gamma_S(t, t_0)$	0.00032	0.00029	0.00028	0.00027	0.00021	0.00017
$\rho(t, t_0)$	-26.1142	-23.8098	-20.8363	-26.1645	-24.4051	-22.6102

$$\begin{aligned}
\Delta_S(t, t_0) &= \frac{\partial P_{\text{cat}}(t, t_0)}{\partial S_t} = - \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{\lambda(\kappa-1)(T-t)}}{n!} \\
&\quad \cdot \int_{\tilde{L}}^{+\infty} e^{-\mu y} \Phi(-\tilde{d}_1(y)) f_Y^{(n)}(y) dy, \\
\Gamma_S(t, t_0) &= \frac{\partial^2 P_{\text{cat}}(t, t_0)}{\partial S_t^2} = \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{\lambda(\kappa-1)(T-t)}}{n!} \\
&\quad \cdot \frac{1}{\sqrt{2\pi} S_t \theta(t, T)} \int_{\tilde{L}}^{+\infty} e^{-\mu y - \tilde{d}_1^2(y)/2} f_Y^{(n)}(y) dy, \\
\rho(t, t_0) &= \frac{\partial P_{\text{cat}}(t, t_0)}{\partial r_t} = \frac{\partial P_{\text{cat}}(t, t_0)}{\partial P(t, T)} \cdot \frac{\partial P(t, T)}{\partial r_t} \\
&= -KB(t, T)P(t, T) \cdot \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{\lambda(\kappa-1)(T-t)}}{n!} \\
&\quad \int_{\tilde{L}}^{+\infty} \Phi(-\tilde{d}_2(y)) f_Y^{(n)}(y) dy.
\end{aligned} \tag{31}$$

Table 3 reports the values of delta, gamma, and rho for the in-the-money, at-the-money, and out-of-the-money CatEPut option under JW and mfBM models with two different trigger ratio levels of $c = 1$ and $c = 2$. First, the values of delta and gamma under the JW model are greater than those of the mfBM model. However, the values of rho under the JW model are smaller than those of the mfBM model. These findings are consistent with the results in Figure 1 since higher option's price leads to lower the values of delta and gamma and higher the values of rho.

Second, the values of delta, gamma, and rho for the CatEPut option decrease from in-the-money to out-of-the-money. Finally, the values of delta, gamma, and rho for the CatEPut option are sensitive to the trigger ratio level c and are decreasing with the trigger ratio level c .

6. Conclusion

This paper extends the analysis of Jaimungal and Wang [2] to make it more realistic by introducing the mfBM model, which includes the fBM model and the BM model in the literature as its special cases, to represent the underlying asset price and interest rate changes. A closed-form analytical expression for the price of the CatEPut option is then derived in the case of deterministic and stochastic interest rates based on the fractional Girsanov formula. Finally, through numerical experiments, the quantitative impacts of different parameters on the option price and hedging parameters of the CatEPut option are discussed.

Some possible extensions should be considered in future research. Firstly, the proposed model can be tested empirically by using the catastrophic data from PCS (Property Claim Service). For example, we can estimate the parameters of catastrophe arrival rate λ and the Hurst index H using the maximum likelihood estimation method. Secondly, the proposed model can also be extended by taking the counterparty risk into account and evaluating other catastrophe-linked financial options.

Appendix

A. The Proof of Proposition 1

Using the risk-neutral pricing theory and the law of iterated conditional expectation, we have

$$\begin{aligned}
P_{\text{cat}}(t, t_0) &= E \left\{ 1_{(L_T - L_{t_0} > L)} E \left[e^{-\int_t^T r_u du} (K - S_T)^+ \middle| L_T - L_t = \sum_{i=1}^{N_{T-t}} Y_i \right] \middle| \mathcal{F}_t \right\} \\
&= I_1 - I_2,
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned} I_1 &= K e^{-\int_t^T r_u du} E \left\{ 1_{(L_T - L_t > L + L_{t_0} - L_t)} E \left[1_{(S_T \leq K)} \middle| L_T - L_t = \sum_{i=1}^{N_{T-t}} Y_i \right] \middle| \mathcal{F}_t \right\} \\ &= K e^{-\int_t^T r_u du} E \left\{ 1_{(\sum_{i=1}^{N_{T-t}} Y_i > L + L_{t_0} - L_t)} Q \left(S_T \leq K \middle| \sum_{i=1}^{N_{T-t}} Y_i \right) \middle| \mathcal{F}_t \right\}, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} I_2 &= E \left\{ 1_{(L_T - L_t > L + L_{t_0} - L_t)} E \left[S_T e^{-\int_t^T r_u du} 1_{(S_T \leq K)} \middle| L_T - L_t = \sum_{i=1}^{N_{T-t}} Y_i \right] \middle| \mathcal{F}_t \right\}, \\ &= E \left\{ 1_{(\sum_{i=1}^{N_{T-t}} Y_i > L + L_{t_0} - L_t)} E \left[S_T e^{-\int_t^T r_u du} 1_{(S_T \leq K)} \middle| \sum_{i=1}^{N_{T-t}} Y_i \right] \middle| \mathcal{F}_t \right\}. \end{aligned} \quad (\text{A.3})$$

From (2), thus, formula I_1 in (A.2) can be rewritten as

$$\begin{aligned} I_1 &= K e^{-\int_t^T r_u du} \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} E \left\{ 1_{(\sum_{i=1}^n Y_i > L + L_{t_0} - L_t)} \cdot Q \left(\ln S_T \leq \ln K \middle| \sum_{i=1}^n Y_i \right) \middle| \mathcal{F}_t \right\} \\ &= K e^{-\int_t^T r_u du} \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} E \left\{ 1_{(\sum_{i=1}^n Y_i > L + L_{t_0} - L_t)} \cdot Q(\sigma_s(M_T^H - M_t^H)) \right. \\ &\quad \left. \leq - \left[\ln \frac{S_t}{K} - \mu \sum_{i=1}^n Y_i + \lambda \kappa(T-t) + \int_t^T r_u du - \frac{1}{2} \sigma_s^2 [\alpha^2(T-t) + \beta^2(T^{2H} - t^{2H})] \right] \middle| \sum_{i=1}^n Y_i \right\} \\ &= \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \int_{\tilde{L}}^{+\infty} K e^{-\int_t^T r_u du} \Phi(-d_2(y)) f_Y^{(n)}(y) dy. \end{aligned} \quad (\text{A.4})$$

Similarly, we rewrite (A.3) as

$$\begin{aligned} I_2 &= S_t E \left\{ 1_{(\sum_{i=1}^{N_{T-t}} Y_i > L + L_{t_0} - L_t)} E \left[\frac{S_T e^{-\int_t^T r_u du}}{S_t} 1_{(S_T \leq K)} \middle| \sum_{i=1}^{N_{T-t}} Y_i \right] \middle| \mathcal{F}_t \right\} \\ &= S_t \tilde{E} \left\{ 1_{(\sum_{i=1}^{N_{T-t}} Y_i > L + L_{t_0} - L_t)} \tilde{Q} \left(\ln S_T \leq \ln K \middle| \sum_{i=1}^{N_{T-t}} Y_i \right) \middle| \mathcal{F}_t \right\}, \end{aligned} \quad (\text{A.5})$$

where $\tilde{E}[\mathcal{F}_t]$ denotes the quasi-conditional expectation with respect to the probability measure $\tilde{Q}$ which is equivalent to Q , and the Radon-Nikodym derivative which induced the measure change from Q to $\tilde{Q}$ is given by the formula

$$\frac{d\tilde{Q}}{dQ} \bigg|_{\mathcal{F}_T} = \frac{S_T e^{-\int_0^T r_u du}}{S_0} = \exp \left\{ -\frac{1}{2} [\alpha^2 T + \beta^2 T^{2H}] + \sigma_s M_T^H \right\}. \quad (\text{A.6})$$

Let $\tilde{M}_t^H = \alpha \tilde{W}_t + \beta \tilde{W}_t^H = M_t^H - \sigma_s (\alpha^2 t + \beta^2 t^{2H})$; using the fractional Girsanov theorem (see Hu and Oksendal [45]),

then $\tilde{M}_t^H$ is also a mfBM with parameters α, β , and H under the new probability measure $\tilde{Q}$. Hence, we can rewrite (2) as

$$\begin{aligned} \ln S_T &= \ln S_t + \left\{ \int_t^T r_u du - \mu(L_T - L_t) + \lambda \kappa(T-t) \right. \\ &\quad \left. + \frac{1}{2} \sigma_s^2 [\alpha^2(T-t) + \beta^2(T^{2H} - t^{2H})] + \sigma_s (\tilde{M}_T^H - \tilde{M}_t^H) \right\}. \end{aligned} \quad (\text{A.7})$$

under $\tilde{Q}$ and obtain that

$$\begin{aligned}
I_2 &= S_t \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} E \left\{ 1_{\left(\sum_{i=1}^n Y_i > L+L_{t_0}-L_t\right)} e^{-\mu \sum_{i=1}^n Y_i + \lambda \kappa(T-t)} \cdot \tilde{Q}(\sigma_s(M_T^H - M_t^H)) \right. \\
&\leq \left. - \left[\ln \frac{S_t}{K} - \mu \sum_{i=1}^n Y_i + \lambda \kappa(T-t) + \int_t^T r_u du \frac{1}{2} \sigma_s^2 [\alpha^2(T-t) + \beta^2(T^{2H} - t^{2H})] \right] \left| \sum_{i=1}^n Y_i \right| \mathcal{F}_t \right\} \\
&= \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \int_{\tilde{L}}^{+\infty} S_t e^{-\mu y + \lambda \kappa(T-t)} \Phi(-d_1(y)) f_Y^{(n)}(y) dy.
\end{aligned} \tag{A.8}$$

From (A.4) and (A.8), the proof is completed.

B. The Proof of Proposition 2

From (16), we write $P_{\text{cat}}(t, t_0)$ as a sum of two terms $\hat{I}_1$ and $\hat{I}_2$:

$$\begin{aligned}
\hat{I}_1 &= KP(t, T) E^T \left[1_{\left(L_T - L_t > L+L_{t_0}-L_t\right)} 1_{(\ln S_T \leq \ln K)} \middle| \mathcal{F}_t \right], \\
\hat{I}_2 &= P(t, T) E^T \left[1_{\left(L_T - L_t > L+L_{t_0}-L_t\right)} S_T 1_{(\ln S_T \leq \ln K)} \middle| \mathcal{F}_t \right].
\end{aligned} \tag{B.1}$$

We denote $\gamma = -[\ln(S_t/KP(t, T)) - \mu \sum_{i=1}^n Y_i + \lambda \kappa(T-t) - (1/2)\theta^2(t, T)]$ and firstly calculate the term $\hat{I}_1$. By using (14), we have

$$\begin{aligned}
\hat{I}_1 &= KP(t, T) E^T \left[1_{\left(L_T - L_t > L+L_{t_0}-L_t\right)} 1_{(\ln S_T \leq \ln K)} \middle| \mathcal{F}_t \right] \\
&= KP(t, T) E^T \left[1_{\left(L_T - L_t > L+L_{t_0}-L_t\right)} E^T \left\{ 1_{(\ln S_T \leq \ln K)} \middle| L_T - L_t \right\} \middle| \mathcal{F}_t \right] \\
&= KP(t, T) \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} E \left\{ 1_{\left(\sum_{i=1}^n Y_i > L+L_{t_0}-L_t\right)} \cdot Q^T \left(\ln S_T \leq \ln K \middle| \sum_{i=1}^n Y_i \right) \middle| \mathcal{F}_t \right\} \\
&= KP(t, T) \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} E \left\{ 1_{\sum_{i=1}^n Y_i > L+L_{t_0}-L_t} \cdot Q^T \left(\int_t^T \sigma_s d\hat{M}_t^H + \int_t^T \sigma_r B(u, T) d\hat{Z}_t^H \leq \gamma \middle| \sum_{i=1}^n Y_i \right) \middle| \mathcal{F}_t \right\} \\
&= \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \int_{\tilde{L}}^{+\infty} KP(t, T) \Phi(-\hat{d}_2(y)) f_Y^{(n)}(y) dy.
\end{aligned} \tag{B.2}$$

Similarly, we have

$$\begin{aligned}
\hat{I}_2 &= P(t, T) E^T \left[1_{\left(L_T - L_t > L+L_{t_0}-L_t\right)} S_T 1_{(\ln S_T \leq \ln K)} \middle| \mathcal{F}_t \right] \\
&= P(t, T) E^T \left[1_{\left(L_T - L_t > L+L_{t_0}-L_t\right)} E^T \left\{ e^{\ln S_T} 1_{(\ln S_T \leq \ln K)} \middle| L_T - L_t \right\} \middle| \mathcal{F}_t \right] \\
&= S_t \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} E^T \left\{ 1_{\left(\sum_{i=1}^n Y_i > L+L_{t_0}-L_t\right)} e^{-\mu \sum_{i=1}^n Y_i + \lambda \kappa(T-t) - (1/2)\theta^2(t, T)} \cdot E^T \right. \\
&\quad \cdot \left. \left[e^{\int_t^T \sigma_s d\hat{M}_t^H + \int_t^T \sigma_r B(u, T) d\hat{Z}_t^H} 1_{\left(\int_t^T \sigma_s d\hat{M}_t^H + \int_t^T \sigma_r B(u, T) d\hat{Z}_t^H \leq \gamma\right)} \middle| \sum_{i=1}^n Y_i \right] \middle| \mathcal{F}_t \right\}.
\end{aligned} \tag{B.3}$$

Using the fact that $E[e^X 1_{(X < \gamma)}] = e^{\delta^2/2} \Phi((\gamma + \delta)/\delta)$ when $X \sim N(0, \delta^2)$, then

$$\begin{aligned} \hat{I}_2 &= S_t \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} E^T \left\{ 1_{\left(\sum_{i=1}^n Y_i > L + L_{t_0} - L_t\right)} \cdot e^{-\mu \sum_{i=1}^n Y_i + \lambda \kappa(T-t)} \Phi\left(\frac{\gamma + \theta(t, T)}{\theta(t, T)}\right) \middle| \mathcal{F}_t \right\} \\ &= \sum_{n=1}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \int_{\tilde{L}}^{+\infty} S_t e^{-\mu y + \lambda \kappa(T-t)} \Phi(-\hat{d}_2(y)) f_Y^{(n)}(y) dy. \end{aligned} \quad (\text{B.4})$$

Hence, (B.2) and (B.4) yield (17).

Data Availability

All data used to support the findings of this study are included within this article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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