Research Article

A High-Order Numerical Method for a Nonlinear System of Second-Order Boundary Value Problems

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This paper is concerned with a high-order numerical scheme for nonlinear systems of second-order boundary value problems (BVPs). First, by utilizing quasi-Newton's method (QNM), the nonlinear system can be transformed into linear ones. Based on the standard Lobatto orthogonal polynomials, we introduce a high-order Lobatto reproducing kernel method (LRKM) to solve these linear equations. Numerical experiments are performed to investigate the reliability and efficiency of the presented method.

1. Introduction

Nonlinear systems of second-order BVPs are widely used in applied physics, mechanical engineering, biology, etc. In this

article, we mainly focus on numerical solutions for such problems. In general, these systems can be described by

$$\begin{cases} u'' + a_1(x)u' + a_2(x)v'' + a_3(x)v' + \mathcal{F}(u, v) = f, & x \in (a, b), \\ v'' + b_1(x)v' + b_2(x)u'' + b_3(x)u' + \mathcal{G}(u, v) = g, & x \in (a, b), \\ u(a) = 0, u(b) = 0, v(a) = 0, v(b) = 0, \end{cases} \quad (1)$$

in which $\mathcal{F}$ and $\mathcal{G}$ represent the nonlinear terms, $a_i(x), b_i(x) \in C[a, b]$ for $i = 1, 2, 3$, and f, g are given functions defined in $[a, b]$.

Concerning the existence and uniqueness of solution of problem (1), we refer to [1–3]. Nevertheless, to the best of our knowledge, research studies on numerical methods for nonlinear systems of second-order BVPs were seldom reported. Recently, several techniques for approximate solutions of such systems have been presented. Arqub and Abo-Hammour [4] suggested a continuous genetic algorithm to deal with second-order linear systems. In [5, 6], the authors combined homotopy perturbation methods and iterative

RKM to solve problem (1). In [7–9], Peng et al. proposed an alternative iterative method called symplectic method to solve nonlinear BVPs. Furthermore, methods such as sinc-collocation methods and Jacobi matrix methods were also considered in [10, 11].

In this work, QNM [12] and simplified reproducing kernel method (SRKM) [13] are combined to design a numerical method for solving nonlinear systems (1). It is worth noting that QNM acts directly on operator equations to linearize nonlinear problems by *Fréchet derivative*. Here we extend QNM to solve nonlinear systems (1). Many improved versions of RKM have been developed to solve

linear second-order BVPs [14–21]. However, the high-order RKM has been seldom discussed. Motivated by designing RKM of high-order accuracy, we establish a reproducing kernel space with polynomial form by applying the well-known Lobatto orthogonal polynomials. NDM converges fast and LRKM avoids the Schmidt orthogonalization process and uses very few points. Therefore, QDM-LRKM is efficient and simple to implement.

The article is built up as follows. Several fundamental definitions about *Fréchet derivative* and Hilbert spaces are recalled in Section 2. The QNM-LRKM is introduced in Section 3. Some numerical examples validate the robustness

and effectiveness in Section 4. Finally, a summary is provided in the last section.

2. Preliminaries

To simplify our discussion and demonstrate the idea clearly, we rewrite system (1) into a system with homogenous boundary conditions and the domain defined on $[-1, 1]$. Actually, by a scaling from $[a, b]$ to $[-1, 1]$, that is, $x = ((b-a)/2)t + ((b+a)/2)$, we can derive a system defined on $[-1, 1]$ as follows:

$$\begin{cases} \hat{u}'' + \hat{a}_1(t)\hat{u}' + \hat{a}_2(t)\hat{v}'' + \hat{a}_3(t)\hat{v}' + \hat{\mathcal{F}}(\hat{u}; \hat{v}) = \hat{f}(t); & t \in (-1, 1), \\ \hat{v}'' + \hat{b}_1(t)\hat{v}' + \hat{b}_2(t)\hat{u}'' + \hat{b}_3(t)\hat{u}' + \hat{\mathcal{G}}(\hat{u}; \hat{v}) = \hat{g}(t); & t \in (-1, 1), \\ \hat{u}(-1) = \hat{u}(1) = 0, & \hat{v}(-1) = \hat{v}(1) = 0, \end{cases} \quad (2)$$

where $\hat{a}_1(t) = ((b-a)/2)a_1(x(t))$, $\hat{a}_2(t) = a_2(x(t))$, $\hat{a}_3(t) = ((b-a)/2)a_3(x(t))$, $\hat{b}_1(t) = ((b-a)/2)b_1(x(t))$, $\hat{b}_2(t) = b_2(x(t))$, $\hat{b}_3(t) = ((b-a)/2)b_3(x(t))$, $\hat{\mathcal{F}}(\hat{u}; \hat{v}) = ((b-a)^2/4)\mathcal{F}(\hat{u}; \hat{v})$, $\hat{\mathcal{G}}(\hat{u}; \hat{v}) = ((b-a)^2/4)\mathcal{G}(\hat{u}; \hat{v})$, $\hat{f}(t) = ((b-a)^2/4)f(x(t))$, and $\hat{g}(t) = ((b-a)^2/4)g(x(t))$.

2.1. Several Reproducing Kernel Spaces and Hilbert Spaces. Denote by $\mathcal{P}_n$ the space of polynomial functions with degrees no more than n . Let P_n and L_n be the well-known Legendre and Lobatto polynomials, respectively. To be more precise,

$$P_0(t) = 1, \quad P_1(t) = t,$$

$$P_n(t) = \frac{2n-1}{n}tP_{n-1}(t) - \frac{n-1}{n}P_{n-2}(t), \quad n = 2, 3, \dots, \quad (3)$$

satisfying the orthogonality condition on $[-1, 1]$, which means

$$(P_m, P_n) = \int_{-1}^1 P_m(t)P_n(t)dt = \begin{cases} \frac{2}{2n+1}, & m \neq n, \\ 0, & m = n. \end{cases} \quad (4)$$

Furthermore, Lobatto polynomials are defined by

$$\begin{aligned} L_0(t) &= \frac{1-t}{2}, \quad L_1(t) = \frac{1+t}{2}, \\ L_n(t) &= \int_{-1}^t P_{n-1}(s)ds = \frac{2}{2n-1}(P_n - P_{n-2}), \\ n &= 2, 3, \dots \end{aligned} \quad (5)$$

Clearly, Lobatto polynomials satisfy

$$L_n(-1) = L_n(1) = 0, \quad n = 2, 3, \dots \quad (6)$$

Let $L^2[-1, 1] = \left\{u \mid \int_{-1}^1 u^2(x)dx < \infty\right\}$ and $W_n[-1, 1]$ be the standard linear inner product space of polynomials with degrees not exceeding n , namely,

$$\langle u, v \rangle_{W_n} = \int_{-1}^1 uv dx, \quad \forall u, v \in W_n[-1, 1], \quad (7)$$

and the norm

$$\|u\|_{W_n[-1, 1]} = \sqrt{\langle u, u \rangle_{W_n}}, \quad \forall u \in W_n[-1, 1]. \quad (8)$$

From Theorem 3.7 of [22], we obtain that the inner space $W_n[-1, 1]$ is a Hilbert space as $W_n[-1, 1]$ is a finite dimension closed subspace of $L^2[-1, 1]$.

Definition 1

$$W_n^0[-1, 1] = \{u \mid u \in W_n[-1, 1], \quad u(-1) = u(1) = 0\}. \quad (9)$$

Let $\{e_i\}_{i=2}^n$ be the standard orthonormal basis functions obtained by the Gram-Schmidt process of $\{L_i\}_{i=2}^n$.

Lemma 1. $W_n^0[-1, 1]$ is a reproducing kernel space with kernel function

$$K(s, t) = \sum_{i=2}^n e_i(t)e_i(s). \quad (10)$$

Definition 2

$$\begin{aligned} \mathbf{W}_n[-1, 1] &= W_n[-1, 1] \oplus W_n[-1, 1] \\ &= \left\{ (u_1, u_2)^T \mid u_1, u_2 \in W_n[-1, 1] \right\}. \end{aligned} \quad (11)$$

We equip the following inner product and norm for $\mathbf{W}_n[-1, 1]$:

$$\begin{aligned}\langle \mathbf{U}, \mathbf{V} \rangle_{\mathbf{W}_n} &= \sum_{i=1}^2 \langle u_i, v_i \rangle_{W_n[-1,1]}, \quad \mathbf{U}, \mathbf{V} \in \mathbf{W}_n[-1,1], \\ \|\mathbf{U}\|_{\mathbf{W}_n}^2 &= \|u_1\|_{W_n[-1,1]}^2 + \|u_2\|_{W_n[-1,1]}^2, \quad \mathbf{U} \in \mathbf{W}_n[-1,1].\end{aligned}\quad (12)$$

Under such definitions, we obtain that $\mathbf{W}_n[-1,1]$ is a Hilbert space. Similarly, we can define a subspace $\mathbf{W}_n^0[-1,1]$ of $\mathbf{W}_n[-1,1]$,

$$\mathbf{W}_n^0[-1,1] = W_n^0[-1,1] \oplus W_n^0[-1,1]. \quad (13)$$

2.2. Fréchet Derivative. Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. $\mathcal{F}$ denotes an operator from X to Y .

Definition 3 (see [23]). We say that the linear operator $\mathcal{A}$ is the Fréchet derivative of $\mathcal{F}$ at $u_0 \in X$ if

$$\lim_{h \rightarrow 0} \frac{\|\mathcal{F}(u_0 + h) - \mathcal{F}(u_0) - \mathcal{A}(h)\|_Y}{\|h\|_X} = 0, \quad (14)$$

and $\mathcal{A}$ is denoted by $\mathcal{F}'(u_0)$.

Let $\mathcal{F}: X_1 \times X_2 \rightarrow Y$, $\mathcal{A}_1: X_1 \times X_2 \rightarrow Y$, and $\mathcal{A}_2: X_1 \times X_2 \rightarrow Y$, where X_1 and X_2 are also Banach spaces with norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$.

Definition 4. We say that the linear operator $\mathcal{A}_1$ is the partial Fréchet derivative of $\mathcal{F}$ on u at $(u_0, v_0) \in X_1 \times X_2$ if

$$\lim_{h \rightarrow 0} \frac{\|\mathcal{F}(u_0 + h, v_0) - \mathcal{F}(u_0, v_0) - \mathcal{A}_1(h, v_0)\|_Y}{\|h\|_{X_1}} = 0, \quad (15)$$

and $\mathcal{A}_1$ is denoted by $\mathcal{F}'_u(u_0, v_0)$.

Similarly, we can define the partial Fréchet derivative of $\mathcal{F}$ at $(u_0, v_0) \in X_1 \times X_2$ with respect to v by $\mathcal{F}'_v(u_0, v_0)$.

3. QNM-SRKM for Solving Equation (2)

In this section, we present the QNM-SRKM to solve the nonlinear system (2).

3.1. Analysis of QNM. By equation (1), we define $\mathcal{L}_{ij}: W_n^0[-1,1] \rightarrow L^2[-1,1]$ as

$$\begin{aligned}\mathcal{L}_{11}: \hat{u} &\mapsto \hat{u}'' + \hat{a}_1 \hat{u}', \mathcal{L}_{12}: \hat{v} \mapsto \hat{a}_2 \hat{v}'' + \hat{a}_3 \hat{v}', \\ \mathcal{L}_{21}: \hat{u} &\mapsto \hat{b}_2 \hat{u}'' + \hat{b}_3 \hat{u}', \mathcal{L}_{22}: \hat{v} \mapsto \hat{v}'' + \hat{b}_1 \hat{v}'.\end{aligned}\quad (16)$$

Let

$$\begin{aligned}\mathcal{L} &= \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}, \\ \mathcal{N}(\mathbf{U}) &= \begin{pmatrix} \hat{\mathcal{F}}(\hat{u}, \hat{v}) \\ \hat{\mathcal{G}}(\hat{u}, \hat{v}) \end{pmatrix}.\end{aligned}\quad (17)$$

One can correspondingly rewrite equation (2) into the following operator form:

$$\begin{cases} \mathcal{L}\mathbf{U} + \mathcal{N}\mathbf{U} = \mathbf{F}, \\ \mathbf{U}(-1) = 0, \mathbf{U}(1) = 0, \end{cases} \quad (18)$$

where $\mathbf{U} = (\hat{u}, \hat{v})^T$ and $\mathbf{F} = (\hat{f}, t\hat{g})^T \in L^2[-1,1]$.

Theorem 1. The Fréchet derivative of $\mathcal{N}$ at $\mathbf{U}_0$ is

$$\mathcal{N}'(\mathbf{U}_0): \mathbf{U} \mapsto \begin{pmatrix} \mathcal{F}_u(u_0, v_0) & \mathcal{F}_v(u_0, v_0) \\ \mathcal{G}_u(u_0, v_0) & \mathcal{G}_v(u_0, v_0) \end{pmatrix} \mathbf{U}. \quad (19)$$

To show Fréchet derivative of the nonlinear operator more clearly, we introduce two examples.

Example 1. Let $\mathcal{N}\mathbf{U} = \begin{pmatrix} xv + u^2 \\ u + x^2v^2 \end{pmatrix}$, then

$$\mathcal{N}'(\mathbf{U}_0)\mathbf{U} = \begin{pmatrix} 2u_0(x) & x\mathcal{F} \\ x\mathcal{F} & 2x^2v_0(x) \end{pmatrix} \mathbf{U}, \text{ where } \mathcal{F} \text{ is identity}$$

operator.

Example 2. Let $\mathcal{N}\mathbf{U} = \begin{pmatrix} u'' + xe^u + v^3 \\ v'' + u^2 + x \sin v \end{pmatrix}$, then

$$\mathcal{N}'(\mathbf{U}_0)\mathbf{U} = \begin{pmatrix} u'' \\ v'' \end{pmatrix} + \begin{pmatrix} xe^{u_0} & 3v_0^2 \\ 2u_0^2 & x \cos v_0 \end{pmatrix} \mathbf{U}.$$

Now, we will introduce the QNM to linearize the operator equation (18).

The Quasi-Newton's Scheme: Finding $\{\mathbf{U}^k\}_{k=1}^\infty$ such that

$$\begin{cases} \mathcal{L}\mathbf{U}^{k+1} + \mathcal{N}'(\mathbf{U}^k)(\mathbf{U}^{k+1} - \mathbf{U}^k) + \mathcal{N}(\mathbf{U}^k) = \mathbf{F}, \\ \mathbf{U}(-1) = 0, \mathbf{U}(1) = 0, \quad k = 0, 1, \dots \end{cases} \quad (20)$$

3.2. Analysis of LRKM. By improving the RKM, an effective and simple algorithm is derived to solve the linear system (20). For convenience, we rewrite (20) into the following form:

$$\begin{cases} \mathcal{L}\mathbf{U} = \mathbf{F}, \\ \mathbf{U}(-1) = 0, \mathbf{U}(1) = 0. \end{cases} \quad (21)$$

Lemma 2. $\mathcal{L}$ is linear and bounded.

Proof. Obviously, $\mathcal{L}_{11}, \mathcal{L}_{12}, \mathcal{L}_{21}$, and $\mathcal{L}_{22}$ are linear operators. For $\forall \mathbf{U} \in \mathbf{W}_n^0$, we have

$$\begin{aligned}\|\mathcal{L}\mathbf{U}\|_{L_2}^2 &= \sum_{i=1}^2 \|\mathcal{L}_{i1}u + \mathcal{L}_{i2}v\|_{L_2}^2 \\ &\leq \sum_{i=1}^2 (\|\mathcal{L}_{i1}\| \cdot \|u\|_{W_n} + \|\mathcal{L}_{i2}\| \cdot \|v\|_{W_n})^2 \\ &\leq 2 \sum_{i=1}^2 (\|\mathcal{L}_{i1}\|^2 + \|\mathcal{L}_{i2}\|^2) \cdot (\|u\|_{W_n}^2 + \|v\|_{W_n}^2) \\ &\leq 2 \sum_{i=1}^2 (\mathcal{L}_{i1}^2 + \|\mathcal{L}_{i2}\|^2) \cdot \|\mathbf{U}\|_{\mathbf{W}_n}^2.\end{aligned}\quad (22)$$

The boundedness of $\{\mathcal{L}_{ij}\}_{i,j=1}^2$ results in the boundedness of the operator $\mathcal{L}$.

Let

$$\begin{aligned}\Psi_{i1} &= \mathcal{L}^*(R(x, x_i), 0)^T, \\ \Psi_{i2} &= \mathcal{L}^*(0, R(x, x_i))^T,\end{aligned}\quad (23)$$

wherein $\{x_i\}_{i=1}^n$ are different points in $[-1, 1]$, $\mathcal{L}^*$ is the conjugate operator of $\mathcal{L}$, and R is the reproducing kernel of $L_2[-1, 1]$.

Lemma 3. $\Psi_{i1}(x) = \begin{pmatrix} \mathcal{L}_{11}K(x, x_i) \\ \mathcal{L}_{12}K(x, x_i) \end{pmatrix}, \Psi_{i2}(x) = \begin{pmatrix} \mathcal{L}_{21}K(x, x_i) \\ \mathcal{L}_{22}K(x, x_i) \end{pmatrix}, i = 1, 2, \dots, n.$

Proof. From the reproducibility of R , we immediately deduce that

$$\begin{aligned}\Psi_{i1}(x) &= \mathcal{L}^* \begin{pmatrix} R(x, x_i) \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{pmatrix} \begin{pmatrix} R(x, x_i) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{L}_{11}^* R(x, x_i) \\ \mathcal{L}_{12}^* R(x, x_i) \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{11}K(x, x_i) \\ \mathcal{L}_{12}K(x, x_i) \end{pmatrix}.\end{aligned}\quad (24)$$

Similarly, we have $\Psi_{i2}(x) = \begin{pmatrix} \mathcal{L}_{21}K(x, x_i) \\ \mathcal{L}_{22}K(x, x_i) \end{pmatrix}, i = 1, 2, \dots, n.$

Lemma 4. $\{\Psi_{11}, \Psi_{21}, \dots, \Psi_{n1}, \Psi_{12}, \Psi_{22}, \dots, \Psi_{n2}\}$ are linearly independent in $\mathbf{W}_n^0[-1, 1]$.

Considering the numerical implement, we apply the SRKM proposed in [12, 13] to solve equation (21). We first denote a finite-dimensional subspace $\mathbf{S}_{2n}$ by

$$\mathbf{S}_{2n} \triangleq \overline{\text{span}\{\Psi_{i2}\}_1^n \cup \{\Psi_{i1}\}_1^n}. \quad (25)$$

Let $\hat{\mathbf{U}}_n \triangleq \mathcal{P}_{2n+4} \hat{\mathbf{U}}$, where $\mathcal{P}_{2n} : \mathbf{W}_\infty \longrightarrow \mathbf{S}_{2n}$ is the projection operator. Then, $\hat{\mathbf{U}}_n$ converges to $\hat{\mathbf{U}}$ in the sense of $\|\cdot\|_{\mathbf{W}_n[-1,1]}$, which means that

$$\|\hat{\mathbf{U}} - \hat{\mathbf{U}}_n\|_{\mathbf{W}_n} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (26)$$

As $\hat{\mathbf{U}}_n \in \mathbf{W}_n[-1, 1]$, there exists undetermined coefficients a_{i1} and a_{i2} ($i = 1, 2, \dots, n$) such that

$$\hat{\mathbf{U}}_n(x) = \sum_{i=1}^n a_{i1} \Psi_{i1}(x) + \sum_{i=1}^n a_{i2} \Psi_{i2}(x). \quad (27)$$

The LRKM reads as follows: Finding $\hat{\mathbf{U}} \in \mathbf{W}_n^0[-1, 1]$ such that

$$\begin{cases} \mathcal{L}_{11}\hat{u}(x_i) + \mathcal{L}_{12}\hat{v}(x_i) = f_1(x_i), & i = 1, 2, \dots, n, \\ \mathcal{L}_{21}\hat{u}(x_i) + \mathcal{L}_{22}\hat{v}(x_i) = f_2(x_i), & i = 1, 2, \dots, n. \end{cases} \quad (28)$$

Once $\hat{\mathbf{U}}_n$ is obtained, one can get an approximate solution $\mathbf{U}_n$ of $\mathbf{U}$ by applying scaling from $[-1, 1]$ to $[a, b]$:

$$\hat{\mathbf{U}}_n(x) = \hat{\mathbf{U}} \left(\frac{2}{b-a}x - \frac{b+a}{b-a} \right). \quad (29)$$

Next, we will discuss the convergence of LRKM.

Theorem 2. $\hat{u}_n(x)$ and $\hat{v}_n(x)$ uniformly converge to $\hat{u}(x)$ and $\hat{v}(x)$ on $[-1, 1]$ as $n \longrightarrow \infty$, respectively.

Proof

$$\begin{aligned}|\hat{u}_n - \hat{u}| &= |(\hat{u}_n(\cdot) - \hat{u}(\cdot), K(\cdot, x))| \leq \|\hat{u}_n - \hat{u}\|_{W_n^2} \|K(x, x)\|_{W_n} \\ &\leq \sqrt{\|\hat{u}_n - \hat{u}\|_{W_n}^2 + \|\hat{v}_n - \hat{v}\|_{W_n}^2} \|K\|_{W_n} = \|\hat{\mathbf{U}}_n - \hat{\mathbf{U}}\|_{\mathbf{W}_n} \|K\|_{W_n}.\end{aligned}\quad (30)$$

From the continuity of K , we have $\|K\|_{W_n} \leq M$. Notice that $\|\hat{\mathbf{U}}_n - \hat{\mathbf{U}}\|_{\mathbf{W}_n} \longrightarrow 0$ as $n \longrightarrow \infty$, and we conclude that

$$|\hat{u}_n(x) - \hat{u}(x)| \leq M \|\hat{\mathbf{U}}_n - \hat{\mathbf{U}}\| \longrightarrow 0. \quad (31)$$

Analogously, $|\hat{v}_n(x) - \hat{v}(x)| \leq M \|\hat{\mathbf{U}}_n - \hat{\mathbf{U}}\| \longrightarrow 0$. The proof is completed.

4. Numerical Experiments

Example 3. Let us consider a linear problem suggested in [5, 10]:

$$\begin{cases} u''(x) + u'(x) + xu(x) + v'(x) + 2xv(x) = f(x), & x \in [0, 1], \\ 2u'(x) + x^2u(x) + v''(x) + v(x) = g(x), & x \in [0, 1], \\ u(0) = u(1) = 0, & v(0) = v(1) = 0. \end{cases} \quad (32)$$

Here, $f(x) = \pi \cos(\pi x) - 2(1+x)\cos(x) + (4x - 2x^2 - 4)\sin(x) + 2x\sin(\pi x)$, $g(x) = (1 - \pi^2)\sin(\pi x) + 2(-x^3 + x^2 + x - 3)\sin(x) - 4x\cos(x)$. The exact solution is $u(x) = 2(1-x)\sin(x)$, $v(x) = \sin(\pi x)$. Applying the LRKM, absolute errors are presented in Table 1 and Figure 1. Besides, we compare our method with RKM [5] and the sinc-collocation method [10] in Table 2. Numerical results show that LRKM has high accuracy in solving the linear equation, but uses the least number of points.

Example 4. Let us consider a nonlinear system proposed in [5, 10]:

$$\begin{cases} u''(x) + xu(x) + 2xv(x) + xu^2(x) = f, & x \in [0, 1], \\ v''(x) + v(x) + x^2u(x) + \sin(x)v^2(x) = g, & x \in [0, 1], \\ u(0) = u(1) = 0, & v(0) = v(1) = 0, \end{cases} \quad (33)$$

where $f = 2x\sin(\pi x) + x^5 - 2x^4 + x^2 - 2$ and $g = x^3(1-x) + (1 - \pi^2)\sin(\pi x) + \sin(x)\sin^2(\pi x)$. The exact solution is $u = x - x^2$; $v = \sin(\pi x)$. Selecting initial functions u_0, v_0 as the polynomial that satisfies the boundary conditions and applying the technique of QNM-LRKM three times, we depict numerical results of $u(x_i)$ and $v(x_i)$ in Table 3 and Figure 2. Numerical comparison with homotopy analysis and collocation method for problem 2 is shown in Table 4. Numerical results indicate that our method uses fewer nodes but obtains the more accurate numerical solution. We take $n = 11$ in LRKM (28) and different iteration steps

TABLE 1: Absolute errors for Example 3.

n	$\ u - u_n\ _{nC}$	$\ u' - u'_n\ _C$	$\ v - v_n\ _C$	$\ v' - v'_n\ _C$
5	3.32×10^{-4}	2.09×10^{-3}	1.51×10^{-3}	1.44×10^{-2}
7	1.80×10^{-6}	1.96×10^{-5}	1.85×10^{-5}	2.18×10^{-4}
9	9.44×10^{-9}	1.20×10^{-7}	1.13×10^{-7}	1.89×10^{-6}
11	4.21×10^{-10}	7.28×10^{-9}	1.28×10^{-9}	2.86×10^{-8}

FIGURE 1: Absolute error $|u - u_9|$ (a) and $|v - v_9|$ (b) for Example 3.

TABLE 2: Comparison with RKM [5] and the sinc-collocation method [10] for Example 3.

x_i	Method [5] ($n = 21$)		Method [10] ($N = 10$)		Our method ($n = 9$)	
	$\ u - u_n\ _{nC}$	$\ v - v_n\ _{nC}$	$\ u - u_n\ _C$	$\ v - v_n\ _C$	$\ u - u_n\ _{nC}$	$\ v - v_n\ _C$
0.08	8.0×10^{-4}	1.9×10^{-3}	3.0×10^{-4}	2.0×10^{-3}	4.03×10^{-10}	5.86×10^{-8}
0.24	1.9×10^{-3}	5.1×10^{-3}	8.5×10^{-5}	9.8×10^{-4}	6.57×10^{-9}	6.68×10^{-10}
0.40	2.4×10^{-3}	6.9×10^{-3}	3.5×10^{-4}	1.1×10^{-3}	5.56×10^{-9}	3.59×10^{-8}
0.56	2.4×10^{-3}	6.9×10^{-3}	2.6×10^{-4}	1.4×10^{-3}	8.16×10^{-9}	3.62×10^{-8}
0.72	1.8×10^{-3}	5.2×10^{-3}	2.0×10^{-4}	5.5×10^{-5}	1.13×10^{-9}	3.68×10^{-8}
0.88	8.0×10^{-4}	2.4×10^{-3}	2.6×10^{-4}	7.7×10^{-4}	2.37×10^{-9}	2.37×10^{-9}
0.96	2.0×10^{-4}	8.0×10^{-4}	2.6×10^{-3}	8.3×10^{-4}	2.24×10^{-9}	3.98×10^{-8}

TABLE 3: Absolute errors for Example 4.

n	$\ u - u_n\ _C$	$\ u' - u'_n\ _C$	$\ v - v_n\ _{nC}$	$\ v' - v'_n\ _C$
5	2.92×10^{-5}	2.55×10^{-4}	1.50×10^{-3}	1.46×10^{-2}
7	3.62×10^{-7}	1.95×10^{-5}	1.88×10^{-5}	2.12×10^{-4}
9	1.84×10^{-9}	9.84×10^{-8}	1.14×10^{-7}	1.88×10^{-6}
11	2.67×10^{-10}	2.43×10^{-9}	2.73×10^{-9}	4.42×10^{-8}

FIGURE 2: Absolute error $|u - u_9|$ (a) and $|v - v_9|$ (b) for Example 4.

TABLE 4: Comparison with homotopy method and collocation method for Example 4.

x_i	Method [10] ($N = 10$)		Method [5] ($n = 21$)		Our method ($n = 9$)	
	$\ u - u\ _{nC}$	$\ v - v\ _{nC}$	$\ u - u_n\ _C$	$\ v - v_n\ _C$	$\ u - u\ _{nC}$	$\ v - v\ _{nC}$
0.08	1.4×10^{-4}	2.4×10^{-4}	5.0×10^{-4}	2.0×10^{-3}	3.76×10^{-10}	5.82×10^{-8}
0.24	4.4×10^{-5}	2.3×10^{-3}	1.4×10^{-3}	5.6×10^{-3}	1.09×10^{-9}	1.36×10^{-9}
0.40	6.7×10^{-5}	8.9×10^{-4}	2.1×10^{-3}	7.9×10^{-3}	1.59×10^{-9}	3.65×10^{-8}
0.56	9.3×10^{-5}	1.4×10^{-3}	2.2×10^{-3}	8.2×10^{-3}	1.82×10^{-9}	3.62×10^{-8}
0.72	4.9×10^{-5}	3.1×10^{-3}	1.8×10^{-3}	6.5×10^{-3}	1.70×10^{-9}	7.29×10^{-8}
0.88	8.6×10^{-5}	1.6×10^{-3}	9.0×10^{-4}	3.1×10^{-3}	8.80×10^{-10}	2.21×10^{-9}
0.96	7.1×10^{-5}	9.8×10^{-4}	3.0×10^{-4}	1.0×10^{-3}	3.35×10^{-10}	4.00×10^{-8}

TABLE 5: Convergence order of NDM for Example 4.

k	$\ u - u\ _C^k$	$ \varepsilon_{k+1}/\varepsilon_k^2 $	$\ v - v^k\ _C$	$ \varepsilon_{k+1}/\varepsilon_k^2 $
1	1.57×10^{-2}		1.27×10^{-2}	
2	4.08×10^{-5}	0.160	1.61×10^{-5}	0.0998
3	2.67×10^{-10}	0.165	2.52×10^{-11}	0.0972

k , and absolute errors are shown in Table 5. Numerical results indicate that the convergence order of NDM is nearly 2.

5. Conclusions

This paper presents an efficient numerical method for solving second-order nonlinear system of BVPs. The algorithm is easy to implement. Numerical results verify that the QNM-LRKM is a reliable numerical technique of high-order accuracy. Precisely, NDM ensures the speed of iteration, while LRKM guarantees the accuracy of the algorithm.

Data Availability

No datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] R. Dalmasso, "An existence and uniqueness theorem for a second order nonlinear system," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 1, pp. 715–722, 2007.
- [2] H. B. Thompson and C. Tisdell, "Boundary value problems for systems of difference equations associated with systems of second-order ordinary differential equations," *Applied Mathematics Letters*, vol. 15, no. 6, pp. 761–766, 2002.
- [3] J. Henderson and R. Luca, "Positive solutions for a system of second-order multi-point boundary value problems," *Applied Mathematics and Computation*, vol. 218, no. 10, pp. 6083–6094, 2012.
- [4] O. A. Arqub and Z. Abo-Hammour, "Numerical solution of systems of second-order boundary value problems using continuous genetic algorithm," *Information Sciences*, vol. 279, pp. 396–415, 2014.
- [5] F. Geng and M. Cui, "Solving a nonlinear system of second order boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 2, pp. 1167–1181, 2007.
- [6] F. Geng and M. Cui, "Homotopy perturbation-reproducing kernel method for nonlinear systems of second order boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 235, no. 8, pp. 2405–2411, 2011.
- [7] M. Li, H. Peng, and W. Zhong, "A symplectic sequence iteration approach for nonlinear optimal control problems with state-control constraints," *Journal of the Franklin Institute*, vol. 352, no. 6, pp. 2381–2406, 2015.
- [8] H. Peng, Q. Gao, Z. Wu, and W. Zhong, "Symplectic approaches for solving two-point boundary-value problems," *Journal of Guidance, Control, and Dynamics*, vol. 35, no. 2, pp. 653–659, 2012.
- [9] H. J. Peng, Q. Gao, Z. G. Wu, and W. X. Zhong, "Symplectic adaptive algorithm for solving nonlinear two-point boundary value problems in astrodynamics," *Celestial Mechanics and Dynamical Astronomy*, vol. 110, no. 4, pp. 319–342, 2011.
- [10] M. Dehghan and A. Saadatmandi, "The numerical solution of a nonlinear system of second-order boundary value problems using the sinc-collocation method," *Mathematical and Computer Modelling*, vol. 46, no. 11–12, pp. 1434–1441, 2007.
- [11] M. R. Eslahchi, M. Dehghan, and S. Ahmadi Asl, "The general Jacobi matrix method for solving some nonlinear ordinary differential equations," *Applied Mathematical Modelling*, vol. 36, no. 8, pp. 3387–3398, 2012.
- [12] M. Q. Xu, Y. Z. Lin, and Y. H. Wang, "A new algorithm for nonlinear fourth order multi-point boundary value problems," *Applied Mathematics and Computation*, vol. 274, pp. 163–168, 2016.
- [13] M.-Q. Xu and Y.-Z. Lin, "Simplified reproducing kernel method for fractional differential equations with delay," *Applied Mathematics Letters*, vol. 52, pp. 156–161, 2016.
- [14] Z. Chen, L. Wu, and Y. Lin, "Exact solution of a class of fractional integro-differential equations with the weakly singular kernel based on a new fractional reproducing kernel space," *Mathematical Methods in the Applied Sciences*, vol. 41, no. 10, pp. 3841–3855, 2018.
- [15] B. Azarnavid and K. Parand, "An iterative reproducing kernel method in Hilbert space for the multi-point boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 328, pp. 151–163, 2018.
- [16] J. Yang, H. Yao, and B. Wu, "An efficient numerical method for variable order fractional functional differential equation," *Applied Mathematics Letters*, vol. 76, pp. 221–226, 2018.

- [17] L. Mei and Y. Lin, "Simplified reproducing kernel method and convergence order for linear Volterra integral equations with variable coefficients," *Journal of Computational and Applied Mathematics*, vol. 346, pp. 390–398, 2019.
- [18] J. Niu, M. Xu, and G. Yao, "An efficient reproducing kernel method for solving the Allen-Cahn equation," *Applied Mathematics Letters*, vol. 89, pp. 78–84, 2019.
- [19] J. Niu, L. Sun, M. Xu, and J. Hou, "A reproducing kernel method for solving heat conduction equations with delay," *Applied Mathematics Letters*, vol. 100, Article ID 106036, 2020.
- [20] M. Xu, Z. Zhao, J. Niu, L. Guo, and Y. Lin, "A simplified reproducing kernel method for 1-D elliptic type interface problems," *Journal of Computational and Applied Mathematics*, vol. 351, pp. 29–40, 2019.
- [21] H. Li, W. Jiang, and W. Li, "Space-time spectral method for the Cattaneo equation with time fractional derivative," *Applied Mathematics and Computation*, vol. 349, pp. 325–336, 2019.
- [22] F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer, New York, NY, USA, 2001.
- [23] P. R. McGillivray and D. W. Olenburg, "Methods for calculating frechet derivatives and sensitivities for the non-linear inversion problem," *Geophysical Prospecting*, vol. 38, no. 5, pp. 499–524, 1990.