Research Article

Archimedean Copula-Based Hesitant Fuzzy Information Aggregation Operators for Multiple Attribute Decision Making

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In view of the good properties of copulas and their effective use in various fuzzy environments, the goal of the current study is to develop a series of aggregation operators for hesitant fuzzy information based on Archimedean copula and cocopula, which are applied to the MADM problems. Firstly, operational laws of hesitant fuzzy elements on the basis of copulas and cocopulas are defined which can show the relevance between hesitant fuzzy values. Secondly, four aggregation operators (AC-HFWA, AC-GHFWA, AC-HFWG, and AC-GHFWG) under hesitant fuzzy environment are developed according to the proposed operational laws. The properties of these operators are also studied in detail, including idempotence, monotonicity, boundedness, etc. Subsequently, five special cases of copula are also given and the special forms of aggregation operator are obtained. In the end, an example is used to illustrate the application of the proposed approach in MADM problems. The influences of different generated functions and parameters are shown, and the feasibility of the proposed method is validated through comparative analyses.

1. Introduction

Multiple attribute decision making (MADM), also known as limited scheme multiobjective decision, is to select the optimal alternatives or ranking decision making problems in the case of considering multiple attributes. It is a vital part of modern decision science; its theories and methods have been widely utilized in engineering, technology, economy, management, military, and many other fields. One of the most important tasks of MADM is to fuse the attribute values given to each alternative by the decision maker and then summarize the decision maker's opinion on each alternative. In this process, a primary issue is to describe the values of criteria. For this issue, many experts proposed to adopt fuzzy sets. MADM problems with different kinds of fuzzy information are handled by utilizing fuzzy set (FS) [1] which is proposed by Zadeh and their various extensions, including the intuitionistic fuzzy set (IFS) [2], interval-valued intuitionistic fuzzy set (IVIFS) [3], hesitant fuzzy set (HFS) [4, 5], Pythagorean fuzzy set (PFS) [6], neutrosophic set (NS) [7], and so on.

In the numerous extensions of the FS, IFS as one of the most important, was introduced by Atanassov [2]. Because it provides a membership degree (MD), a nonmembership degree (NMD), and a hesitancy degree (HD) to each element, IFS is better at handling uncertainty and vagueness than FS. Since its emergence, IFS has attracted more and more researchers' attention. However, when giving the membership degree of an element, the difficulty of establishing the membership degree is not because we have a margin of error or some possibility distribution on the possibility values but because we have several possible values. For such cases, Torra and Narukawa [4] proposed hesitant fuzzy set (HFS) and indicated that the envelope of a hesitant fuzzy element (HFE) is an intuitionistic fuzzy value (IFV). So, all the operations on IFS can be suitable for HFS, and many research studies of IFS can be extended to HFS.

The aggregation operator, which fuses multiple information sources, plays a key role in the realization of collective opinions in MADM. In order to deal with information in different fuzzy environments, various

aggregation operators are proposed. Weighted average (WA) operator and weighted geometry (WG) operator are the most commonly used integration operators in classical decision science theory. In the process of MADM, they have been deeply studied by scholars [8–12], which have been extended to the integration of different types of decision information, such as ordered weighted averaging operator (OWA) and ordered weighted geometry operator (OWG). Based on the defined operations for IFS, Xia and Xu [13] presented eight hesitant fuzzy aggregation operators, such as hesitant fuzzy weighted averaging (HFWA) operator, hesitant fuzzy weighted geometric (HFWG) operator, and so on. According to the operators mentioned above, many scholars investigated many operators to solve MCDM problems under hesitant fuzzy environment [14–21]. Qin et al. [22] developed some hesitant fuzzy aggregation operators based on Frank operations, such as HFFWA operator, HFFOWA operator, and so on. Yu et al. [23] studied a set of hesitant fuzzy Einstein aggregation operators, such as HFECOA operator, HFECOG operator, HFEPWA operator, and HFEPWG operator. Using the technique of obtaining values in the interval, Du et al. [24] proposed the generalized hesitant fuzzy harmonic mean operators including GHFHHM operator, GHFOWHM operator, and GHFHMM operator. Li and Chen [25] presented two new aggregation operators: belief structure hesitant fuzzy induced ordered weighted averaging operator and belief structure hesitant fuzzy induced ordered weighted geometric operator. Although the research and application of the integration operator have been well developed, the decision problem based on the integration operator has certain complexity, so it is necessary to conduct in-depth research on it and explore new information integration methods.

In the aforementioned aggregation operators under hesitant fuzzy environment, the operational laws of any two HFEs are built on the t-norms (TCs) and t-conorms (TCs). Commonly, TNs are applied to integrate MD of fuzzy sets, while copulas are tools to deal with probability distributions. Besides, there exist also TNs which are copulas and vice versa. Thus, the application of copulas in fuzzy sets has important practical significance. Copulas [26] can not only reveal the dependence among attributes but also prevent information loss in the midst of aggregation. There are two distinguishing features of copula: (1) copulas and cocopulas are flexible because decision makers can select different types of copulas and cocopulas to define the operations under fuzzy environment, and the results obtained from these operations are closed; (2) copula functions are flexible to capture the correlations among attributes in MADMs. Based on the two obvious characteristic, copulas have been applied to some MADMs. In the light of Archimedean copula, Tao et al. [27] studied a new computational model for unbalanced linguistic variables. Chen et al. [28] defined new aggregation operators in linguistic neutrosophic set based on copula and applied them to settle MCDM problems.

In this paper, based on the current research, the copulas are generalized to the HFS, and two kinds of hesitating fuzzy information integration operators based on copulas are proposed, which are applied to the MADM problems. For the goals, the structure of this work is arranged as follows. Some notions on hesitant fuzzy set and copulas are reviewed

firstly in Section 2. The hesitant fuzzy weighted averaging operator-based Archimedean copulas (AC-HFWA) are defined in Section 3; before AC-HFWA is given, the operations of hesitant fuzzy elements based on Archimedean copula are also defined. After AC-HFWA is given, the generalized hesitant fuzzy weighted averaging operator-based Archimedean copulas (AC-GHFWA) are introduced, and the properties of AC-HFWA and AC-HFWG are investigated along with the different cases. The hesitant fuzzy weighted geometry operator-based Archimedean copulas (AC-HFWG) are defined in Section 4; before AC-HFWG is given, the operations of hesitant fuzzy elements based on Archimedean copula are also defined. After AC-HFWG is given, the generalized hesitant fuzzy weighted geometry operator-based Archimedean copulas (AC-GHFWG) are introduced, and the properties of AC-HFWA and AC-HFWG are investigated along with the different cases. In Section 5, the algorithm of MADM with hesitant fuzzy information based on AC-HFWA/AC-HFWG is constructed firstly; next, case analysis will be carried out and some comparisons with existing approaches in the hesitant fuzzy environment and merits of the proposed MADM approach based on AC-HFWA/AC-HFWG operators are analysed, and the conclusion will be obtained in Section 6.

2. Preliminaries

In this section, we will retrospect the related concepts of HFS and copula and cocopula; these notions are the basis of this work.

2.1. Hesitant Fuzzy Sets

Definition 1 (see [5]). Let S be a finite reference set. A hesitant fuzzy set G on S in terms of a function when applied to S returns a subset of $[0, 1]$ denoted by

$$G = \{\langle s, g(h) \rangle \mid \forall s \in S\}, \quad (1)$$

where $g(h)$ is a collection of numbers h_i from $[0, 1]$, indicating the possible membership degrees of $\forall s \in S$ to G . We call $g(h)$ a hesitant fuzzy element (HFE) and G the set of all HFEs.

To compare the HFEs, the comparison laws are defined as follows [5].

Definition 2 (see [5]). For a HFE $g(h) = \cup_{i=1}^{\sharp g} \{h_i\}$, $\mu(g) = (1/\sharp g) \sum_{i=1}^{\sharp g} h_i$ is called the score function of $g(h)$, where $\sharp g$ is the number of possible elements in $g(h)$.

For two HFEs $g_1(h)$ and $g_2(h)$,

If $\mu(g_1) > \mu(g_2)$, then $g_1 \succ g_2$;

If $\mu(g_1) = \mu(g_2)$, then $g_1 = g_2$.

2.2. Copulas and Cocopulas

Definition 3 (see [26]). A two-dimensional function $\Omega: [0, 1]^2 \rightarrow [0, 1]$ is called a copula, if the following conditions are met:

- (1) $\Omega(m, 1) = \Omega(1, m) = m$, $\Omega(m, 0) = \Omega(0, m) = 0$
 (2) $\Omega(m_1, n_1) - \Omega(m_2, n_1) - \Omega(m_1, n_2) + \Omega(m_2, n_2) \geq 0$

where $m, m_1, m_2, n_1, n_2 \in [0, 1]$ and $m_1 \leq m_2, n_1 \leq n_2$.

Definition 4 (see [29]). A copula Ω is named as an Archimedean copula, if there is a strictly decreasing and continuous function $\varsigma(\delta): [0, 1] \rightarrow [0, \infty]$ with $\varsigma(1) = 0$, and σ from $[0, \infty]$ to $[0, 1]$ is defined as follows:

$$\sigma(\delta) = \begin{cases} \varsigma^{-1}(\delta), & \delta \in [0, \varsigma(0)]; \\ 0, & \delta \in [\varsigma(0), +\infty). \end{cases} \quad (2)$$

For all $(\delta, \varepsilon) \in [0, 1]^2$, we have

$$\sigma(\delta, \varepsilon) = \sigma(\varsigma(\delta) + \varsigma(\varepsilon)). \quad (3)$$

If Ω is strictly increasing on $[0, 1]^2$, $\varsigma(0) = +\infty$, and σ coincides with ς^{-1} on $[0, +\infty]$, then Ω is written as [30]

$$\Omega(\delta, \varepsilon) = \varsigma^{-1}(\varsigma(\delta) + \varsigma(\varepsilon)), \quad (4)$$

and the function ς is called a strict generator and Ω is called a strict Archimedean copula.

Definition 5 (see [31]). Let Ω be a copula, and the cocopula is introduced as follows:

$$\Omega^*(\delta, \varepsilon) = 1 - \Omega(1 - \delta, 1 - \varepsilon). \quad (5)$$

If Ω is a strict Archimedean copula, Ω^* is also changed to be

$$\Omega^*(\delta, \varepsilon) = 1 - \Omega(1 - \delta, 1 - \varepsilon) = 1 - \varsigma^{-1}(\varsigma(1 - \delta) + \varsigma(1 - \varepsilon)). \quad (6)$$

In order to introduce some new operations based on copulas and cocopulas mentioned above, the following conclusion is given firstly.

Theorem 1. For $\forall \delta, \varepsilon \in [0, 1]$, then $0 \leq \Omega(\delta, \varepsilon) \leq 1$, $0 \leq \Omega^*(\delta, \varepsilon) \leq 1$.

Proof. If $0 \leq \delta \leq \varepsilon \leq 1$, then $0 \leq 1 - \varepsilon \leq 1 - \delta \leq 1$. As ς is strictly decreasing and $\varsigma(1) = 0$, $\varsigma(0) = +\infty$,

$$\begin{aligned} 0 &\leq \varsigma(\varepsilon) \leq \varsigma(\delta) \leq +\infty, \\ 0 &\leq \varsigma(1 - \delta) \leq \varsigma(1 - \varepsilon) \leq +\infty. \end{aligned} \quad (7)$$

So,

$$\begin{aligned} \varsigma(\delta) &\leq \varsigma(\delta) + \varsigma(\varepsilon) \leq 2\varsigma(\delta) \leq +\infty, \\ \varsigma(1 - \varepsilon) &\leq \varsigma(1 - \delta) + \varsigma(1 - \varepsilon) \leq 2\varsigma(1 - \varepsilon) \leq +\infty. \end{aligned} \quad (8)$$

We have

$$0 \leq \varsigma^{-1}(\varsigma(\delta) + \varsigma(\varepsilon)) \leq \delta \leq \varepsilon \leq 1 - \varsigma^{-1}(\varsigma(1 - \delta) + \varsigma(1 - \varepsilon)) \leq 1. \quad (9)$$

Thus, Theorem 1 holds. $\square$

Definition 6. Let $\delta, \varepsilon \in [0, 1]$; the algebra operations based on copula and cocopula are defined as follows:

$$\begin{aligned} (1) \delta \oplus \varepsilon &= \Omega^*(\delta, \varepsilon) = 1 - \varsigma^{-1}(\varsigma(1 - \delta) + \varsigma(1 - \varepsilon)), \\ (2) \delta \otimes \varepsilon &= \Omega(\delta, \varepsilon) = \varsigma^{-1}(\varsigma(\delta) + \varsigma(\varepsilon)). \end{aligned} \quad (10)$$

It is easy to verify that $\oplus$ and $\otimes$ satisfy associative law, that is, for $\forall \delta, \varepsilon, \nu \in [0, 1]$,

$$\begin{aligned} (\delta \oplus \varepsilon) \oplus \nu &= \delta \oplus (\varepsilon \oplus \nu), \\ (\delta \otimes \varepsilon) \otimes \nu &= \delta \otimes (\varepsilon \otimes \nu). \end{aligned} \quad (11)$$

Theorem 2. For $\forall \delta \in [0, 1]$, $\rho \geq 0$, we have $\rho\delta = 1 - \varsigma^{-1}(\rho\varsigma(1 - \delta))$, $\delta^\rho = \varsigma^{-1}(\rho\varsigma(\delta))$.

3. Archimedean Copula-Based Hesitant Fuzzy Weighted Averaging Operator (AC-HFWA)

In this part, we will put forward the Archimedean copula-based HF weighted averaging operator (AC-HFWA). Before AC-HFWA is introduced, the new operations of HFE based on copula will be defined, and then some properties of AC-HFWA are also investigated.

3.1. New Operations for HFEs Based on Copulas. We will give a new version of operational rules based on copulas and cocopulas.

Definition 7. Let $g_1(h) = \cup_{m_1=1}^{\sharp g_1} \{h_{1m_1}\}$, $g_2(h) = \cup_{m_2=1}^{\sharp g_2} \{h_{2m_2}\}$, and $g(h) = \cup_{i=1}^{\sharp g} \{h_i\}$ be three HFEs and $\rho \geq 0$; the novel operational rules of HFEs are given as follows:

$$\begin{aligned} g_1 \oplus g_2 &= \bigcup_{\substack{h_{1m_1} \in g_1 \\ h_{2m_2} \in g_2}} \left\{ 1 - \varsigma^{-1}(\varsigma(1 - h_{1m_1}) + \varsigma(1 - h_{2m_2})) \mid m_1 = 1, 2, \dots, \sharp g_1, m_2 = 1, 2, \dots, \sharp g_2 \right\}, \\ g_1 \otimes g_2 &= \bigcup_{\substack{h_{1m_1} \in g_1 \\ h_{2m_2} \in g_2}} \left\{ \varsigma^{-1}(\varsigma(h_{1m_1}) + \varsigma(h_{2m_2})) \mid m_1 = 1, 2, \dots, \sharp g_1, m_2 = 1, 2, \dots, \sharp g_2 \right\}, \\ \rho g &= \bigcup_{h_i \in g} \left\{ 1 - \varsigma^{-1}(\rho\varsigma(1 - h_i)) \mid i = 1, 2, \dots, \sharp g \right\}, \\ g^\rho &= \bigcup_{h_i \in g} \left\{ \varsigma^{-1}(\rho\varsigma(h_i)) \mid i = 1, 2, \dots, \sharp g \right\}. \end{aligned} \quad (12)$$

From the above definition, the following conclusions can be easily drawn.

Theorem 3. Let g_1, g_2 , and g_3 be three HFEs and $a, b, c \in R^+$; then, we have

$$\begin{aligned} (1) & g_1 \oplus g_2 = g_2 \oplus g_1, \\ (2) & (g_1 \oplus g_2) \oplus g_3 = g_1 \oplus (g_2 \oplus g_3), \\ (3) & ag_1 \oplus bg_1 = (a + b)g_1, \\ (4) & a(bg_1 \oplus cg_2) = abg_1 \oplus acg_2, \\ (5) & a(bg_1) = abg_1, \\ (6) & g_1 \otimes g_2 = g_2 \otimes g_1, \\ (7) & (g_1 \otimes g_2) \otimes g_3 = g_1 \otimes (g_2 \otimes g_3). \end{aligned} \quad (13)$$

The algorithms can be used to fuse the HF information and investigate their ideal properties which is the focus of the following sections.

3.2. AC-HFWA. In this section, the AC-HFWA will be introduced and the proposed operations of HFEs based on copula as well as the properties of AC-HFWA are investigated.

Definition 8. Let $G = \{g_1, g_2, \dots, g_n\}$ be a set of n HFEs and Φ be a function on G , $\Phi: [0, 1]^n \rightarrow [0, 1]$; then, $\Phi(G) = \cup\{\Phi(g_1, g_2, \dots, g_n)\}$.

Definition 9. Let $g_i(h) = \cup_{m_i=1}^{\sharp g_i} \{h_{im_i} \mid i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. Archimedean copula-based hesitant fuzzy weighted averaging operator (AC-HFWA) is defined as follows:

$$\text{AC-HFWA}(g_1, g_2, \dots, g_n) = \omega_1 g_1 \oplus \omega_2 g_2 \oplus \dots \oplus \omega_n g_n. \quad (14)$$

Theorem 4. Let $g_i(h) = \cup_{m_i=1}^{\sharp g_i} \{h_{im_i} \mid i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$; then,

$$\text{AC-HFWA}(g_1, g_2, \dots, g_n) = \bigoplus_{i=1}^n \omega_i g_i = \bigcup_{h_{im_i} \in g_i} \left\{ 1 - \tau^{-1} \left(\sum_{i=1}^n \omega_i \tau(1 - h_{im_i}) \right) \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \quad (15)$$

Proof. For $n = 2$, we have

$$\begin{aligned} \text{AC-HFWA}(g_1, g_2) &= \omega_1 g_1 \oplus \omega_2 g_2 \\ &= \bigcup_{\substack{h_{1m_1} \in g_1 \\ h_{2m_2} \in g_2}} \left\{ 1 - \tau^{-1} (\omega_1 \tau(1 - h_{1m_1}) + \omega_2 \tau(1 - h_{2m_2})) \mid m_1 = 1, 2, \dots, \sharp g_1, m_2 = 1, 2, \dots, \sharp g_2 \right\}. \end{aligned} \quad (16)$$

Suppose that equation (15) holds for $n = k$, that is,

$$\begin{aligned} \text{AC-HFWA}(g_1, g_2, \dots, g_k) &= \omega_1 g_1 \oplus \omega_2 g_2 \oplus \dots \oplus \omega_k g_k \\ &= \bigcup_{h_{im_i} \in g_i} \left\{ 1 - \tau^{-1} \left(\sum_{i=1}^k \omega_i \tau(1 - h_{im_i}) \right) \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \end{aligned} \quad (17)$$

Then,

$$\begin{aligned} \text{AC-HFWA}(g_1, g_2, \dots, g_k, g_{k+1}) &= \bigoplus_{i=1}^k \omega_i g_i \oplus \omega_{k+1} g_{k+1} \\ &= \bigcup_{h_{im_i} \in g_i} \left\{ 1 - \tau^{-1} \left(\sum_{i=1}^k \omega_i \tau(1 - h_{im_i}) \right) \mid m_i = 1, 2, \dots, \sharp g_i \right\} \\ &\quad \oplus \bigcup_{h_{k+1,m_{k+1}} \in g_{k+1}} \left\{ 1 - \tau^{-1} (\omega_{k+1} \tau(1 - h_{k+1,m_{k+1}})) \mid m_{k+1} = 1, 2, \dots, \sharp g_{k+1} \right\} \\ &= \bigcup_{h_{im_i} \in g_i} \left\{ 1 - \tau^{-1} \left(\sum_{i=1}^k \omega_i \tau(1 - h_{im_i}) + \omega_{k+1} \tau(1 - h_{k+1,m_{k+1}}) \right) \mid m_i = 1, 2, \dots, \sharp g_i \right\} \\ &= \bigcup_{h_{im_i} \in g_i} \left\{ 1 - \tau^{-1} \left(\sum_{i=1}^{k+1} \omega_i \tau(1 - h_{im_i}) \right) \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \end{aligned} \quad (18)$$

Equation (15) holds for $n = k + 1$. Thus, equation (15) holds for all n . $\square$

Theorem 5. Let $g_i(h) = \cup_{m_i=1}^{h_{g_i}} \{h_{im_i} | i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$; then,

- (1) (Idempotency) If $g_1 = g_2 = \dots = g_n = \{h\}$, $AC-HFWA(g_1, g_2, \dots, g_n) = \{h\}$.
- (2) (Monotonicity) Let $g_i^*(h) = \cup_{m_i=1}^{h_{g_i}^*} \{h_{im_i}^* | i = 1, 2, \dots, n\}$; if $h_{im_i} \leq h_{im_i}^*$,
 $AC-HFWA(g_1, g_2, \dots, g_n) \leq AC-HFWA(g_1^*, g_2^*, \dots, g_n^*)$. (19)

- (3) (Boundedness) If $h^- = \min_{i=1,2,\dots,n} \{h_{im_i}\}$ and $h^+ = \max_{i=1,2,\dots,n} \{h_{im_i}\}$,
 $h^- \leq AC-HFWA(g_1, g_2, \dots, g_n) \leq h^+$. (20)

Proof (1) $AC-HFWA(g_1, g_2, \dots, g_n) = \cup \{1 - \zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(1-h)) | h \in g_i, i = 1, 2, \dots, n\} = \{h\}$.
 (2) If $h_{im_i} \leq h_{im_i}^*$, $\zeta(1-h_{im_i}) \leq \zeta(1-h_{im_i}^*)$ and $\sum_{i=1}^n \omega_i \zeta(1-h_{im_i}) \leq \sum_{i=1}^n \omega_i \zeta(1-h_{im_i}^*)$.
 Then, $\zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(1-h_{im_i})) \geq \zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(1-h_{im_i}^*))$ and $1 - \zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(1-h_{im_i})) \leq 1 - \zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(1-h_{im_i}^*))$.
 So, $AC-HFWA(g_1, g_2, \dots, g_n) \leq AC-HFWA(g_1^*, g_2^*, \dots, g_n^*)$.
 (3) Suppose $h^- = \min_{i=1,2,\dots,n} \{h_{im_i}\}$ and $h^+ = \max_{i=1,2,\dots,n} \{h_{im_i}\}$.
 Therefore, $1 - h^+ \leq 1 - h_{im_i}$, $1 - h^- \geq 1 - h_{im_i}$, for all i and m_i .

Since ζ is strictly decreasing, ζ^{-1} is also strictly decreasing.

Then, $\zeta(1-h^-) \leq \zeta(1-h_{im_i}) \leq \zeta(1-h^+)$, $\forall i = 1, 2, \dots, n$, and so

$$\sum_{i=1}^n \omega_i \zeta(1-h^-) \leq \sum_{i=1}^n \omega_i \zeta(1-h_{im_i}) \leq \sum_{i=1}^n \omega_i \zeta(1-h^+). \quad (21)$$

That is, $\zeta(1-h^-) \leq \sum_{i=1}^n \omega_i \zeta(1-h_{im_i}) \leq \zeta(1-h^+)$.

Therefore, $\zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(1-h_{im_i})) \leq \zeta^{-1}(\zeta(1-h^-))$, $h_0 \leq 1 - \zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(1-h_{im_i})) \leq h^+$. $\square$

Definition 10. Let $g_i(h) = \cup_{m_i=1}^{h_{g_i}} \{h_{im_i} | i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. The Archimedean copula-based generalized hesitant fuzzy averaging operator (AC-GHFWA) is given by

$$AC-GHFWA_\theta(g_1, g_2, \dots, g_n) = (\omega_1 g_1^\theta \oplus \omega_2 g_2^\theta \oplus \dots \oplus \omega_n g_n^\theta)^{1/\theta} = \left(\bigoplus_{i=1}^n \omega_i g_i^\theta \right)^{1/\theta}. \quad (22)$$

Especially, when $\theta = 1$, the AC-GHFWA operator becomes the AC-HFWA operator.

The following theorems are easily obtained from Theorem 4 and the operations of HFEs.

Theorem 6. Let $g_i(h) = \cup_{m_i=1}^{h_{g_i}} \{h_{im_i} | i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$; then,

$$AC-GHFWA_\theta(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ -\zeta^{-1} \left(\frac{1}{\theta} \left(\zeta \left(1 - \zeta^{-1} \left(\sum_{i=1}^n \omega_i \left(\zeta \left(1 - \zeta^{-1}(\theta \zeta(h_{im_i})) \right) \right) \right) \right) \right) \right) \middle| m_i = 1, 2, \dots, h_{g_i} \right\}. \quad (23)$$

Similar to Theorem 5, the properties of AC-GHFWA can be obtained easily.

Theorem 7. Let $g_i(h) = \cup_{m_i=1}^{h_{g_i}} \{h_{im_i} | i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$; then,

- (1) (Idempotency) If $g_1 = g_2 = \dots = g_n = \{h\}$, $AC-GHFWA_\theta(g_1, g_2, \dots, g_n) = \{h\}$.
- (2) (Monotonicity) Let $g_i^*(h) = \cup_{m_i=1}^{h_{g_i}^*} \{h_{im_i}^* | i = 1, 2, \dots, n\}$; if $h_{im_i} \leq h_{im_i}^*$,
 $AC-GHFWA_\theta(g_1, g_2, \dots, g_n) \leq AC-GHFWA_\theta(g_1^*, g_2^*, \dots, g_n^*)$. (24)

- (3) (Boundedness) If $h^- = \min_{i=1,2,\dots,n} \{h_{im_i}\}$ and $h^+ = \max_{i=1,2,\dots,n} \{h_{im_i}\}$,

$$h^- \leq AC-GHFWA_\theta(g_1, g_2, \dots, g_n) \leq h^+. \quad (25)$$

3.3. Different Forms of AC-HFWA. We can see from Theorem 4 that some specific AC-HFWAs can be obtained when ζ is assigned different generators.

Case 1. If $\zeta(t) = (-\ln t)^\kappa$, $\kappa \geq 1$, then $\zeta^{-1}(t) = e^{-t^{1/\kappa}}$. So,
 $\delta \oplus \varepsilon = 1 - e^{-((- \ln(1-\delta))^\kappa + (- \ln(1-\varepsilon))^\kappa)^{1/\kappa}}$,
 $\delta \otimes \varepsilon = 1 - e^{-((- \ln(\delta))^\kappa + (- \ln(\varepsilon))^\kappa)^{1/\kappa}}$.

$$\text{AC-HFWA}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ 1 - e^{-\left(\sum_{i=1}^n \omega_i (-\ln(1-h_{im_i}))^\lambda\right)^{1/\lambda}} \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \quad (26)$$

Specifically, when $\kappa = 1$, $\zeta(t) = -\ln t$, then $\delta \oplus \varepsilon = 1 - (1 - \delta)(1 - \varepsilon)$, $\delta \otimes \varepsilon = \delta \varepsilon$, and the AC-HFWA becomes the following:

$$\begin{aligned} \text{HFWA}(g_1, g_2, \dots, g_n) &= \bigcup_{h_{im_i} \in g_i} \left\{ 1 - \prod_{i=1}^n (1 - h_{im_i})^{\omega_i} \mid m_i = 1, 2, \dots, \sharp g_i \right\}, \\ \text{GHFWA}_\theta(g_1, g_2, \dots, g_n) &= \bigcup_{h_{im_i} \in g_i} \left\{ \left(1 - \prod_{i=1}^n (1 - h_{im_i})^{\omega_i} \right)^{1/\theta} \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \end{aligned} \quad (27)$$

They are the HF operators defined by Xia and Xu [13].

Case 2. If $\zeta(t) = t^{-\kappa} - 1$, $\kappa > 0$, then $\zeta^{-1}(t) = (t + 1)^{-(1/\kappa)}$. So, $\delta \oplus \varepsilon = 1 - ((1 - \delta)^{-\kappa} + (1 - \varepsilon)^{-\kappa} - 1)^{-1/\kappa}$, $\delta \otimes \varepsilon = (\delta^{-\kappa} + \varepsilon^{-\kappa} - 1)^{-1/\kappa}$.

$$\text{AC-HFWA}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ 1 - \left(\sum_{i=1}^n \omega_i (1 - h_{im_i})^{-\kappa} \right)^{-1/\kappa} \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \quad (28)$$

Case 3. If $\zeta(t) = -\ln((e^{-\kappa t} - 1)/(e^{-\kappa} - 1))$, $\kappa \neq 0$, then $\zeta^{-1}(t) = -(1/\kappa)\ln(e^{-t}(e^{-\kappa} - 1) + 1)$. So, $\delta \oplus \varepsilon = 1 +$

$(1/\kappa)\ln(((e^{-\kappa(1-\delta)} - 1)(e^{-\kappa(1-\varepsilon)} - 1)/(e^{-\kappa} - 1)) + 1)$, $\delta \otimes \varepsilon = -(1/\kappa)\ln(((e^{-\kappa\delta} - 1)(e^{-\kappa\varepsilon} - 1)/(e^{-\kappa} - 1)) + 1)$.

$$\text{AC-HFWA}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ 1 + \frac{1}{\kappa} \ln \left(\prod_{i=1}^n \left(e^{-\kappa(1-h_{im_i})} - 1 \right)^{\omega_i} + 1 \right) \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \quad (29)$$

Case 4. If $\zeta(t) = \ln((1 - \kappa(1 - t))/t)$, $-1 \leq \kappa < 1$, then $\zeta^{-1}(t) = ((1 - \kappa)/(e^t - \kappa))$. So, $\delta \oplus \varepsilon = 1 - ((1 - \delta)(1 - \varepsilon)/(1 - \kappa\delta\varepsilon))$, $\delta \otimes \varepsilon = (\delta\varepsilon/(1 - \kappa(1 - \delta)(1 - \varepsilon)))$.

$$\text{AC-HFWA}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ \frac{\prod_{i=1}^n (1 - \kappa h_{im_i})^{\omega_i} - \prod_{i=1}^n (1 - h_{im_i})^{\omega_i}}{\prod_{i=1}^n (1 - \kappa h_{im_i})^{\omega_i} - \kappa \prod_{i=1}^n (1 - h_{im_i})^{\omega_i}} \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \quad (30)$$

Case 5. If $\zeta(t) = -\ln(1 - (1 - t)^\kappa)$, $\kappa \geq 1$, then $\zeta^{-1}(t) = 1 - (1 - e^{-t})^{1/\kappa}$. So $\delta \oplus \varepsilon = (\delta^\kappa + \varepsilon^\kappa - \delta^\kappa \varepsilon^\kappa)^{1/\kappa}$, $\delta \otimes \varepsilon = 1 - ((1 - \delta)^\kappa + (1 - \varepsilon)^\kappa - (1 - \delta)^\kappa (1 - \varepsilon)^\kappa)^{1/\kappa}$.

$$\text{AC-HFWA}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ \left(1 - \prod_{i=1}^n (1 - h_{im_i}^\kappa)^{\omega_i} \right)^{1/\kappa} \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \quad (31)$$

4. Archimedean Copula-Based Hesitant Fuzzy Weighted Geometric Operator (AC-HFWG)

In this section, the Archimedean copula-based hesitant fuzzy weighted geometric operator (AC-HFWG) will be introduced, and some special forms of AC-HFWG operators will be discussed when the generator ς takes different functions.

4.1. AC-HFWG

Definition 11. Let $g_i(h) = \cup_{m_i=1}^{\sharp g_i} \{h_{im_i} \mid i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. The Archimedean copula-based hesitant fuzzy weighted geometric operator (AC-HFWG) is defined as follows:

$$\text{AC-HFWG}(g_1, g_2, \dots, g_n) = g_1^{\omega_1} \otimes g_2^{\omega_2} \otimes \dots \otimes g_n^{\omega_n}. \quad (32)$$

Theorem 8. Let $g_i(h) = \cup_{m_i=1}^{\sharp g_i} \{h_{im_i} \mid i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$; then,

$$\text{AC-HFWG}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ \varsigma^{-1} \left(\sum_{i=1}^n \omega_i \varsigma(h_{im_i}) \right) \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \quad (33)$$

Proof. For $n = 2$, we have

$$\begin{aligned} \text{AC-HFWG}(g_1, g_2) &= g_1^{\omega_1} \otimes g_2^{\omega_2} \\ &= \bigcup_{h_{1m_1} \in g_1, h_{2m_2} \in g_2} \left\{ \varsigma^{-1} \left(\omega_1 \varsigma(h_{1m_1}) + \omega_2 \varsigma(h_{2m_2}) \right) \mid m_1 = 1, 2, \dots, \sharp g_1, m_2 = 1, 2, \dots, \sharp g_2 \right\}. \end{aligned} \quad (34)$$

Suppose that equation (33) holds for $n = k$, that is,

$$\text{AC-HFWG}(g_1, g_2, \dots, g_k) = \bigcup_{h_{im_i} \in g_i} \left\{ \varsigma^{-1} \left(\sum_{i=1}^k \omega_i \varsigma(h_{im_i}) \right) \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \quad (35)$$

Then,

$$\text{AC-HFWG}(g_1, g_2, \dots, g_k, g_{k+1}) = \bigotimes_{i=1}^k g_i^{\omega_i} \oplus g_{k+1}^{\omega_{k+1}} = \bigcup_{h_{im_i} \in g_i} \left\{ \varsigma^{-1} \left(\sum_{i=1}^{k+1} \omega_i \varsigma(h_{im_i}) \right) \mid m_i = 1, 2, \dots, \sharp g_i \right\}. \quad (36)$$

Equation (33) holds for $n = k + 1$. Thus, equation (33) holds for all n . $\square$

Theorem 9. Let $g_i(h) = \cup_{m_i=1}^{\sharp g_i} \{h_{im_i} \mid i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$; then,

(1) If $g_1 = g_2 = \dots = g_n = \{h\}$, $\text{AC-HFWG}(g_1, g_2, \dots, g_n) = \{h\}$.

(2) Let $g_i^*(h) = \cup_{m_i=1}^{\sharp g_i^*} \{h_{im_i}^* \mid i = 1, 2, \dots, n\}$; if $h_{im_i} \leq h_{im_i}^*$,

$$\text{AC-HFWG}(g_1, g_2, \dots, g_n) \leq \text{AC-HFWG}(g_1^*, g_2^*, \dots, g_n^*). \quad (37)$$

(3) If $h^- = \min_{i=1,2,\dots,n} \{h_{im_i}\}$ and $h^+ = \max_{i=1,2,\dots,n} \{h_{im_i}\}$,

$$h^- \leq \text{AC-HFWG}(g_1, g_2, \dots, g_n) \leq h^+. \quad (38)$$

Proof. Suppose $h^- = \min_{i=1,2,\dots,n} \{h_{im_i}\}$ and $h^+ = \max_{i=1,2,\dots,n} \{h_{im_i}\}$.

Since ς is strictly decreasing, ς^{-1} is also strictly decreasing.

Then, $\varsigma(h^+) \leq \varsigma(h_{im_i}) \leq \varsigma(h^-)$, $\forall i = 1, 2, \dots, n$.
 $\sum_{i=1}^n \omega_i \varsigma(h^+) \leq \sum_{i=1}^n \omega_i \varsigma(h_{im_i}) \leq \sum_{i=1}^n \omega_i \varsigma(h^-)$, $\varsigma(h^+) \leq \sum_{i=1}^n \omega_i \varsigma(h_{im_i}) \leq \varsigma(h^-)$, so $\varsigma^{-1}(\varsigma(h^-)) \leq \varsigma^{-1}(\sum_{i=1}^n \omega_i \varsigma(h_{im_i})) \leq \varsigma^{-1}(\varsigma(h^+))h^- \leq \varsigma^{-1}(\sum_{i=1}^n \omega_i \varsigma(h_{im_i})) \leq h^+$. $\square$

Definition 12. Let $g_i(h) = \cup_{m_i=1}^{\natural g_i} \{h_{im_i} \mid i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$; then, the generalized hesitant fuzzy weighted geometric operator based on Archimedean copulas (AC-CHFWG) is defined as follows:

$$\text{AC-GHFWG}_{\theta}(g_1, g_2, \dots, g_n) = \frac{1}{\theta} ((\theta g_1)^{\omega_1} \otimes (\theta g_2)^{\omega_2} \otimes \dots \otimes (\theta g_n)^{\omega_n}) = \frac{1}{\theta} \left(\bigotimes_{i=1}^n (\theta g_i)^{\omega_i} \right). \quad (39)$$

Especially, when $\theta = 1$, the AC-GHFWG operator becomes the AC-HFWG operator.

Theorem 10. Let $g_i(h) = \cup_{m_i=1}^{\natural g_i} \{h_{im_i} \mid i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$; then,

$$\text{AC-GHFWG}_{\theta}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ 1 - \tau^{-1} \left(\frac{1}{\theta} \left(\tau \left(1 - \tau^{-1} \left(\sum_{i=1}^n \omega_i \left(\tau \left(1 - \tau^{-1} (\theta \tau(1 - h_{im_i})) \right) \right) \right) \right) \right) \right) \right\} \mid m_i = 1, 2, \dots, \natural g_i \}. \quad (40)$$

Theorem 11. Let $g_i(h) = \cup_{m_i=1}^{\natural g_i} \{h_{im_i} \mid i = 1, 2, \dots, n\}$, ω_i be the weight vector of g_i with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$; then,

(1) If $g_1 = g_2 = \dots = g_n = \{h\}$, $\text{AC-GHFWG}_{\theta}(g_1, g_2, \dots, g_n) = \{h\}$.

(2) Let $g_i^*(h) = \cup_{m_i=1}^{\natural g_i^*} \{h_{im_i}^* \mid i = 1, 2, \dots, n\}$; if $h_{im_i} \leq h_{im_i}^*$,

$$\text{AC-GHFWG}_{\theta}(g_1, g_2, \dots, g_n) \leq \text{AC-GHFWG}_{\theta}(g_1^*, g_2^*, \dots, g_n^*). \quad (41)$$

(3) If $h^- = \min_{i=1,2,\dots,n} \{h_{im_i}\}$ and $h^+ = \max_{i=1,2,\dots,n} \{h_{im_i}\}$,
 $h^- \leq \text{AC-GHFWG}_{\theta}(g_1, g_2, \dots, g_n) \leq h^+$. (42)

4.2. Different Forms of AC-HFWG Operators. We can see from Theorem 8 that some specific AC-HFWGs can be obtained when ς is assigned different generators.

Case 1. If $\varsigma(t) = (-\ln t)^{\kappa}$, $\kappa \geq 1$, then $\varsigma^{-1}(t) = e^{-t^{1/\kappa}}$. So,
 $\delta \oplus \varepsilon = 1 - e^{-((-\ln(1-\delta))^{\kappa} + (-\ln(1-\varepsilon))^{\kappa})^{1/\kappa}}$,
 $\delta \otimes \varepsilon = e^{-((-\ln \delta)^{\kappa} + (-\ln \varepsilon)^{\kappa})^{1/\kappa}}$.

$$\text{AC-HFWG}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ e^{-\left(\sum_{i=1}^n \omega_i (-\ln h_{im_i})^{\kappa} \right)^{1/\kappa}} \mid m_i = 1, 2, \dots, \natural g_i \right\}. \quad (43)$$

Specifically, when $\kappa = 1$ and $\varsigma(t) = -\ln t$, then $\delta \oplus \varepsilon = 1 - (1 - \delta)(1 - \varepsilon)$ and $\delta \otimes \varepsilon = \delta \varepsilon$, and the AC-HFWG operator reduces to HFWG and GHFWG [13]:

$$\begin{aligned} \text{HFWG}(g_1, g_2, \dots, g_n) &= \bigcup_{h_{im_i} \in g_i} \left\{ \prod_{i=1}^n h_{im_i}^{\omega_i} \mid m_i = 1, 2, \dots, \natural g_i \right\}, \\ \text{GHFWG}_{\theta}(g_1, g_2, \dots, g_n) &= \bigcup_{h_{im_i} \in g_i} \left\{ 1 - \left(1 - \prod_{i=1}^n \left(1 - (1 - h_{im_i})^{\theta} \right)^{\omega_i} \right)^{1/\theta} \mid m_i = 1, 2, \dots, \natural g_i \right\}. \end{aligned} \quad (44)$$

Case 2. If $\zeta(t) = t^{-\kappa} - 1$, $\kappa > 0$, then $\zeta^{-1}(t) = (t+1)^{-(1/\kappa)}$. So, $\delta \oplus \varepsilon = 1 - ((1-\delta)^{-\kappa} + (1-\varepsilon)^{-\kappa} - 1)^{-(1/\kappa)}$, $\delta \otimes \varepsilon = (\delta^{-\kappa} + \varepsilon^{-\kappa} - 1)^{-1/\kappa}$.

$$\text{AC-HFWG}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ \left(\sum_{i=1}^n \omega_i h_{im_i}^{-\kappa} \right)^{-(1/\kappa)} \middle| m_i = 1, 2, \dots, \mathfrak{h}g_i \right\}. \quad (45)$$

Case 3. If $\zeta(t) = -\ln((e^{-\kappa t} - 1)/(e^{-\kappa} - 1))$, $\kappa \neq 0$, then $\zeta^{-1}(t) = -(1/\kappa)\ln(e^{-t}(e^{-\kappa} - 1) + 1)$. So, $\delta \oplus \varepsilon = 1 +$

$$(1/\kappa)\ln(((e^{-\kappa(1-\delta)} - 1)(e^{-\kappa(1-\varepsilon)} - 1)/(e^{-\kappa} - 1)) + 1),$$

$$\delta \otimes \varepsilon = -(1/\kappa)\ln(((e^{-\kappa\delta} - 1)(e^{-\kappa\varepsilon} - 1)/(e^{-\kappa} - 1)) + 1).$$

$$\text{AC-HFWG}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ \frac{1}{\kappa} \ln \left(\prod_{i=1}^n (e^{-\kappa h_{im_i}} - 1)^{\omega_i} + 1 \right) \middle| m_i = 1, 2, \dots, \mathfrak{h}g_i \right\}. \quad (46)$$

Case 4. If $\zeta(t) = \ln((1 - \kappa(1 - t))/t)$, $-1 \leq \kappa < 1$, then $\zeta^{-1}(t) = ((1 - \kappa)/(e^t - \kappa))$. So, $\delta \oplus \varepsilon = 1 - ((1 - \delta)(1 - \varepsilon)/(1 - \kappa\delta\varepsilon))$, $\delta \otimes \varepsilon = (\delta\varepsilon/(1 - \kappa(1 - \delta)(1 - \varepsilon)))$.

$$\text{AC-HFWG}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ \frac{(1 - \kappa) \prod_{i=1}^n h_{im_i}^{\omega_i}}{\prod_{i=1}^n (1 - \kappa(1 - h_{im_i}))^{\omega_i} - \kappa \prod_{i=1}^n h_{im_i}^{\omega_i}} \middle| m_i = 1, 2, \dots, \mathfrak{h}g_i \right\}. \quad (47)$$

Case 5. If $\zeta(t) = -\ln(1 - (1 - t)^\kappa)$, $\kappa \geq 1$, then $\zeta^{-1}(t) = 1 - (1 - e^{-t})^{1/\kappa}$. So, $\delta \oplus \varepsilon = (\delta^\kappa + \varepsilon^\kappa - \delta^\kappa \varepsilon^\kappa)^{1/\kappa}$, $\delta \otimes \varepsilon = 1 - ((1 - \delta)^\kappa + (1 - \varepsilon)^\kappa - (1 - \delta)^\kappa (1 - \varepsilon)^\kappa)^{1/\kappa}$.

$$\text{AC-HFWG}(g_1, g_2, \dots, g_n) = \bigcup_{h_{im_i} \in g_i} \left\{ 1 - \left(1 - \prod_{i=1}^n (1 - (1 - h_{im_i})^\kappa)^{\omega_i} \right)^{1/\kappa} \middle| m_i = 1, 2, \dots, \mathfrak{h}g_i \right\}. \quad (48)$$

4.3. The Properties of AC-HFWA and AC-HFWG. It is seen from above discussion that AC-HFWA and AC-HFWG are functions with respect to the parameter which is from the generator ζ . In this section, we will introduce the properties of the AC-HFWA and AC-HFWG operator regarding to the parameter κ .

Theorem 12. Let $\zeta(t)$ be the generator function of copula, and it takes five cases proposed in Section 4.2; then $\mu(\text{AC-HFWA}(\kappa))$

is an increasing function of κ , $\mu(\text{AC-HFWG}(\kappa))$ is a decreasing function of κ , and $\mu(\text{AC-HFWA}(g_1, g_2, \dots, g_n)) \geq \mu(\text{AC-HFWG}(g_1, g_2, \dots, g_n))$.

Proof. (Case 1) When $\zeta(t) = (-\ln t)^\kappa$ with $\kappa \geq 1$. By Definition 6, we have

$$\mu(\text{AC-HFWA}(\kappa)) = \frac{1}{\mathfrak{h}g_1 \mathfrak{h}g_2 \dots \mathfrak{h}g_n} \sum_{m_1=1}^{\mathfrak{h}g_1} \sum_{m_2=1}^{\mathfrak{h}g_2} \dots \sum_{m_n=1}^{\mathfrak{h}g_n} \left(1 - e^{-\left(\sum_{i=1}^n \omega_i (-\ln(1 - h_{im_i}))^\kappa \right)^{1/\kappa}} \right), \quad (49)$$

$$\mu(\text{AC-HFWG}(\kappa)) = \frac{1}{\mathfrak{h}g_1 \mathfrak{h}g_2 \dots \mathfrak{h}g_n} \sum_{m_1=1}^{\mathfrak{h}g_1} \sum_{m_2=1}^{\mathfrak{h}g_2} \dots \sum_{m_n=1}^{\mathfrak{h}g_n} \left(e^{-\left(\sum_{i=1}^n \omega_i (-\ln(1 - h_{im_i}))^\kappa \right)^{1/\kappa}} \right).$$

Suppose $1 \leq \kappa_1 < \kappa_2$; according to reference [10], $(\sum_{i=1}^n \omega_i a_i^\kappa)^{1/\kappa}$ is an increasing function of κ . So,

$$\begin{aligned} \left(\sum_{i=1}^n \omega_i (-\ln(1-h_{im_i}))^{\kappa_1} \right)^{1/\kappa_1} &\leq \left(\sum_{i=1}^n \omega_i (-\ln(1-h_{im_i}))^{\kappa_2} \right)^{1/\kappa_2}, \\ \left(\sum_{i=1}^n \omega_i (-\ln h_{im_i})^{\kappa_1} \right)^{1/\kappa_1} &\leq \left(\sum_{i=1}^n \omega_i (-\ln h_{im_i})^{\kappa_2} \right)^{1/\kappa_2}. \end{aligned} \quad (50)$$

Furthermore,

$$\begin{aligned} 1 - e^{-\left(\sum_{i=1}^n \omega_i (-\ln(1-h_{im_i}))^{\kappa_1} \right)^{1/\kappa_1}} &\leq 1 - e^{-\left(\sum_{i=1}^n \omega_i (-\ln(1-h_{im_i}))^{\kappa_2} \right)^{1/\kappa_2}}, \\ e^{-\left(\sum_{i=1}^n \omega_i (-\ln(1-h_{im_i}))^{\kappa_1} \right)^{1/\kappa_1}} &\geq e^{-\left(\sum_{i=1}^n \omega_i (-\ln(1-h_{im_i}))^{\kappa_2} \right)^{1/\kappa_2}}. \end{aligned} \quad (51)$$

Therefore, $\mu(\text{AC-HFWA}(\kappa_1)) \leq \mu(\text{AC-HFWA}(\kappa_2))$, $\mu(\text{AC-HFWG}(\kappa_1)) \geq \mu(\text{AC-HFWG}(\kappa_2))$. Because $\kappa \geq 1$,

$$\begin{aligned} e^{-\left(\sum_{i=1}^n \omega_i (-\ln(1-h_{im_i}))^\kappa \right)^{1/\kappa}} + e^{-\left(\sum_{i=1}^n \omega_i (-\ln h_{im_i})^\kappa \right)^{1/\kappa}} &\leq e^{-\left(\sum_{i=1}^n \omega_i (-\ln(1-h_{im_i})) \right)} + e^{-\left(\sum_{i=1}^n \omega_i (-\ln h_{im_i}) \right)} \\ &= \prod_{i=1}^n (1-h_{im_i})^{\omega_i} + \prod_{i=1}^n h_{im_i}^{\omega_i} \leq \sum_{i=1}^n \omega_i (1-h_{im_i}) + \sum_{i=1}^n \omega_i h_{im_i} = \sum_{i=1}^n \omega_i = 1. \end{aligned} \quad (52)$$

So, $1 - e^{-\left(\sum_{i=1}^n \omega_i (-\ln(1-h_{im_i}))^\kappa \right)^{1/\kappa}} \geq e^{-\left(\sum_{i=1}^n \omega_i (-\ln h_{im_i})^\kappa \right)^{1/\kappa}}$.

That is, $\mu(\text{AC-HFWA}(g_1, g_2, \dots, g_n)) \geq \mu(\text{AC-HFWG}(g_1, g_2, \dots, g_n))$.

So Theorem 12 holds under Case 1.

Case 2. When $\zeta(t) = t^{-\kappa} - 1$, $\kappa > 0$. Firstly,

$$\begin{aligned} \mu(\text{AC-HFWA}(\kappa)) &= \frac{1}{\natural g_1 \natural g_2 \cdots \natural g_n} \sum_{m_1=1}^{\natural g_1} \sum_{m_2=1}^{\natural g_2} \cdots \sum_{m_n=1}^{\natural g_n} \left(1 - \zeta^{-1} \left(\sum_{i=1}^n \omega_i \zeta(1-h_{im_i}) \right) \right), \\ \mu(\text{AC-HFWG}(\kappa)) &= \frac{1}{\natural g_1 \natural g_2 \cdots \natural g_n} \sum_{m_1=1}^{\natural g_1} \sum_{m_2=1}^{\natural g_2} \cdots \sum_{m_n=1}^{\natural g_n} \left(\zeta^{-1} \left(\sum_{i=1}^k \omega_i \zeta(h_{im_i}) \right) \right). \end{aligned} \quad (53)$$

Secondly, because $\zeta(\kappa, t) = t^{-\kappa} - 1$, $\kappa > 0$, $0 < t < 1$, $(\partial \zeta / \partial \kappa) = t^{-\kappa} \ln t$, $t^{-\kappa} > 0$, $\ln t < 0$, and then $(\partial \zeta / \partial \kappa) < 0$, $\zeta(\kappa, t)$ is decreasing with respect to κ .

$\zeta^{-1}(\kappa, t) = (t+1)^{-(1/\kappa)}$, $\kappa > 0$, $0 < t < 1$, so $(\partial \zeta^{-1} / \partial \kappa) = (t+1)^{-(1/\kappa)} \ln(t+1)(1/\kappa^2)$, $(t+1)^{-(1/\kappa)} > 0$, $\ln(t+1) > 0$, $(1/\kappa^2) > 0$, and then $(\partial \zeta^{-1} / \partial \kappa) > 0$, $\zeta^{-1}(\kappa, t)$ is decreasing with respect to κ .

Thus, $1 - \zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(1-h_{im_i}))$ is decreasing with respect to κ , and $\zeta^{-1}(\sum_{i=1}^k \omega_i \zeta(h_{im_i}))$ is decreasing with respect to κ . Suppose $1 \leq \kappa_1 < \kappa_2$; we have

$\mu(\text{AC-HFWA}(\kappa_1)) \leq \mu(\text{AC-HFWA}(\kappa_2))$ and $\mu(\text{AC-HFWG}(\kappa_1)) \geq \mu(\text{AC-HFWG}(\kappa_2))$.

Lastly,

$$\begin{aligned}
\left(\sum_{i=1}^n \omega_i (1 - h_{im_i})^{-\kappa} \right)^{-(1/\kappa)} + \left(\sum_{i=1}^n \omega_i h_{im_i}^{-\kappa} \right)^{-(1/\kappa)} &< \lim_{\kappa \rightarrow 0} \left(\sum_{i=1}^n \omega_i (1 - h_{im_i})^{-\kappa} \right)^{-(1/\kappa)} + \lim_{\kappa \rightarrow 0} \left(\sum_{i=1}^n \omega_i h_{im_i}^{-\kappa} \right)^{-(1/\kappa)} \\
&= e^{\sum_{i=1}^n \omega_i \ln(1 - h_{im_i})} + e^{\sum_{i=1}^n \omega_i \ln(h_{im_i})} = \prod_{i=1}^n (1 - h_{im_i})^{\omega_i} + \prod_{i=1}^n h_{im_i}^{\omega_i} \\
&\leq \sum_{i=1}^n \omega_i (1 - h_{im_i}) + \sum_{i=1}^n \omega_i h_{im_i} = \sum_{i=1}^n \omega_i = 1.
\end{aligned} \tag{54}$$

That is, Theorem 12 holds under Case 2.

Case 3. When $\zeta(t) = -\ln((e^{-\kappa t} - 1)/(e^{-\kappa} - 1))$, $\kappa \neq 0$. Firstly,

$$\begin{aligned}
\mu(\text{AC-HFWA}(\kappa)) &= \frac{1}{\mathfrak{g}_1 \mathfrak{g}_2 \cdots \mathfrak{g}_n} \sum_{m_1=1}^{\mathfrak{g}_1} \sum_{m_2=1}^{\mathfrak{g}_2} \cdots \sum_{m_n=1}^{\mathfrak{g}_n} \left(1 - \zeta^{-1} \left(\sum_{i=1}^n \omega_i \zeta(1 - h_{im_i}) \right) \right), \\
\mu(\text{AC-HFWG}(\kappa)) &= \frac{1}{\mathfrak{g}_1 \mathfrak{g}_2 \cdots \mathfrak{g}_n} \sum_{m_1=1}^{\mathfrak{g}_1} \sum_{m_2=1}^{\mathfrak{g}_2} \cdots \sum_{m_n=1}^{\mathfrak{g}_n} \left(\zeta^{-1} \left(\sum_{i=1}^k \omega_i \zeta(h_{im_i}) \right) \right).
\end{aligned} \tag{55}$$

Secondly,

$$\zeta(\kappa, t) = -\ln((e^{-\kappa t} - 1)/(e^{-\kappa} - 1)), \quad \kappa \neq 0, \quad 0 < t < 1.$$

If $\zeta = -\ln \zeta_1$, $\zeta_1 = ((\zeta_2^t - 1)/(\zeta_2 - 1))$, $\zeta_2 = e^{-\kappa}$, $\kappa \neq 0$, $0 < t < 1$.

Because ζ is decreasing with respect to ζ_1 , ζ_1 is decreasing with respect to ζ_2 , and ζ_2 is decreasing with respect to κ , $\zeta(\kappa, t)$ is decreasing with respect to κ .

$$\zeta^{-1}(\kappa, t) = -\frac{1}{\kappa} \ln(e^{-t}(e^{-\kappa} - 1) + 1), \quad \kappa \neq 0, \quad 0 < t < 1. \tag{56}$$

Suppose $\zeta^{-1} = -\ln \psi_1$, $\psi_1 = \psi_2^{1/\kappa}$, $\psi_2 = e^{-t} \psi_3 + 1$, $\zeta_3 = e^{-\kappa} - 1$, $\kappa \neq 0$, $0 < t < 1$.

Because ζ^{-1} is decreasing with respect to ψ_1 , ψ_1 is decreasing with respect to ψ_2 , and ψ_2 is decreasing with respect to ψ_3 , ψ_3 is decreasing with respect to κ . Then, $\zeta^{-1}(\kappa, t)$ is decreasing with respect to κ .

Thus, $1 - \zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(1 - h_{im_i}))$ is decreasing with respect to κ and $\zeta^{-1}(\sum_{i=1}^k \omega_i \zeta(h_{im_i}))$ is decreasing with respect to κ .

Suppose $1 \leq \kappa_1 < \kappa_2$.

Therefore, $\mu(\text{AC-HFWA}(\kappa_1)) \leq \mu(\text{AC-HFWA}(\kappa_2))$ and $\mu(\text{AC-HFWG}(\kappa_1)) \geq \mu(\text{AC-HFWG}(\kappa_2))$.

Lastly, when $-\infty < \kappa < +\infty$,

$$\begin{aligned}
&-\frac{1}{\kappa} \ln \left(\prod_{i=1}^n (e^{-\kappa h_{im_i}} - 1)^{\omega_i} + 1 \right) \\
&\leq \lim_{\kappa \rightarrow -\infty} \frac{\ln(\prod_{i=1}^n (e^{-\kappa h_{im_i}} - 1)^{\omega_i} + 1)}{-\kappa} = \lim_{\kappa \rightarrow -\infty} \frac{(\prod_{i=1}^n (e^{-\kappa h_{im_i}} - 1)^{\omega_i} / (\prod_{i=1}^n (e^{-\kappa h_{im_i}} - 1)^{\omega_i} + 1)) (\sum_{i=1}^n \omega_i (-h_{im_i}) (e^{-\kappa h_{im_i}} / (e^{-\kappa h_{im_i}} - 1)))}{-1} \\
&= \lim_{\kappa \rightarrow -\infty} \left(\frac{\prod_{i=1}^n (e^{-\kappa h_{im_i}} - 1)^{\omega_i}}{\prod_{i=1}^n (e^{-\kappa h_{im_i}} - 1)^{\omega_i} + 1} \right) \left(\sum_{i=1}^n \omega_i h_{im_i} \frac{e^{-\kappa h_{im_i}}}{e^{-\kappa h_{im_i}} - 1} \right) = \sum_{i=1}^n \omega_i h_{im_i}, \\
&-\frac{1}{\kappa} \ln \left(\prod_{i=1}^n (e^{-\kappa(1-h_{im_i})} - 1)^{\omega_i} + 1 \right) \leq \sum_{i=1}^n \omega_i (1 - h_{im_i}).
\end{aligned} \tag{57}$$

So, $-(1/\kappa) \ln(\prod_{i=1}^n (e^{-\kappa(1-h_{im_i})} - 1)^{\omega_i} + 1) - (1/\kappa) \ln(\prod_{i=1}^n (e^{-\kappa h_{im_i}} - 1)^{\omega_i} + 1) \leq \sum_{i=1}^n \omega_i (1 - h_{im_i}) + \sum_{i=1}^n \omega_i h_{im_i} = 1$.

That is, Theorem 12 holds under Case 3.

Case 4. When $\zeta(t) = \ln((1 - \kappa(1 - t))/t)$, $-1 \leq \kappa < 1$. Firstly,

$$\mu(\text{AC-HFWA}(\kappa)) = \frac{1}{\mathfrak{g}_1 \mathfrak{g}_2 \cdots \mathfrak{g}_n} \sum_{m_1=1}^{\mathfrak{g}_1} \sum_{m_2=1}^{\mathfrak{g}_2} \cdots \sum_{m_n=1}^{\mathfrak{g}_n} \left(1 - \zeta^{-1} \left(\sum_{i=1}^n \omega_i \zeta(1 - h_{im_i}) \right) \right),$$

$$\mu(\text{AC-HFWG}(\kappa)) = \frac{1}{\mathfrak{g}_1 \mathfrak{g}_2 \cdots \mathfrak{g}_n} \sum_{m_1=1}^{\mathfrak{g}_1} \sum_{m_2=1}^{\mathfrak{g}_2} \cdots \sum_{m_n=1}^{\mathfrak{g}_n} \left(\zeta^{-1} \left(\sum_{i=1}^k \omega_i \zeta(h_{im_i}) \right) \right).$$

Secondly, because $\zeta(\kappa, t) = \ln((1 - \kappa(1 - t))/t)$, $-1 \leq \kappa < 1$, $0 < t < 1$, so $(\partial\zeta/\partial\kappa) = (t(t-1)/(1 - \kappa(1 - t)))$, $t(t-1) < 0$.

Suppose $\zeta_1(\kappa, t) = 1 - \kappa(1 - t)$, $-1 \leq \kappa < 1$, $0 < t < 1$, $(\partial\zeta_1/\partial\kappa) = t - 1 < 0$, $\zeta_1(\kappa, t)$ is decreasing with respect to κ .

So, $\zeta_1(\kappa, t) > \zeta_1(1, t) = t > 0$; then, $(\partial\zeta/\partial\kappa) < 0$, and $\zeta(\kappa, t)$ is decreasing with respect to κ .

As $\zeta^{-1}(\kappa, t) = (1 - \kappa)/(e^t - \kappa)$, $-1 \leq \kappa < 1$, $0 < t < 1$, $(\partial\zeta^{-1}/\partial\kappa) = ((e^t + 1)/(e^t - \kappa)^2)$, $e^t + 1 > 0$, $(e^t - \kappa)^2 > 0$;

then $(\partial\zeta^{-1}/\partial\kappa) > 0$, $\zeta^{-1}(\kappa, t)$ is decreasing with respect to κ . Thus, $1 - \zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(1 - h_{im_i}))$ is decreasing with respect to κ and $\zeta^{-1}(\sum_{i=1}^k \omega_i \zeta(h_{im_i}))$ is decreasing with respect to κ .

Suppose $1 \leq \kappa_1 < \kappa_2$. We have $\mu(\text{AC-HFWA}(\kappa_1)) \leq \mu(\text{AC-HFWA}(\kappa_2))$ and $\mu(\text{AC-HFWG}(\kappa_1)) \geq \mu(\text{AC-HFWG}(\kappa_2))$.

Lastly, because $y = x^{\omega_i}$, $\omega_i \in (0, 1)$ is decreasing with respect to κ , $(1 - h_i^2)^{\omega_i} \geq 1$, $(1 - (1 - h_i)^2)^{\omega_i} \geq 1$; then,

$$\prod_{i=1}^n (1 + h_{im_i})^{\omega_i} + \prod_{i=1}^n (1 - h_{im_i})^{\omega_i} \geq 2 \left(\prod_{i=1}^n (1 + h_{im_i})^{\omega_i} \prod_{i=1}^n (1 - h_{im_i})^{\omega_i} \right)^{1/2} = 2 \left(\prod_{i=1}^n (1 - h_{im_i}^2)^{\omega_i} \right)^{1/2} \geq 2$$

$$\prod_{i=1}^n (2 - h_{im_i})^{\omega_i} + \prod_{i=1}^n h_{im_i}^{\omega_i} \geq 2 \left(\prod_{i=1}^n (2 - h_{im_i})^{\omega_i} \prod_{i=1}^n h_{im_i}^{\omega_i} \right)^{1/2} = 2 \left(\prod_{i=1}^n (1 - (1 - h_{im_i})^2)^{\omega_i} \right)^{1/2} \geq 2.$$

When $-1 \leq \kappa < 1$, we have

$$\left(\frac{\prod_{i=1}^n (1 - h_{im_i})^{\omega_i} - \kappa \prod_{i=1}^n (1 - h_{im_i})^{\omega_i}}{\prod_{i=1}^n (1 - \kappa h_{im_i})^{\omega_i} - \kappa \prod_{i=1}^n (1 - h_{im_i})^{\omega_i}} \right) + \left(\frac{(1 - \kappa) \prod_{i=1}^n h_{im_i}^{\omega_i}}{\prod_{i=1}^n (1 - \kappa(1 - h_{im_i}))^{\omega_i} - \kappa \prod_{i=1}^n h_{im_i}^{\omega_i}} \right)$$

$$\leq \left(\frac{\prod_{i=1}^n (1 - h_{im_i})^{\omega_i} + \prod_{i=1}^n (1 - h_{im_i})^{\omega_i}}{\prod_{i=1}^n (1 + h_{im_i})^{\omega_i} + \prod_{i=1}^n (1 - h_{im_i})^{\omega_i}} \right) + \left(\frac{2 \prod_{i=1}^n h_{im_i}^{\omega_i}}{\prod_{i=1}^n (1 + (1 - h_{im_i}))^{\omega_i} + \prod_{i=1}^n h_{im_i}^{\omega_i}} \right)$$

$$= \left(\frac{2 \prod_{i=1}^n (1 - h_{im_i})^{\omega_i}}{\prod_{i=1}^n (1 + h_{im_i})^{\omega_i} + \prod_{i=1}^n (1 - h_{im_i})^{\omega_i}} \right) + \left(\frac{2 \prod_{i=1}^n h_{im_i}^{\omega_i}}{\prod_{i=1}^n (2 - h_{im_i})^{\omega_i} + \prod_{i=1}^n h_{im_i}^{\omega_i}} \right)$$

$$\leq \prod_{i=1}^n (1 - h_{im_i})^{\omega_i} + \prod_{i=1}^n h_{im_i}^{\omega_i} \leq \sum_{i=1}^n \omega_i (1 - h_{im_i}) + \sum_{i=1}^n \omega_i h_{im_i} = \sum_{i=1}^n \omega_i = 1.$$

So,

$$\frac{\prod_{i=1}^n (1 - \kappa h_{im_i})^{\omega_i} - \prod_{i=1}^n (1 - h_{im_i})^{\omega_i}}{\prod_{i=1}^n (1 - \kappa h_{im_i})^{\omega_i} - \kappa \prod_{i=1}^n (1 - h_{im_i})^{\omega_i}} \geq \frac{(1 - \kappa) \prod_{i=1}^n h_{im_i}^{\omega_i}}{\prod_{i=1}^n (1 - \kappa(1 - h_{im_i}))^{\omega_i} - \kappa \prod_{i=1}^n h_{im_i}^{\omega_i}}.$$

Therefore, $\mu(\text{AC-HFWA}(g_1, g_2, \dots, g_n)) \geq \mu(\text{AC-HFWG}(g_1, g_2, \dots, g_n))$. That is, Theorem 12 holds under Case 4.

Case 5. When $\zeta(t) = -\ln(1 - (1-t)^\kappa)$, $\kappa \geq 1$.

Firstly,

$$\begin{aligned}\mu(\text{AC-HFWA}(\kappa)) &= \frac{1}{\sharp g_1 \sharp g_2 \cdots \sharp g_n} \sum_{m_1=1}^{\sharp g_1} \sum_{m_2=1}^{\sharp g_2} \cdots \sum_{m_n=1}^{\sharp g_n} \left(1 - \zeta^{-1} \left(\sum_{i=1}^n \omega_i \zeta(1 - h_{im_i}) \right) \right), \\ \mu(\text{AC-HFWG}(\kappa)) &= \frac{1}{\sharp g_1 \sharp g_2 \cdots \sharp g_n} \sum_{m_1=1}^{\sharp g_1} \sum_{m_2=1}^{\sharp g_2} \cdots \sum_{m_n=1}^{\sharp g_n} \left(\zeta^{-1} \left(\sum_{i=1}^k \omega_i \zeta(h_{im_i}) \right) \right).\end{aligned}\quad (62)$$

Secondly, because $\zeta(\kappa, t) = -\ln(1 - (1-t)^\kappa)$, $\kappa \geq 1$, $0 < t < 1$, $(\partial \zeta / \partial \kappa) = ((1-t)^\kappa \ln(1-t) / (1 - (1-t)^\kappa))$, $\ln(1-t) < 0$.

Suppose $\zeta_1(\kappa, t) = (1-t)^\kappa$, $\kappa \geq 1$, $0 < t < 1$, $(\partial \zeta_1 / \partial \kappa) = (1-t)^\kappa \ln(1-t) < 0$, $\zeta_1(\kappa, t)$ is decreasing with respect to κ , so $0 < \zeta_1(\kappa, t) \leq \zeta_1(1, t) = 1-t < 1$, $1 - (1-t)^\kappa > 0$.

Then, $(\partial \zeta / \partial \kappa) < 0$, $\zeta(\kappa, t)$ is decreasing with respect to κ . $\zeta^{-1}(\kappa, t) = 1 - (1 - e^{-t})^{1/\kappa}$, $\kappa \geq 1$, $0 < t < 1$, so $(\partial \zeta^{-1} / \partial \kappa) = (1 - e^{-t})^{1/\kappa} \ln(1 - e^{-t}) (-1/\kappa^2)$, $(1 - e^{-t})^{1/\kappa} > 0$,

$\ln(1 - e^{-t}) < 0$, $-(1/\kappa^2) < 0$; then, $(\partial \zeta^{-1} / \partial \kappa) > 0$, $\zeta^{-1}(\kappa, t)$ is decreasing with respect to κ .

Thus, $1 - \zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(1 - h_{im_i}))$ is decreasing with respect to κ and $\zeta^{-1}(\sum_{i=1}^n \omega_i \zeta(h_{im_i}))$ is decreasing with respect to κ .

Lastly, suppose $1 \leq \kappa_1 < \kappa_2$, Therefore, $\mu(\text{AC-HFWA}(\kappa_1)) \leq \mu(\text{AC-HFWA}(\kappa_2))$, $\mu(\text{AC-HFWG}(\kappa_1)) \geq \mu(\text{AC-HFWG}(\kappa_2))$.

Because $\kappa \geq 1$,

$$\begin{aligned}& \left(\left(\prod_{i=1}^n (1 - h_{im_i}^\kappa)^{\omega_i} \right)^{1/\kappa} \right) + \left(1 - \left(1 - \prod_{i=1}^n (1 - (1 - h_{im_i}^\kappa)^{\omega_i}) \right)^{1/\kappa} \right) \\ & \leq \left(\left(\prod_{i=1}^n (1 - h_{im_i}^1)^{\omega_i} \right)^1 \right) + \left(1 - \left(1 - \prod_{i=1}^n (1 - (1 - h_{im_i}^1)^{\omega_i}) \right)^1 \right) \\ & = \prod_{i=1}^n (1 - h_{im_i})^{\omega_i} + \prod_{i=1}^n h_{im_i}^{\omega_i} \leq \sum_{i=1}^n \omega_i (1 - h_{im_i}) + \sum_{i=1}^n \omega_i h_{im_i} = \sum_{i=1}^n \omega_i = 1.\end{aligned}\quad (63)$$

So, $1 - e^{-\left(\sum_{i=1}^n \omega_i (-\ln(1 - h_{im_i}^\kappa))^{1/\kappa}\right)} \geq e^{-\left(\sum_{i=1}^n \omega_i (-\ln(h_{im_i}^\kappa))^{1/\kappa}\right)}$, $\mu(\text{AC-HFWA}(g_1, g_2, \dots, g_n)) \geq \mu(\text{AC-HFWG}(g_1, g_2, \dots, g_n))$.

That is, Theorem 12 holds under Case 5. $\square$

5. MADM Approach Based on AC-HFWA and AC-HFWG

From the above analysis, a novel decision making way will be given to address MADM problems under HF environment based on the proposed operators.

Let $Y = \{\Psi_i (i = 1, 2, \dots, m)\}$ be a collection of alternatives and $C = \{G_j (j = 1, 2, \dots, n)\}$ be the set of attributes, whose weight vector is $W = (\omega_1, \omega_2, \dots, \omega_n)^T$, satisfying $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. If DMs provide several values for the alternative Y_i under the attribute $G_j (j = 1, 2, \dots, n)$ with anonymity, these values can be considered as a HFE h_{ij} . In that case, if two DMs give the same value, then the value emerges only once in h_{ij} . In what follows, the specific algorithm for MADM problems under HF environment will be designed.

Step 1. The DMs provide their evaluations about the alternative Y_i under the attribute G_j , denoted by the HFEs h_{ij} , $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

Step 2. Use the proposed AC-HFWA (AC-HFWG) to fuse the HFE $y_i (i = 1, 2, \dots, m)$ for $Y_i (i = 1, 2, \dots, m)$.

Step 3. The score values $\mu(y_i) (i = 1, 2, \dots, m)$ of y_i are calculated using Definition 6 and compared.

Step 4. The order of the attributes $Y_i (i = 1, 2, \dots, m)$ is given by the size of $\mu(y_i) (i = 1, 2, \dots, m)$.

5.1. Illustrative Example. An example of stock investment will be given to illustrate the application of the proposed method. Assume that there are four short-term stocks Y_1, Y_2, Y_3, Y_4 . The following four attributes (G_1, G_2, G_3, G_4) should be considered: G_1 —net value of each share; G_2 —earnings per share; G_3 —capital reserve per share; and G_4 —accounts receivable. Assume the weight vector of each attributes is $W = (0.2, 0.3, 0.15, 0.35)^T$.

Next, we will use the developed method to find the ranking of the alternatives and the optimal choice.

Step 1. The decision matrix is given by DM and is shown in Table 1.

Step 2. Use the AC-HFWA operator to fuse the HEE $y_i (i = 1, \dots, 4)$ for $Y_i (i = 1, \dots, 4)$.

Take Y_4 as an example, and let $\zeta(t) = (-\ln t)^{1.2}$ in Case 1; we have

$$y_4 = \text{AC-HFWA}(g_1, g_2, g_3, g_4) = \bigoplus_{j=1}^4 \omega_j g_j = \bigcup_{h_{jm_j} \in g_j} \left\{ 1 - e^{-\left(\sum_{j=1}^4 \omega_j (1 - h_{jm_j})^{1.2} \right)^{5/6}} \mid m_j = 1, 2, \dots, 4 \right\}$$

$$= \{0.5870, 0.6139, 0.6021, 0.6278, 0.6229, 0.6470, 0.7190, 0.7361, 0.7286, 0.6117, 0.6367, 0.6257, 0.6496, 0.6450, 0.6675, 0.7347, 0.7508, 0.7437, 0.6303, 0.6538, 0.6435, 0.6660, 0.6617, 0.6829, 0.7466, 0.7618, 0.7551, 0.7451, 0.7419, 0.7574, 0.7591, 0.7561, 0.7707, 0.7697, 0.7669, 0.7808\}.$$
(64)

Step 3. Compute the score values $\mu(y_i) (i = 1, \dots, 4)$ of $y_i (i = 1, \dots, 4)$ by Definition 6. $\mu(y_4) = 0.6944$. Similarly,

$$\mu(y_1) = 0.5761, \mu(y_2) = 0.6322, \mu(y_3) = 0.5149.$$

Step 4. Rank the alternatives $Y_i (i = 1, \dots, 4)$ and select the desirable one in terms of comparison rules. Since $\mu(y_4) > \mu(y_2) > \mu(y_1) > \mu(y_3)$, we can obtain the rank of alternatives as $Y_4 > Y_2 > Y_1 > Y_3$, and Y_4 is the best alternative.

the increase in κ , the ranking results change from $Y_4 > Y_1 > Y_3 > Y_2$ to $Y_4 > Y_3 > Y_1 > Y_2$ and finally settle at $Y_3 > Y_4 > Y_1 > Y_2$, which indicates that AC-HFWG has certain stability. In the decision process, DMs can confirm the value of κ in accordance with their preferences.

Through the analysis of Tables 3 and 4, we can find that the score values calculated by the AC-HFWG operator decrease with the increase of parameter κ . For the same generator and parameter κ , the values obtained by the AC-HFWA operator are always greater than those obtained by the AC-HFWG operator, which is consistent with Theorem 12.

5.2. Sensitivity Analysis

5.2.1. The Effect of Parameters on the Results. We take Case 1 as an example to analyse the effect of parameter changes on the result. The results are shown in Table 2 and Figure 1.

It is easy to see from Table 2 that the scores of alternatives by the AC-HFWA operator increase as the value of the parameter κ ranges from 1 to 5. This is consistent with Theorem 12. But the order has not changed, and the best alternative is consistent.

5.2.2. The Effect of Different Types of Generators on the Results in AC-HFWA. The ordering results of alternatives using other generators proposed in present work are listed in Table 3. Figure 2 shows more intuitively how the score function varies with parameters.

The results show that the order of alternatives obtained by applying different generators and parameters is different, the ranking order of alternatives is almost the same, and the desirable one is always Y_4 , which indicates that AC-HFWA operator is highly stable.

5.2.3. The Effect of Different Types of Generators on the Results in AC-HFWG. In step 2, if we utilize the AC-HFWG operator instead of the AC-HFWA operator to calculate the values of the alternatives, the results are shown in Table 4.

Figure 3 shows the variation of the values with parameter κ . We can find that the ranking of the alternatives may change when κ changes in the AC-HFWG operator. With

5.2.4. The Effect of Parameter θ on the Results in AC-GHFWA and AC-GHFWG. We employ the AC-GHFWA operator and AC-GHFWG operator to aggregate the values of the alternatives, taking Case 1 as an example. The results are listed in Tables 5 and 6. Figures 4 and 5 show the variation of the score function with κ and θ .

The results indicate that the scores of alternatives by the AC-GHFWA operator increase as the parameter κ goes from 0 to 10 and the parameter θ ranges from 0 to 10, and the ranking of alternatives has not changed. But the score values of alternatives reduce, and the order of alternatives obtained by the AC-GHFWG operator changes significantly.

5.3. Comparisons with Existing Approach. The above results indicate the effectiveness of our method, which can solve MADM problems. To further prove the validity of our method, this section compares the existing methods with our proposed method. The previous methods include hesitant fuzzy weighted averaging (HFWA) operator (or HFWG operator) by Xia and Xu [13], hesitant fuzzy Bonferroni means (HFBM) operator by Zhu and Xu [32], and hesitant fuzzy Frank weighted average (HFFWA) operator (or HFFWG operator) by Qin et al. [22]. Using the data in reference [22] and different operators, Table 7 is obtained. The order of these operators is $A_6 > A_2 > A_3 > A_5 > A_1 > A_4$, except for the HFBM operator which is $A_6 > A_2 > A_3 > A_1 > A_5 > A_4$.

TABLE 1: HF decision matrix [13].

Alternatives	G_1	G_2	G_3	G_4
Y_1	{0.2, 0.4, 0.7}	{0.2, 0.6, 0.8}	{0.2, 0.3, 0.6, 0.7, 0.9}	{0.3, 0.4, 0.5, 0.7, 0.8}
Y_2	{0.2, 0.4, 0.7, 0.9}	{0.1, 0.2, 0.4, 0.5}	{0.3, 0.4, 0.6, 0.9}	{0.5, 0.6, 0.8, 0.9}
Y_3	{0.3, 0.5, 0.6, 0.7}	{0.2, 0.4, 0.5, 0.6}	{0.3, 0.5, 0.7, 0.8}	{0.2, 0.5, 0.6, 0.7}
Y_4	{0.3, 0.5, 0.6}	{0.2, 0.4}	{0.5, 0.6, 0.7}	{0.8, 0.9}

TABLE 2: The ordering results obtained by AC-HFWA in Case 1.

Parameter κ	$\mu(Y_1)$	$\mu(Y_2)$	$\mu(Y_3)$	$\mu(Y_4)$	Ranking order
1	0.5612	0.6009	0.5178	0.6524	$Y_4 > Y_2 > Y_1 > Y_3$
1.2	0.5720	0.6160	0.5249	0.6678	$Y_4 > Y_2 > Y_1 > Y_3$
2	0.6084	0.6621	0.5498	0.7156	$Y_4 > Y_2 > Y_1 > Y_3$
2.5	0.6258	0.6823	0.5626	0.7365	$Y_4 > Y_2 > Y_1 > Y_3$
3	0.6399	0.6981	0.5736	0.7525	$Y_4 > Y_2 > Y_1 > Y_3$
5	0.6757	0.7356	0.6045	0.7893	$Y_4 > Y_2 > Y_1 > Y_3$

FIGURE 1: The score values obtained by AC-HFWA in Case 1.

TABLE 3: The ordering results obtained by AC-HFWA in Cases 2–5.

Functions	Parameter λ	$\mu(Y_1)$	$\mu(Y_2)$	$\mu(Y_3)$	$\mu(Y_4)$	Ranking order
Case 2	1	0.6059	0.6681	0.5425	0.7256	$Y_4 > Y_2 > Y_1 > Y_3$
	2	0.6411	0.7110	0.5656	0.7694	$Y_4 > Y_2 > Y_1 > Y_3$
	3	0.6650	0.7351	0.5851	0.7932	$Y_4 > Y_2 > Y_1 > Y_3$
Case 3	1	0.5710	0.6143	0.5241	0.6665	$Y_4 > Y_2 > Y_1 > Y_3$
	2	0.5815	0.6278	0.5311	0.6806	$Y_4 > Y_2 > Y_1 > Y_3$
	3	0.5922	0.6410	0.5386	0.6943	$Y_4 > Y_2 > Y_1 > Y_3$
Case 4	0.25	0.5666	0.6084	0.5212	0.6604	$Y_4 > Y_2 > Y_1 > Y_3$
	0.5	0.5739	0.6188	0.5256	0.6717	$Y_4 > Y_2 > Y_1 > Y_3$
	0.75	0.5849	0.6349	0.5320	0.6894	$Y_4 > Y_2 > Y_1 > Y_3$
Case 5	1	0.5612	0.6009	0.5178	0.6524	$Y_4 > Y_2 > Y_1 > Y_3$
	2	0.5830	0.6279	0.5337	0.6781	$Y_4 > Y_2 > Y_1 > Y_3$
	3	0.6022	0.6494	0.5484	0.6996	$Y_4 > Y_2 > Y_1 > Y_3$

(1) From the previous argument, it is obvious to find that Xia's operators [13] and Qin's operators [22] are special cases of our operators. When $\kappa = 1$ in Case 1 of our proposed operators, AC-HFWA reduces to HFWA and AC-HFWG reduces to HFWG. When $\kappa = 1$ in Case 3 of our proposed operators, AC-HFWA becomes HFWA and AC-HFWG becomes HFWG. The operational rules of Xia's method are algebraic t-norm and algebraic t-conorm, and the

operational rules of Qin's method are Frank t-norm and Frank t-conorm, which are all special forms of copula and cocopula. Therefore, our proposed operators are more general. Furthermore, the proposed approach will supply more choice for DMs in real MADM problems.

(2) The results of Zhu's operator [32] are basically the same as ours, but Zhu's calculations were more complicated. Although we all introduce parameters,

FIGURE 2: The score functions obtained by AC-HFWA.

which can resize the aggregate value on account of actual decisions, the regularity of our parameter changes is stronger than Zhu's method. The AC-HFWA operator has an ideal property of monotonically increasing with respect to the parameter, and the AC-HFWG operator has an ideal property of

monotonically decreasing with respect to the parameter, which provides a basis for DMs to select appropriate values according to their risk appetite. If the DM is risk preference, we can choose the largest parameter possible, and if the DM is risk aversion, we can choose the smallest parameter possible.

TABLE 4: The ordering results obtained by AC-HFWG.

Functions	Parameter λ	$\mu(Y_1)$	$\mu(Y_2)$	$\mu(Y_3)$	$\mu(Y_4)$	Ranking order
Case 1	1.1	0.4687	0.4541	0.4627	0.5042	$Y_4 > Y_1 > Y_3 > Y_2$
	2	0.4247	0.3970	0.4358	0.4437	$Y_4 > Y_3 > Y_1 > Y_2$
	8	0.3249	0.2964	0.3605	0.3365	$Y_3 > Y_4 > Y_1 > Y_2$
Case 2	1	0.4319	0.4010	0.4389	0.4573	$Y_4 > Y_3 > Y_1 > Y_2$
	2	0.3962	0.3601	0.4159	0.4147	$Y_3 > Y_4 > Y_1 > Y_2$
	3	0.3699	0.3346	0.3982	0.3858	$Y_3 > Y_4 > Y_1 > Y_2$
Case 3	1	0.4642	0.4482	0.4599	0.5257	$Y_4 > Y_1 > Y_3 > Y_2$
	3	0.4417	0.4190	0.4462	0.5212	$Y_4 > Y_3 > Y_1 > Y_2$
	8	0.3904	0.3628	0.4124	0.4036	$Y_3 > Y_4 > Y_1 > Y_2$
Case 4	-0.9	0.4866	0.4803	0.4732	0.5317	$Y_4 > Y_1 > Y_2 > Y_3$
	0.25	0.4689	0.4547	0.4627	0.5051	$Y_4 > Y_1 > Y_3 > Y_2$
	0.75	0.4507	0.4290	0.4511	0.4804	$Y_4 > Y_3 > Y_1 > Y_2$
Case 5	1	0.4744	0.4625	0.4661	0.5210	$Y_4 > Y_1 > Y_3 > Y_2$
	2	0.4503	0.4295	0.4526	0.5157	$Y_4 > Y_3 > Y_1 > Y_2$
	8	0.3645	0.3399	0.3922	0.3750	$Y_3 > Y_4 > Y_1 > Y_2$

FIGURE 3: Continued.

FIGURE 3: The score functions obtained by AC-HFWG.

TABLE 5: The ordering results obtained by AC-GHFWA in Case 1.

Parameter κ	Parameter θ	$\mu(Y_1)$	$\mu(Y_2)$	$\mu(Y_3)$	$\mu(Y_4)$	Ranking order
1	1	0.5612	0.6009	0.5178	0.6524	$Y_4 > Y_2 > Y_1 > Y_3$
	5	0.6317	0.6806	0.5722	0.7313	$Y_4 > Y_2 > Y_1 > Y_3$
	10	0.6701	0.7236	0.6079	0.7745	$Y_4 > Y_2 > Y_1 > Y_3$
5	1	0.6757	0.7356	0.6045	0.7893	$Y_4 > Y_2 > Y_1 > Y_3$
	5	0.6852	0.7442	0.6142	0.7972	$Y_4 > Y_2 > Y_1 > Y_3$
	10	0.6892	0.7479	0.6186	0.8005	$Y_4 > Y_2 > Y_1 > Y_3$
10	1	0.7081	0.7688	0.6386	0.8199	$Y_4 > Y_2 > Y_1 > Y_3$
	5	0.7112	0.7712	0.6420	0.8219	$Y_4 > Y_2 > Y_1 > Y_3$
	10	0.7126	0.7723	0.6435	0.8227	$Y_4 > Y_2 > Y_1 > Y_3$

TABLE 6: The ordering results obtained by AC-GHFWG in Case 1.

Parameter κ	Parameter θ	$\mu(Y_1)$	$\mu(Y_2)$	$\mu(Y_3)$	$\mu(Y_4)$	Ranking order
1	1	0.4744	0.4625	0.4661	0.5131	$Y_4 > Y_1 > Y_3 > Y_2$
	5	0.3962	0.3706	0.4171	0.4083	$Y_3 > Y_4 > Y_1 > Y_2$
	10	0.3562	0.3262	0.3808	0.3607	$Y_3 > Y_4 > Y_1 > Y_2$
5	1	0.3523	0.3222	0.3836	0.3636	$Y_3 > Y_4 > Y_1 > Y_2$
	5	0.3417	0.3124	0.3746	0.3527	$Y_3 > Y_4 > Y_1 > Y_2$
	10	0.3372	0.3083	0.3706	0.3481	$Y_3 > Y_4 > Y_1 > Y_2$
10	1	0.3200	0.2870	0.3517	0.3266	$Y_3 > Y_4 > Y_1 > Y_2$
	5	0.3165	0.2841	0.3484	0.3234	$Y_3 > Y_4 > Y_1 > Y_2$
	10	0.3150	0.2829	0.3470	0.3220	$Y_3 > Y_4 > Y_1 > Y_2$

FIGURE 4: Continued.

FIGURE 4: The score functions obtained by AC-GHFWA in Case 1.

FIGURE 5: The score functions obtained by AC-GHFWG in Case 1.

TABLE 7: The scores obtained by different operators.

Operator	Parameter	$\mu (A_1)$	$\mu (A_2)$	$\mu (A_3)$	$\mu (A_4)$	$\mu (A_5)$	$\mu (A_6)$
HFWA [13]	None	0.4722	0.6098	0.5988	0.4370	0.5479	0.6969
HFWG [13]	None	0.4033	0.5064	0.4896	0.3354	0.4611	0.6472
HFBM [32]	$p = 2, q = 1$	0.5372	0.5758	0.5576	0.3973	0.5275	0.7072
HFFWA [22]	$\lambda = 2$	0.3691	0.5322	0.4782	0.3759	0.4708	0.6031
HFFWG [22]	$\lambda = 2$	0.5419	0.6335	0.6232	0.4712	0.6029	0.7475
AC-HFWA (Case 2)	$\lambda = 1$	0.5069	0.6647	0.5985	0.4916	0.5888	0.7274
AC-HFWG (Case 2)	$\lambda = 1$	0.3706	0.4616	0.4574	0.2873	0.4158	0.6308

6. Conclusions

From the above analysis, while copulas and cocomulas are described as different functions with different parameters, there are many different HF information aggregation operators, which can be considered as a reflection of DM's preferences. These operators include specific cases which enable us to select the one that best fits with our interests in respective decision environments. This is the main advantage of these operators. Of course, they also have some shortcomings, which provide us with ideas for the following work.

We will apply the operators to deal with hesitant fuzzy linguistic [33], hesitant Pythagorean fuzzy sets [34], and multiple attribute group decision making [35] in the future. In order to reduce computation, we will also consider to simplify the operation of HFS and redefine the distance between HFEs [36] and score function [37]. Besides, the application of those operators to different decision-making methods will be developed, such as pattern recognition, information retrieval, and data mining.

The operators studied in this paper are based on the known weights. How to use the information of the data itself to determine the weight is also our next work. Liu and Liu [38] studied the generalized intuitionistic trapezoid fuzzy power averaging operator, which is the basis of our introduction of power mean operators into hesitant fuzzy sets. Wang and Li [39] developed power Bonferroni mean (PBM) operator to integrate Pythagorean fuzzy information, which gives us the inspiration to research new aggregation operators by combining copulas with PBM. Wu et al. [40] extended the best-worst method (BWM) to interval type-2 fuzzy sets, which inspires us to also consider using BWM in more fuzzy environments to determine weights.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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