Research Article

Multivariate Inverted Kumaraswamy Distribution: Derivation and Estimation

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Industrial revolution leads to the manufacturing of multicomponent products; to guarantee the sufficiency of the product and consumer satisfaction, the producer has to study the lifetime of the products. This leads to the use of bivariate and multivariate lifetime distributions in reliability engineering. The most popular and applicable is Marshall–Olkin family of distributions. In this paper, a new bivariate lifetime distribution which is the bivariate inverted Kumaraswamy (BIK) distribution is found and its properties are illustrated. Estimation using both maximum likelihood and Bayesian approaches is accomplished. Using different selection criteria, it is found that BIK provides the best performance compared with other bivariate distributions like bivariate exponential and bivariate inverse Weibull distributions. As a generalization, the multivariate inverted Kumaraswamy (MIK) distribution is derived. Few studies have been conducted on the multivariate Marshall–Olkin lifetime distributions. To the best of our knowledge, none of them handle estimation process. In this paper, we developed an algorithm to show how to estimate the unknown parameters of MIK using both maximum likelihood and Bayesian approaches. This algorithm could be applied in estimating other Marshall–Olkin multivariate lifetime distributions.

1. Introduction

Global competition in combination with using higher manufacturing technologies results in producing two or multicomponent products. This led to the use of bivariate and multivariate distributions in reliability engineering. Different families of distributions were constructed. One of the most commonly used is the Marshall–Olkin (MO) family. It is widely used due to its flexibility in considering different situations of failures (i.e., the first component has lifetime smaller, greater, or equal to the lifetime of other components).

In the literature, several lifetime distributions were derived as members of the bivariate Marshall–Olkin family. Marshall and Olkin [1] presented a bivariate exponential distribution with exponential marginals and loss of memory property. Using the same strategy, Kundu and Dey [2], Bareto-Souza and Lemonte [3], Muhammed [4], and Alqallaf and Kundu [5] introduced the bivariate Weibull, bivariate Kumaraswamy, bivariate generalized Burr, and bivariate inverse generalized exponential distributions, respectively. Using the maximum instead of the minimum in the MO scheme, Kundu and Gupta [6, 7] introduced the bivariate generalized exponential and bivariate proportional reversed hazard distributions, respectively. Moreover, Sarhan et al. [8] presented bivariate generalized linear failure rate distribution. Recently, Muhammed [9] introduced bivariate inverse Weibull (BIW) distribution.

Sometimes, the use of bivariate distributions may not be sufficient and there exists a need for multivariate distributions. For example, in air fighter jets, the natural lifetime since being manufactured and the flying time since being put into service are recorded. Studying the reliability of the air fighter jets using only two variables may not be good enough. One should take into consideration the lifetime of the engine, the wing, and the fuselage. This leads to the use of multivariate distribution. For more details, see Li and Li [10]. There is no much work performed in the multivariate case. Sarhan et al. [8] and Kundu

and Gupta [11] derived the multivariate generalized linear failure rate and multivariate inverse Weibull distributions, respectively. To the best of our knowledge, there is no work dealing with estimating the unknown parameters for multivariate Marshall–Olkin family.

Several authors tackled the estimation problem for bivariate MO distributions. For example, Kundu and Gupta [6], Muhammed [9], Aly et al. [12], Eliwa and El-Morshedy [13], El-Morshedy et al. [14], and Sarhan [4, 5, 15] estimated the unknown parameters using maximum likelihood approach for different bivariate lifetime distributions. On the other hand, Hanagal and Ahmadi [16], Kundu and Gupta [17], and Lin et al. [11, 13–15, 18] applied Bayesian approach for estimating certain bivariate lifetime distributions.

The univariate inverted Kumaraswamy (IK) distribution has several applications in different fields (see Abd Al-Fattah et al. [19] and Abdul Hammed et al. [20]), for example, in medical research, life testing problems, and stress-strength analysis. Also, in reliability and biological studies, IK distribution may be used to model failure rates. Due to its expected wide applicability, we are interested in deriving bivariate inverted Kumaraswamy (BIK) distribution. BIK could be applied in different fields like sports, engineering, and medicine as will be explained using three different real datasets. We expect better performance of BIK than other bivariate distributions. No one has derived the distribution before or found its mathematical properties.

The main purpose of this paper is to introduce BIK as a new Marshall–Olkin bivariate distribution in order to be applied efficiently in several fields. As a generalization, the multivariate inverted Kumaraswamy (MIK) distribution is derived. To the best of our knowledge, there is no work dealing with estimating the unknown parameters for multivariate Marshall–Olkin family. Here, estimation of MIK parameters is found using both maximum likelihood and Bayesian approaches. This work could be applied to all Marshall–Olkin multivariate distributions.

The paper is organized as follows. In Section 2, the bivariate inverted Kumaraswamy distribution is derived, and the cumulative distribution function and probability density function are presented. Also, the marginals and conditional distributions of the proposed model are obtained. Moreover, the product moments and the moment generating function are derived. In Section 3, the maximum likelihood estimators of the model parameters, asymptotic Fisher information matrix, and Bayesian estimators are obtained. Multivariate inverted Kumaraswamy distribution and its properties are illustrated in Section 4. The maximum likelihood and Bayesian estimators of the parameters under multivariate case are obtained in Section 5. Numerical analysis using both simulation, and real datasets are presented in Section 6. Finally, the paper is concluded in Section 7.

2. Bivariate Inverted Kumaraswamy Distribution

In this section, we will derive the bivariate inverted Kumaraswamy distribution as a new member in the MO family. Its properties such as the marginal and conditional distributions, joint moment generating function, and product moments are studied.

2.1. Derivation of the Bivariate Inverted Kumaraswamy Distribution.
The probability density function (pdf) and the cumulative distribution function (cdf) of the univariate inverted Kumaraswamy distribution (IK), respectively, are as follows (for more details, see [19]):

$$f_{IK}(x, \beta, \alpha) = \alpha \beta (1 + x)^{-(\alpha+1)} (1 - (1 + x)^{-\alpha})^{\beta-1}, \quad x > 0, \ \alpha, \beta > 0, \tag{1}$$

$$F_{IK}(x, \beta, \alpha) = (1 - (1 + x)^{-\alpha})^{\beta}, \quad x > 0, \ \alpha, \beta > 0. \tag{2}$$

Assume that U_i, $i = 1, 2, 3$, are three random variables, such that U_i follows IK (β_i, α). Define $X_1 = \max(U_1, U_2)$ and $X_2 = \max(U_2, U_3)$. Hence, the bivariate vector (X_1, X_2) follows BIK with shape parameters $\beta_i, \alpha, i = 1, 2, 3$. The joint cdf of (X_1, X_2) has the following form:

$$\begin{aligned} F_{X_1, X_2}(x_1, x_2) &= P(X_1 \le x_1, X_2 \le x_2) \\ &= P(\max(U_1, U_2) \le x_1, \max(U_2, U_3) \le x_2), \\ &= P(U_1 \le x_1, U_2 \le x_2, U_3 \le \min(x_1, x_2)), \\ &= P(U_1 \le x_1, U_2 \le x_2, U_3 \le z), \\ &= F_{IK}(x_1, \beta_1, \alpha) F_{IK}(x_2, \beta_2, \alpha) F_{IK}(z, \beta_3, \alpha), \end{aligned} \tag{3}$$

where $z = \min(x_1, x_2)$. The joint cdf can also be written as follows:

$$F_{X_1, X_2}(x_1, x_2) = \begin{cases} F_{IK}(x_1, \beta_1 + \beta_3, \alpha) F_{IK}(x_2, \beta_2, \alpha), & \text{if } x_1 < x_2, \\ F_{IK}(x_1, \beta_1, \alpha) F_{IK}(x_2, \beta_2 + \beta_3, \alpha), & \text{if } x_2 < x_1, \\ F_{IK}(x, \beta_1 + \beta_2 + \beta_3, \alpha), & \text{if } x_1 = x_2 = x, \end{cases} \tag{4}$$

where F_{IK} is as illustrated in equation (2).

Proposition 1. *The joint pdf of (X_1, X_2) has the following form:*

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & \text{if } x_1 < x_2, \\ f_2(x_1, x_2), & \text{if } x_2 < x_1, \\ f_3(x), & \text{if } x_1 = x_2 = x, \end{cases} \tag{5}$$

where

$$f_1(x_1, x_2) = f_{IK}(x_1, \beta_1 + \beta_3, \alpha) f_{IK}(x_2, \beta_2, \alpha),$$

$$f_2(x_1, x_2) = f_{IK}(x_1, \beta_1, \alpha) f_{IK}(x_2, \beta_2 + \beta_3, \alpha), \tag{6}$$

$$f_3(x) = \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3} f_{IK}(x, \beta_1 + \beta_2 + \beta_3, \alpha),$$

in which f_{IK} is as illustrated in equation (1).

Proof. See Appendix A.1.

The joint pdf of (X_1, X_2) can be expressed as a mixture of absolutely continuous part and singular part as follows. □

Proposition 2. *If* (X_1, X_2) *follows BIK* $(\beta_1, \beta_2, \beta_3, \alpha)$, *then*

$$F_{X_1, X_2}(x_1, x_2) = \frac{\beta_1 + \beta_2}{\beta_1 + \beta_2 + \beta_3} F_a(x_1, x_2) + \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3} F_s(z),$$

(7)

where

$$F_a(x_1, x_2) = \frac{\beta_1 + \beta_2 + \beta_3}{\beta_1 + \beta_2} \left(1 - (1 + x_1)^{-\alpha}\right)^{\beta_1} \left(1 - (1 + x_2)^{-\alpha}\right)^{\beta_2} \left(1 - (1 + z)^{-\alpha}\right)^{\beta_3}$$

$$- \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3} \left(1 - (1 + z)^{-\alpha}\right)^{\beta_1 + \beta_2 + \beta_3},$$

(8)

$$F_s(z) = \left(1 - (1 + z)^{-\alpha}\right)^{\beta_1 + \beta_2 + \beta_3},$$

where $z = \min\{x_1, x_2\}$, $F_s(.)$ *and* $F_a(.,.)$ *are the singular and absolute parts, respectively.*

Proof. See Appendix A.1. □

Corollary 1. *The joint pdf of* (X_1, X_2) *can be written as follows:*

$$f_{X_1, X_2}(x_1, x_2) = \frac{\beta_1 + \beta_2}{\beta_1 + \beta_2 + \beta_3} f_a(x_1, x_2) + \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3} f_s(z),$$

(9)

where

$$f_a(x_1, x_2) = \frac{\beta_1 + \beta_2 + \beta_3}{\beta_1 + \beta_2} \begin{cases} f_1(x_1, x_2) \text{ if } x_1 < x_2 \\ f_2(x_1, x_2) \text{ if } x_2 < x_1 \end{cases},$$

(10)

$$f_s(z) = f_{IK}(z, \beta_1 + \beta_2 + \beta_3).$$

The absolutely continuous part of BIK can be unimodal depending on the values of $\beta_1, \beta_2, \beta_3$, and α. $f_a(x_1, x_2)$ is unimodal if $\beta_1 + \beta_3 < \beta_2$ (under the case $x_1 < x_2$) or $\beta_2 + \beta_3 < \beta_1$ (under the case $x_1 > x_2$). The respective modes are

$$\left\{ \left[\frac{\alpha(\beta_1 + \beta_3) + 1}{\alpha + 1}\right]^{1/\alpha} - 1, \left[\frac{\alpha\beta_2 + 1}{\alpha + 1}\right]^{1/\alpha} - 1 \right\},$$

$$\left\{ \left[\frac{\alpha\beta_1 + 1}{\alpha + 1}\right]^{1/\alpha} - 1, \left[\frac{\alpha(\beta_2 + \beta_3) + 1}{\alpha + 1}\right]^{1/\alpha} - 1 \right\}.$$

(11)

2.2. Properties of Bivariate Inverted Kumaraswamy Distribution. In this section, we illustrate different properties of BIK distribution. We provide marginal, conditional distributions, joint moment generating function, and product moments.

Proposition 3 (Marginal and conditional distributions). *If* (X_1, X_2) *follows BIK* $(\beta_1, \beta_2, \beta_3, \alpha)$, *then*

(a) X_1 *follows IK* $(\beta_1 + \beta_3, \alpha)$ *and* X_2 *follows IK* $(\beta_2 + \beta_3, \alpha)$.
(b) $\max\{X_1, X_2\}$ *follows IK* $(\beta_1 + \beta_2 + \beta_3, \alpha)$.

(c) $F_{X_1|X_2 \leq x_2}(x_1) =$
$$\begin{cases} \left(1 - (1 + x_1)^{-\alpha}\right)^{\beta_1 + \beta_3} \left(1 - (1 + x_2)^{-\alpha}\right)^{-\beta_3}, & \text{if } x_1 \leq x_2, \\ \left(1 - (1 + x_1)^{-\alpha}\right)^{\beta_1}, & \text{if } x_2 < x_1. \end{cases}$$

Proof. See Appendix A.1. □

Proposition 4 (Moment generating function). *If* (X_1, X_2) *follows BIK* $(\beta_1, \beta_2, \beta_3, \alpha)$, *then the joint moment generating function of* X_1 *and* X_2 *is given by* $M(t_1, t_2) = l_1 + l_2 + l_3$, *where*

$$l_1 = \beta_2 (\beta_1 + \beta_3) \sum_{i=0}^{\infty} \frac{t_2^i}{i!} \sum_{k_2=0}^{i} \binom{i}{k_2} (-1)^{i-k_2} \sum_{j=0}^{\infty} \frac{t_1^i}{j!} \sum_{k_1=0}^{j} \binom{j}{k_1} (-1)^{i-k_1}$$

$$\cdot X \left\{ B\left(1 - \frac{k_2}{\alpha}, \beta_2\right) B\left(1 - \frac{k_1}{\alpha}, \beta_1 + \beta_3\right) - \frac{\alpha}{\alpha - k_1} B\left(2 - \frac{k_2 + k_1}{\alpha}, \beta_2\right) X_3 F_2 \left(2 - \frac{k_2 + k_1}{\alpha}, 1 - \frac{k_1}{\alpha}, 1 - \beta_1 - \beta_3; 2 - \frac{k_1}{\alpha}; 2 + \beta_2 - \frac{k_2 + k_1}{\alpha}; 1\right) \right\},$$

$$l_2 = \beta_1 (\beta_2 + \beta_3) \sum_{j=0}^{\infty} \frac{t_1^j}{j!} \sum_{k_1=0}^{j} \binom{j}{k_1} (-1)^{j-k_1} \sum_{i=0}^{\infty} \frac{t_2^i}{i!} \sum_{k_2=0}^{i} \binom{i}{k_2} (-1)^{i-k_2}$$

$$\cdot X \left\{ B\left(1 - \frac{k_1}{\alpha}, \beta_1\right) B\left(1 - \frac{k_2}{\alpha}, \beta_2 + \beta_3\right) - \frac{\alpha}{\alpha - k_2} B\left(2 - \frac{k_1 + k_2}{\alpha}, \beta_1\right) X_3 F_2 \left(2 - \frac{k_1 + k_2}{\alpha}, 1 - \frac{k_2}{\alpha}, 1 - \beta_2 - \beta_3; 2 - \frac{k_2}{\alpha}; 2 + \beta_1 - \frac{k_1 + k_2}{\alpha}; 1\right) \right\},$$

$$l_3 = \beta_1 \sum_{i=0}^{\infty} \frac{(t_1 + t_2)^i}{i!} \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} B\left(1 - \frac{k}{\alpha}, \beta_1 + \beta_2 + \beta_3\right),$$

$$(12)$$

where $\quad {}_pF_q(b_1, \ldots, b_q; c_1, \ldots, c_q; u) = \sum_{k=0}^{\infty} ((b_1)_k, \ldots, (b_p)_k / (c_1)_k, \ldots, (c_q)_k)(u^k/k!)$, $(b_i) = \Gamma(b+i)/\Gamma(b)$, $i = 1, \ldots, p$, $(c_j) = \Gamma(c+j)/\Gamma(c)$, $j = 1, \ldots, q$, *p and q are nonnegative integers, and B (.,.) is beta function.*

Proposition 5 (Product moments). *If* (X_1, X_2) *follows BIK* $(\beta_1, \beta_2, \beta_3, \alpha)$, *then the product moments of* X_1 *and* X_2 *are given by* $E(X_1^s X_2^r) = J_1 + J_2 + J_3$, *where*

Proof. See Appendix A.1.   □

$$J_1 = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} \beta_2 (\beta_1 + \beta_3) \left[B\left(1 - \frac{j}{\alpha}, \beta_1 + \beta_3\right) B\left(1 - \frac{i}{\alpha}, \beta_2\right) \right.$$

$$\left. - \frac{\alpha}{\alpha - j} B\left(2 - \frac{i+j}{\alpha}, \beta_2\right) {}_3F_2\left(2 - \frac{i+j}{\alpha}, 1 - \frac{j}{\alpha}, 1 - \beta_1 - \beta_3; 2 - \frac{j}{\alpha}, 2 - \frac{i+j}{\alpha} + \beta_2; 1\right) \right\},$$

$$J_2 = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} \beta_1 (\beta_2 + \beta_3)$$

$$\cdot \left[B\left(1 - \frac{i}{\alpha}, \beta_2 + \beta_3\right) B\left(1 - \frac{j}{\alpha}, \beta_1\right) - \frac{\alpha}{\alpha - i} B\left(2 - \frac{i+j}{\alpha}, \beta_1\right) \right.$$

$$\left. \cdot {}_3F_2\left(2 - \frac{j+i}{\alpha}, 1 - \frac{i}{\alpha}, 1 - \beta_2 - \beta_3; 2 - \frac{i}{\alpha}, 2 - \frac{j+i}{\alpha} + \beta_1; 1\right) \right\}, J_3 = \sum_{k=0}^{r+s} (-1)^{r+s-k} \binom{r+s}{k} \beta_3 B\left(1 - \frac{k}{\alpha}, \beta_1 + \beta_2 + \beta_3\right) B\left(1 - \frac{k}{\alpha}, \beta_2 + \beta_3\right).$$

$$(13)$$

Proof. See Appendix A.1.   □

3. Estimation of Bivariate Inverted Kumaraswamy Distribution

In this section, we estimate the unknown parameters using both maximum likelihood and Bayesian approaches.

3.1. Using Maximum Likelihood Approach. In this section, we derive the maximum likelihood (ML) estimators of the unknown parameters of BIK distribution. Suppose $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ is a random sample of size n from BIK $(\beta_1, \beta_2, \beta_3, \alpha)$; then, the ML estimators of the unknown parameters are obtained as follows.

The log likelihood function of the sample of size n is given by $\quad \log L(\underline{\theta}) = \sum_{i\epsilon I_1} \log f_1(x_{1i}, x_{2i}) + \sum_{i\epsilon I_2} \log f_2(x_{1i}, x_{2i}) + \sum_{i\epsilon I_3} \log f_3(x_i)$, where

$$I_1 = \{i; x_{1i} < x_{2i}\},$$

$$I_2 = \{i; x_{1i} > x_{2i}\},$$

$$I_3 = \{i; x_{1i} = x_{2i} = x_i\},$$

$$I = I_1 \bigcup I_2 \bigcup I_3,$$

$$|I_1| = n_1,$$

$$|I_2| = n_2,$$

$$|I_3| = n_3,$$

$$n_1 + n_2 + n_3 = n,$$

(14)

where $|I_j|$ denotes the cardinality of the set I_j, for $j = 1, 2, 3$. Thus,

$$
\begin{aligned}
\log L(\underline{\theta}) = {} & (2n_1 + 2n_2 + n_3)\log\alpha + n_1\log(\beta_1 + \beta_3) + n_1\log(\beta_2) + n_2 log(\beta_1) + n_2\log(\beta_2 + \beta_3) + n_3\log(\beta_3) \\
& - (\alpha + 1)\left\{ \sum_{i\epsilon I_1 \bigcup I_2} \log(1 + x_{1i}) + \sum_{i\epsilon I_1 \bigcup I_2} \log(1 + x_{2i}) + \sum_{i\epsilon I_3} \log(1 + x_i) \right\} \\
& + (\beta_1 + \beta_3 - 1)\sum_{i\epsilon I_1} \log\left(1 - (1 + x_{1i})^{-\alpha}\right) \\
& + (\beta_2 - 1)\sum_{i\epsilon I_1} \log\left(1 - (1 + x_{2i})^{-\alpha}\right) \\
& + (\beta_1 - 1)\sum_{i\epsilon I_2} \log\left(1 - (1 + x_{1i})^{-\alpha}\right) \\
& + (\beta_2 + \beta_3 - 1)\sum_{i\epsilon I_2} \log\left(1 - (1 + x_{2i})^{-\alpha}\right) \\
& + (\beta_1 + \beta_2 + \beta_3 - 1)\sum_{i\epsilon I_3} \log\left(1 - (1 + x_i)^{-\alpha}\right),
\end{aligned}
$$

(15)

where $\underline{\theta} = (\alpha, \beta_1, \beta_2, \beta_3)$.

The first derivatives of the log likelihood with respect to the unknown parameters and also the observed Fisher information matrix are obtained in Appendix A.2. The ML estimates $\widehat{\underline{\theta}} = (\widehat{\alpha}, \widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3)$ of $\underline{\theta} = (\alpha, \beta_1, \beta_2, \beta_3)$ are numerically obtained in Section 6.

3.2. Using Bayesian Approach. Let (X_1, X_2) follow BIK $(\underline{\theta})$, where $\underline{\theta} = (\alpha, \beta_1, \beta_2, \beta_3)$ is the vector of unknown parameters. The posterior pdf of parameters can be obtained as follows:

$$P(\underline{\theta} | (X_1, X_2)) \propto L((X_1, X_2); \underline{\theta})P(\underline{\theta}), \quad (16)$$

where $P(\underline{\theta})$ is the prior distribution.

Pena and Gupta [21] considered Bayesian estimation of the parameters for Marshall–Olkin bivariate exponential distribution (BVE), in series and parallel systems. They obtained posterior mode using gamma Dirichlet distribution as prior distribution. Angali et al. [22] considered Bayesian estimation for BVE using gamma prior.

Similar to [22], we considered a gamma prior distribution with the following pdf:

$$
\begin{aligned}
P(\beta_i, a_i, b_i) &\propto \beta_i^{a_i - 1}\exp(-b_i\beta_i), \\
P(\alpha, a_4, b_4) &\propto \alpha^{a_4 - 1}\exp(-b_4\alpha),
\end{aligned}
$$

(17)

where $i = 1, 2, 3$ and a_i, b_i, a_4, b_4 are the hyperparameters. The posterior pdf has the following form:

$$P(\underline{\theta} \mid (X_1, X_2)) \propto \prod_{i=1}^{n_1} \alpha^2 (\beta_1 + \beta_3)\beta_2 (1 + x_{1i})^{-(\alpha+1)} (1 + x_{2i})^{-(\alpha+1)}$$

$$X \left(1 - (1 + x_{1i})^{-\alpha}\right)^{\beta_1 + \beta_3 - 1} \left(1 - (1 + x_{2i})^{-\alpha}\right)^{\beta_2 - 1}$$

$$\prod_{i=1}^{n_2} \alpha^2 (\beta_2 + \beta_3)\beta_1 (1 + x_{1i})^{-(\alpha+1)} (1 + x_{2i})^{-(\alpha+1)}$$

$$X \left(1 - (1 + x_{1i})^{-\alpha}\right)^{\beta_1 - 1} \left(1 - (1 + x_{2i})^{-\alpha}\right)^{\beta_2 + \beta_3 - 1}$$

$$\prod_{i=1}^{n_3} \alpha\beta_3 (1 + x_i)^{-(\alpha+1)} \left(1 - (1 + x_i)^{-\alpha}\right)^{\beta_1 + \beta_2 + \beta_3 - 1}$$

$$X \prod_{i=1}^{3} \beta_i^{a_i - 1} \exp(-b_i \beta_i) \alpha^{a_4 - 1} \exp(-b_4 \alpha). \tag{18}$$

It is observed that Bayesian estimators under square error loss function cannot be obtained in explicit forms. Therefore, we obtain the posterior mean using MCMC technique which is illustrated in Section 6.

4. Multivariate Inverted Kumaraswamy Distribution

In literature, there is no much work that dealt with multivariate MO distributions. Sarhan et al. [8] derived the multivariate generalized linear failure rate distribution. Kundu and Gupta [11] derived the multivariate inverse Weibull distribution. Here, we will introduce a new multivariate MO distribution, which is the multivariate inverted Kumaraswamy (MIK) distribution. It is a generalization of the BIK considered in Section 2. We expect that MIK will be of great importance for several applied fields. In the following two sections, the derivation of MIK is explained and some of its properties are studied.

4.1. Derivation of MIK. In this section, we will derive the cdf and pdf of MIK.

Let $U_1, \ldots, U_{m+1}$ be $m + 1$ independent random variables such that $U_i \sim \mathrm{IK}(\beta_i, \alpha)$, for $i = 1, \ldots, m + 1$. Define $X_j = \max\{U_j, U_{m+1}\}$, $j = 1, \ldots, m$. Then, $X = (X_1, \ldots, X_m)^T$ is an m-variate IK with parameters $(\beta_1, \ldots, \beta_{m+1}, \alpha)$, and it will be denoted by MIK $(m, \beta_1, \ldots, \beta_{m+1}, \alpha)$. We have the following results regarding MIK distribution.

Proposition 6. *If* $\underline{X} = (X_1, \ldots, X_m) \sim MIK(m, \beta_1, \ldots, \beta_{m+1}, \alpha)$, *then the joint cdf of* $\underline{X}$ *for* $x_1 > 0, \ldots, x_m > 0$ *is*

$$F_{\underline{X}}(\underline{x}) = \prod_{i=1}^{m+1} F_{\mathrm{IK}}(x_i, \beta_i, \alpha), \tag{19}$$

where $\underline{x} = (x_1, \ldots, x_m)$ *and* $x_{m+1} = \min\{x_1, \ldots, x_m\}$.

Proof. We prove by generalizing the same method illustrated in Section 2.

Similar to the bivariate case, the MIK distribution can be written as, for $m > 1$,

$$F_{\underline{X}}(\underline{x}) = k F_a(\underline{x}) + (1 - k)F_s(\underline{x}), \tag{20}$$

where $0 < k < 1$ and F_a and F_s denote the absolute and continuous parts, respectively. The corresponding pdf can also be written as

$$f_{\underline{X}}(\underline{x}) = k f_a(\underline{x}) + (1 - k)f_s(\underline{x}). \tag{21}$$

The absolutely continuous part of $f_a(\underline{x})$ can be obtained from

$$\frac{\partial^m F_{\underline{X}}(x_1, \ldots, x_m)}{\partial x_1, \ldots, \partial x_m}, \tag{22}$$

where $\underline{x} = (x_1, \ldots, x_m)^T$ belongs to the set where $F_{\underline{X}}(\underline{x})$ is absolutely continuous, if and only if $x_i's$ are different. For each $\underline{x}$, where x_i are different, there exists a permutation $\rho = \{i_1, \ldots, i_m\}$ such that $x_{i1} < \cdots, < x_{im}$.

Define $f_\rho(\underline{x}) = f_{\mathrm{IK}}(x_{i1}, \beta_{i1} + \beta_{m+1}, \alpha) f_{\mathrm{IK}}(x_{i2}, \beta_{i2}, \alpha), \ldots, f_{\mathrm{IK}}(x_{im}, \beta_{im}, \alpha)$. Then, for $x_{i1} < \cdots, < x_{im}$, $\partial^m F_{\underline{X}}(x_1, \ldots, x_m)/\partial x_1, \ldots, \partial x_m = k f_a(x_1, \ldots, x_m) = f_\rho(x_{i1}, \ldots, x_{im})$, where k can be obtained as follows:

$$k = \sum_\rho J_\rho, \tag{23}$$

where

$$\int_{R^m} f_a(x_1, \ldots, x_m)dx_1, \ldots, dx_m = \sum_\rho \int_{x_{im}=0}^{\infty}$$
$$\cdot \int_{x_{im-1}=0}^{x_{im}}, \ldots, \int_{x_{i1}=0}^{x_{i2}} f_\rho(x_1, \ldots, x_m)dx_{i1}, \ldots, dx_{im}$$
$$= \sum_\rho J_\rho. \tag{24}$$

Since

$$\int_{x_{i_1}=0}^{x_{i_2}} f_\rho(x_1,\ldots,x_m)\mathrm{d}x_{i_1} = F_{\mathrm{IK}}\left(x_{i_2};\beta_{i_1}+\beta_{m+1},\alpha\right)\prod_{j=2}^{m} f_{\mathrm{IK}}\left(x_{i_j},\beta_{i_j},\alpha\right),$$

$$\int_{x_{i_2}=0}^{x_{i_3}}\int_{x_{i_1}=0}^{x_{i_2}} f_\rho(x_1,\ldots,x_m)\mathrm{d}x_{i1}\mathrm{d}x_{i2} = \frac{\beta_{i_2}}{\beta_{i_1}+\beta_{i_2}+\beta_{m+1}}\,F_{\mathrm{IK}}\left(x_{i_3};\beta_{i_1}+\beta_{i_2}+\beta_{m+1},\alpha\right)\prod_{j=3}^{m} f_{\mathrm{IK}}\left(x_{i_j},\beta_{i_j},\alpha\right),$$

$$\vdots$$

$$J_\rho = \frac{\beta_{i_2}}{\beta_{i_1}+\beta_{i_2}+\beta_{m+1}}X\frac{\beta_{i_3}}{\beta_{i_1}+\beta_{i_2}+\beta_{i_3}+\beta_{m+1}}X,\ldots,X\frac{\beta_{im}}{\beta_{i_1}+\ldots+\beta_{i_m}+\beta_{m+1}},$$

(25)

then

$$k = \sum_\rho \frac{\beta_{i_2}}{\beta_{i_1}+\beta_{i_2}+\beta_{m+1}}X\frac{\beta_{i3}}{\beta_{i_1}+\beta_{i_2}+\beta_{i_3}+\beta_{m+1}}X,\ldots,X\frac{\beta_{im}}{\beta_{i_1}+\ldots+\beta_{i_m}+\beta_{m+1}},$$

(26)

and for all $x_{i_1}<,\cdots,<x_{i_m}$, $f_a(\underline{x})=1/kf_\rho(\underline{x})$.

Now, let $I_l=\{i_1,\ldots,i_l\}\subset I=\{1,\ldots,m\}$ such that $i_1<,\cdots,<i_l$. $f_{\underline{X}}(\underline{x})$ can be written as

$$f_{\underline{X}}(\underline{x})=kf_a(\underline{x})+\sum_{l=2}^{m}\sum_{I_l\subset I}k_lf_{I_l}(\underline{x}), \qquad (27)$$

where f_{I_l} is a pdf with respect to $(m-l+1)$ dimensional Lebesgue measure on the hyperplane $A_{I_l}=\{\underline{x}\in\mathfrak{R}^m: x_{i1}=,\cdots,=x_{il}\}$. The exact meaning of $f_{\underline{X}}(\underline{x})$ is as follows.

For any Borel measurable set $B\in\mathfrak{R}^m$,

$$p(x\in B)=k\int_B f_a(\underline{x})+\sum_{l=2}^{m}\sum_{I_l\subset I}k_{I_l}\int_{B_{Il}} f_{I_l}(\underline{x}), \qquad (28)$$

where $B_{I_l}=B\cap A_{I_l}$ is the projection of set B onto $(m-l+1)$ dimensional hyperplane A_{I_l}.

Now, we provide k_{I_l} and $f_{I_l}(\underline{x})$. Note that if $\underline{x}\epsilon A_{I_l}$, then $\underline{x}$ has the following form:

$$\underline{x}=(x_1\ldots,x_{i_1-1},x^*,x_{i_1+1},\ldots,x_{i_2-1},x^*,x_{i_2+1},\ldots,x_{i_l-1},x^*,x_{i_l+1},\ldots,x_m). \qquad (29)$$

For a given $\underline{x}\in\mathfrak{R}^m$, we define g_{I_l} from the $(m-l+1)$ dimensional hyperplane A_{I_l} to $\mathfrak{R}$ as follows:

$$g_{I_l}(\underline{x})=f_{\mathrm{IK}}(x^*,\beta_{m+1},\alpha)F_{\mathrm{IK}}\left(x^*,\sum_{i\in I_l}\beta_i,\alpha\right)\prod_{i\in I-I_l}f_{\mathrm{Ik}}(x_i,\beta_i,\alpha), \qquad (30)$$

if $x_i>x^*$ for $i\in I-I_l$ and zero otherwise. Similar to k, we can obtain k_{I_l} as follows:

$$k_{I_l}=\int_{A_{I_l}}g_{I_l}(\underline{x})\mathrm{d}\underline{x}\sum_{\rho_{I-Il}}\int_{x_{jm-l}=0}^{\infty}\int_{x_{jm-l-1}=0}^{x_{jm-l}}\int_{x_{j1}=0}^{x_{j2}}$$

$$\cdot g_{I_l}(\underline{x})\mathrm{d}x^*\mathrm{d}x_{j1},\ldots,\mathrm{d}x_{jm-l}$$

$$=\sum_{\rho_{I-I_l}}\frac{\beta_{m+1}}{\sum_{i\in I_l}\beta_i+\beta_{m+1}}*\frac{\beta_{j_1}}{\sum_{i\in I}\beta_i+\beta_{m+1}+\beta_{j_1}}*\frac{\beta_{j_{m-l}}}{\sum_{i\in I}\beta_i+\beta_{m+1}},$$

$$f_{I_l}(\underline{x})=\frac{1}{k_{I_l}}g_{I_l}(\underline{x}).$$

(31)

$\square$

4.2. Properties of MIK.

In this section, we will get the marginal and conditional distributions of MIK.

Proposition 7. If $\underline{X}=(X_1,\ldots,X_m)\sim MIK(m,\beta_1,\ldots,\beta_m,\beta_{m+1},\alpha)$, then

(a) $X_1\sim IK(\alpha,\beta_1+\beta_{m+1}),\ldots,X_m\sim IK(\alpha,\beta_m+\beta_{m+1})$.

(b) The conditional distribution of $(X_1,\ldots,X_s)$ given $\{X_{s+1}\leq x_{s+1},\ldots,X_m\leq x_m\}$ is

$$P\left(X_1 \le x_1, \ldots, X_s \le x_s | X_{s+1} \le x_{s+1}, \ldots, X_m \le x_m\right) = \begin{cases} \prod_{j=1}^{s} F_{\mathrm{IK}}\left(x_j, \beta_j, \alpha\right) \text{ if } z = v \\ \prod_{j=1}^{s} F_{\mathrm{IK}}\left(x_j, \beta_j, \alpha\right) F_{\mathrm{IK}}\left(z, \beta_{m+1}, \alpha\right) F_{IK}\left(v, \beta_{m+1}, \alpha\right) \text{ if } z < v \end{cases}, \quad (32)$$

where $z = \min\{x_1, \ldots, x_s\}$, $v = \min\{x_{s+1}, \ldots, x_m\}$.

(c) For $2 \le s \le m$, $(X_1, \ldots, X_s) \sim MIK(s, \beta_1, \ldots, \beta_s, \beta_{m+1}, \alpha)$.

(d) If $T = \max\{X_1, \ldots, X_m\}$, then $T \sim IK(\beta_1 +, \cdots, + \beta_{m+1}, \alpha)$.

Proof. (a, c) B taking the limit for the joint cdf. (b, d) Could be directly obtained from the definition of the multivariate inverted Kumaraswamy distribution. ∎

5. Estimation of the Multivariate Inverted Kumaraswamy Distribution

Although estimating the unknown parameters of a certain multivariate MO distribution is very important, no one in the literature was interested in it. Therefore, in this section, we will consider the process of estimation for MIK parameters. The proposed techniques could be applied for any multivariate MO distribution. Here, we will apply both maximum likelihood and Bayesian approaches.

5.1. Using Maximum Likelihood Approach. In this section, for simplification, we consider the case when we have three random variables X_1, X_2, and X_3. Applying Proposition 6, we have the following cdf:

$$F_{\underline{X}}\left(\underline{x}\right) = \prod_{i=1}^{4} F_{\mathrm{IK}}\left(x_i, \beta_i, \alpha\right). \quad (33)$$

The cdf can be rewritten as

$$F_{X_1, X_2, X_3}\left(x_1, x_2, x_3\right) = \begin{cases} F_1\left(x_1, x_2, x_3\right), & \text{if } x_1 < x_2 < x_3 \text{ or } x_1 < x_3 < x_2, \\ F_2\left(x_1, x_2, x_3\right), & \text{if } x_2 < x_1 < x_3 \text{ or } x_2 < x_3 < x_1, \\ F_3\left(x_1, x_2, x_3\right), & \text{if } x_3 < x_1 < x_2 \text{ or } x_3 < x_2 < x_1 \\ F_4\left(x, x_3\right), & \text{if } x_1 = x_2 = x < x_3, \\ F_5\left(x, x_2\right), & \text{if } x_1 = x_3 = x < x_2, \\ F_6\left(x, x_2\right), & \text{if } x_2 = x_3 = x < x_1, \\ F_7\left(x\right), & \text{if } x_1 = x_2 = x_3 = x, \\ F_8\left(x, x_3\right), & \text{if } x_1 = x_2 = x > x_3, \\ F_9\left(x, x_2\right), & \text{if } x_1 = x_3 = x > x_2, \\ F_{10}\left(x, x_1\right), & \text{if } x_2 = x_3 = x > x_1, \end{cases} \quad (34)$$

where

$$
\begin{aligned}
F_1\left(x_1, x_2, x_3\right) &= F_{\mathrm{IK}}\left(x_1, \beta_1 + \beta_4, \alpha\right) F_{\mathrm{IK}}\left(x_2, \beta_2, \alpha\right) F_{\mathrm{IK}}\left(x_3, \beta_3, \alpha\right), \\
F_2\left(x_1, x_2, x_3\right) &= F_{\mathrm{IK}}\left(x_1, \beta_1, \alpha\right) F_{\mathrm{IK}}\left(x_2, \beta_2 + \beta_4, \alpha\right) F_{\mathrm{IK}}\left(x_3, \beta_3, \alpha\right), \\
F_3\left(x_1, x_2, x_3\right) &= F_{\mathrm{IK}}\left(x_1, \beta_1, \alpha\right) F_{\mathrm{IK}}\left(x_2, \beta_2, \alpha\right) F_{\mathrm{IK}}\left(x_3, \beta_3 + \beta_4, \alpha\right), \\
F_4\left(x, x_3\right) &= F_{\mathrm{IK}}\left(x, \beta_1 + \beta_2 + \beta_4, \alpha\right) F_{\mathrm{IK}}\left(x_3, \beta_3, \alpha\right), \\
F_5\left(x, x_2\right) &= F_{\mathrm{IK}}\left(x, \beta_1 + \beta_3 + \beta_4, \alpha\right) F_{\mathrm{IK}}\left(x_2, \beta_2, \alpha\right), \\
F_6\left(x, x_2\right) &= F_{\mathrm{IK}}\left(x, \beta_2 + \beta_3 + \beta_4, \alpha\right) F_{\mathrm{IK}}\left(x_1, \beta_1, \alpha\right), \\
F_7\left(x\right) &= F_{\mathrm{IK}}\left(x, \beta_1 + \beta_2 + \beta_3 + \beta_4, \alpha\right), \\
F_8\left(x, x_3\right) &= F_{\mathrm{IK}}\left(x, \beta_1 + \beta_2, \alpha\right) F_{\mathrm{IK}}\left(x_3, \beta_3 + \beta_4, \alpha\right), \\
F_9\left(x, x_2\right) &= F_{\mathrm{IK}}\left(x, \beta_1 + \beta_3, \alpha\right) F_{\mathrm{IK}}\left(x_2, \beta_2 + \beta_4, \alpha\right), \\
F_{10}\left(x, x_1\right) &= F_{\mathrm{IK}}\left(x, \beta_2 + \beta_3, \alpha\right) F_{\mathrm{IK}}\left(x_1, \beta_1 + \beta_4, \alpha\right).
\end{aligned}
\quad (35)
$$

The pdf can be obtained by taking the derivative, except for f_{10} where we should take into consideration that the sum of all probabilities equals one.

Now, suppose $\{(x_{11}, x_{21}, x_{31}), \ldots, (x_{1n}, x_{2n}, x_{3n})\}$ is a random sample of size n from MIK$(\alpha, \beta_1, \beta_2, \beta_3, \beta_4)$; the problem is to find the ML estimators of the unknown parameters. Consider the following notation:

$$
\begin{aligned}
I_1 &= \{i; x_{1i} < x_{2i} < x_{3i}\}, \\
I_2 &= \{i; x_{1i} < x_{3i} < x_{2i}\}, \\
I_3 &= \{i; x_{2i} < x_{1i} < x_{3i}\}, \\
I_4 &= \{i; x_{2i} < x_{3i} < x_{1i}\}, \\
I_5 &= \{i; x_{3i} < x_{1i} < x_{2i}\}, \\
I_6 &= \{i; x_{3i} < x_{2i} < x_{1i}\}, \\
I_7 &= \{i; x_{1i} = x_{2i} = x_i < x_{3i}\}, \\
I_8 &= \{i; x_{1i} = x_{3i} = x_i < x_{2i}\}, \\
I_9 &= \{i; x_{2i} = x_{3i} = x_i < x_{1i}\}, \\
I_{10} &= \{i; x_{1i} = x_{2i} = x_{3i} = x_i\}, \\
I_{11} &= \{i; x_{1i} = x_{2i} = x_i > x_{3i}\}, \\
I_{12} &= \{i; x_{1i} = x_{3i} = x_i > x_{2i}\}, \\
I_{13} &= \{i; x_{2i} = x_{3i} = x_i > x_{1i}\}, \\
|I_j| &= n_j, \; j = 1, 2, \ldots, 13, \; \sum_{j=1}^{13} n_j = n,
\end{aligned}
\quad (36)
$$

where $|I_j|$ denotes the cardinality of the set I_j, for $j = 1, 2, \ldots, 13$.

The log likelihood function of the sample of size n is given by

$$\log(L(\underline{\theta})) = \sum_{i \in I_1 \cup I_2} \log \left\{ \alpha^3 (\beta_1 + \beta_4) \beta_2 \beta_3 (1 + x_{1i})^{-(\alpha+1)} (1 - (1 + x_{1i})^{-\alpha})^{\beta_1 + \beta_4 - 1} X (1 + x_{2i})^{-(\alpha+1)} \right.$$
$$\left. \cdot (1 - (1 + x_{2i})^{-\alpha})^{\beta_2 - 1} (1 + x_{3i})^{-(\alpha+1)} X (1 - (1 + x_{3i})^{-\alpha})^{\beta_3 - 1} \right\}$$
$$+ \sum_{i \in I_3 \cup I_4} \log \left\{ \alpha^3 (\beta_2 + \beta_4) \beta_1 \beta_3 (1 + x_{1i})^{-(\alpha+1)} (1 - (1 + x_{1i})^{-\alpha})^{\beta_1 - 1} X (1 + x_{2i})^{-(\alpha+1)} (1 - (1 + x_{2i})^{-\alpha})^{\beta_2 + \beta_4 - 1} (1 + x_{3i})^{-(\alpha+1)} \right.$$
$$\left. \cdot X (1 - (1 + x_{3i})^{-\alpha})^{\beta_3 - 1} \right\}$$
$$+ \sum_{i \in I_5 \cup I_6} \log \left\{ \alpha^3 (\beta_3 + \beta_4) \beta_1 \beta_2 (1 + x_{1i})^{-(\alpha+1)} (1 - (1 + x_{1i})^{-\alpha})^{\beta_1 - 1} X (1 + x_{2i})^{-(\alpha+1)} (1 - (1 + x_{2i})^{-\alpha})^{\beta_2 - 1} (1 + x_{3i})^{-(\alpha+1)} \right.$$
$$\left. \cdot X (1 - (1 + x_{3i})^{-\alpha})^{\beta_3 + \beta_4 - 1} \right\}$$
$$+ \sum_{i \in I_7} \log \left\{ \alpha^2 (\beta_1 + \beta_2 + \beta_4) \beta_3 (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_1 + \beta_2 + \beta_4 - 1} X (1 + x_{3i})^{-(\alpha+1)} (1 - (1 + x_{3i})^{-\alpha})^{\beta_3 - 1} \right\}$$
$$+ \sum_{i \in I_8} \log \left\{ \alpha^2 (\beta_1 + \beta_3 + \beta_4) \beta_2 (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_1 + \beta_3 + \beta_4 - 1} X (1 + x_{2i})^{-(\alpha+1)} (1 - (1 + x_{2i})^{-\alpha})^{\beta_2 - 1} \right\}$$
$$+ \sum_{i \in I_9} \log \left\{ \alpha^2 (\beta_2 + \beta_3 + \beta_4) \beta_1 (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_2 + \beta_3 + \beta_4 - 1} X (1 + x_{1i})^{-(\alpha+1)} (1 - (1 + x_{1i})^{-\alpha})^{\beta_1 - 1} \right\}$$
$$+ \sum_{i \in I_{10}} \log \left\{ \alpha S (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_1 + \beta_2 + \beta_3 + \beta_4 - 1} \right\}$$
$$+ \sum_{i \in I_{11}} \log \left\{ \alpha^2 (\beta_1 + \beta_2) (\beta_3 + \beta_4) (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_1 + \beta_2 - 1} X (1 + x_{3i})^{-(\alpha+1)} (1 - (1 + x_{3i})^{-\alpha})^{\beta_3 + \beta_4 - 1} \right\}$$
$$+ \sum_{i \in I_{12}} \log \left\{ \alpha^2 (\beta_1 + \beta_3) (\beta_2 + \beta_4) (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_1 + \beta_3 - 1} X (1 + x_{2i})^{-(\alpha+1)} (1 - (1 + x_{2i})^{-\alpha})^{\beta_2 + \beta_4 - 1} \right\}$$
$$+ \sum_{i \in I_{13}} \log \left\{ \alpha^2 (\beta_2 + \beta_3) (\beta_1 + \beta_4) (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_2 + \beta_3 - 1} X (1 + x_{1i})^{-(\alpha+1)} (1 - (1 + x_{1i})^{-\alpha})^{\beta_1 + \beta_4 - 1} \right\},$$

$$(37)$$

where $\underline{\theta} = (\alpha, \beta_1, \beta_2, \beta_3, \beta_4)$.

$$S = (\beta_1 + \beta_2 + \beta_3 + \beta_4) - \frac{\beta_3(\beta_1 + \beta_2)}{\beta_1 + \beta_2 + \beta_4} - \frac{\beta_2(\beta_1 + \beta_3)}{\beta_1 + \beta_3 + \beta_4} - \frac{\beta_1(\beta_2 + \beta_3)}{\beta_3 + \beta_2 + \beta_4}$$
$$- 3(\beta_1 + \beta_2 + \beta_3).$$

$$(38)$$

It is seen that the ML estimates could not be obtained in explicit forms, and hence we need to use numerical analysis to obtain them.

5.2. Using Bayesian Approach. Let (X_1, X_2, X_3) be three random variables from MIK $(\underline{\theta})$, where $\underline{\theta} = (\alpha, \beta_1, \beta_2, \beta_3, \beta_4)$ is the vector of unknown parameters. The posterior pdf can be obtained as follows:

$$P(\underline{\theta} \mid (X_1, X_2, X_3)) \propto L((X_1, X_2, X_3); \underline{\theta}) P(\underline{\theta}), \qquad (39)$$

where $P(\underline{\theta})$ is the prior distribution.

We considered a gamma prior distribution with the following pdf:

$$P(\beta_i, a_i, b_i) \propto \beta_i^{a_i - 1} exp(-b_i \beta_i),$$
$$P(\alpha, a_4, b_4) \propto \alpha^{a_4 - 1} \exp(-b_4 \alpha), \qquad (40)$$

where $i = 1, 2, 3, 4$ and a_i, b_i, a_4, b_4 are the hyperparameters. The posterior pdf has the following form:

$$P(\underline{\theta} \mid \underline{X}) \propto \prod_{i \in I_1 \bigcup I_2} \alpha^3 (\beta_1 + \beta_4) \beta_2 \beta_3 (1 + x_{1i})^{-(\alpha+1)} (1 - (1 + x_{1i})^{-\alpha})^{\beta_1 + \beta_4 - 1}$$

$$\cdot X (1 + x_{2i})^{-(\alpha+1)} (1 - (1 + x_{2i})^{-\alpha})^{\beta_2 - 1} (1 + x_{3i})^{-(\alpha+1)} X (1 - (1 + x_{3i})^{-\alpha})^{\beta_3 - 1}$$

$$\cdot X \prod_{i \in I_3 \bigcup I_4} \alpha^3 (\beta_2 + \beta_4) \beta_1 \beta_3 (1 + x_{1i})^{-(\alpha+1)} (1 - (1 + x_{1i})^{-\alpha})^{\beta_1 - 1}$$

$$\cdot X (1 + x_{2i})^{-(\alpha+1)} (1 - (1 + x_{2i})^{-\alpha})^{\beta_2 + \beta_4 - 1} (1 + x_{3i})^{-(\alpha+1)} X (1 - (1 + x_{3i})^{-\alpha})^{\beta_3 - 1}$$

$$\cdot X \prod_{i \in I_5 \bigcup I_6} \alpha^3 (\beta_3 + \beta_4) \beta_1 \beta_2 (1 + x_{1i})^{-(\alpha+1)} (1 - (1 + x_{1i})^{-\alpha})^{\beta_1 - 1}$$

$$\cdot X (1 + x_{2i})^{-(\alpha+1)} (1 - (1 + x_{2i})^{-\alpha})^{\beta_2 - 1} (1 + x_{3i})^{-(\alpha+1)} X (1 - (1 + x_{3i})^{-\alpha})^{\beta_3 + \beta_4 - 1}$$

$$\cdot X \prod_{i \in I_7} \alpha^2 (\beta_1 + \beta_2 + \beta_4) \beta_3 (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_1 + \beta_2 + \beta_4 - 1} X (1 + x_{3i})^{-(\alpha+1)} (1 - (1 + x_{3i})^{-\alpha})^{\beta_3 - 1}$$

$$\cdot X \prod_{i \in I_8} \alpha^2 (\beta_1 + \beta_3 + \beta_4) \beta_2 (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_1 + \beta_3 + \beta_4 - 1} X (1 + x_{2i})^{-(\alpha+1)} (1 - (1 + x_{2i})^{-\alpha})^{\beta_2 - 1}$$

$$\cdot X \prod_{i \in I_9} \alpha^2 (\beta_2 + \beta_3 + \beta_4) \beta_1 (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_2 + \beta_3 + \beta_4 - 1} X (1 + x_{1i})^{-(\alpha+1)} (1 - (1 + x_{1i})^{-\alpha})^{\beta_1 - 1}$$

$$\cdot X \prod_{i \in I_{10}} \alpha S (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_1 + \beta_2 + \beta_3 + \beta_4 - 1} X \prod_{i \in I_{11}} \alpha^2 (\beta_1 + \beta_2) (\beta_3 + \beta_4) (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_1 + \beta_2 - 1}$$

$$\cdot X (1 + x_{3i})^{-(\alpha+1)} (1 - (1 + x_{3i})^{-\alpha})^{\beta_3 + \beta_4 - 1}$$

$$\cdot X \prod_{i \in I_{12}} \alpha^2 (\beta_1 + \beta_3) (\beta_2 + \beta_4) (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_1 + \beta_3 - 1} X (1 + x_{2i})^{-(\alpha+1)} (1 - (1 + x_{2i})^{-\alpha})^{\beta_2 + \beta_4 - 1}$$

$$\cdot X \prod_{i \in I_{13}} \alpha^2 (\beta_2 + \beta_3) (\beta_1 + \beta_4) (1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_2 + \beta_3 - 1} X (1 + x_{1i})^{-(\alpha+1)} (1 - (1 + x_{1i})^{-\alpha})^{\beta_1 + \beta_4 - 1}$$

$$\cdot X \prod_{i=1}^{4} \beta_i^{a_i - 1} \exp(-b_i \beta_i) \alpha^{a_4 - 1} \exp(-b_4 \alpha).$$

$$(41)$$

As in the bivariate case, Bayesian estimation will be obtained numerically using MCMC which is illustrated in the next section.

6. Numerical Analysis

In this section, both simulation and MCMC techniques are carried out to investigate the performance of the derived BIK and MIK distributions. The estimation is performed using both ML and Bayesian approaches. Three real datasets are analyzed in case of BIK and another one for the case of MIK.

6.1. For BIK Distribution. In this section, we perform a simulation study to get the estimates of the unknown parameters of BIK distribution. Also, three real datasets are analyzed.

6.1.1. A Simulation Study. Here, we present an algorithm to generate BIK distribution (Algorithm 1). To perform a simulation study, we first need to select initial values for the

parameters. Here, the following eight different populations are considered:

   (i) Case 1: $\alpha = 0.2, \beta_1 = 0.2, \beta_2 = 0.2, \beta_3 = 0.2$.

   (ii) Case 2: $\alpha = 0.5, \beta_1 = 0.5, \beta_2 = 0.5, \beta_3 = 0.5$.

   (iii) Case 3: $\alpha = 0.8, \beta_1 = 0.8, \beta_2 = 0.8, \beta_3 = 0.8$.

   (iv) Case 4: $\alpha = 1.5, \beta_1 = 1.2, \beta_2 = 1.3, \beta_3 = 2.5$.

   (v) Case 5: $\alpha = 2, \beta_1 = 2, \beta_2 = 2, \beta_3 = 2$.

   (vi) Case 6: $\alpha = 2, \beta_1 = 2.5, \beta_2 = 3, \beta_3 = 3.5$.

   (vii) Case 7: $\alpha = 2.5, \beta_1 = 2.5, \beta_2 = 3, \beta_3 = 2.5$.

   (viii) Case 8: $\alpha = 3.5, \beta_1 = 4, \beta_2 = 3.5, \beta_3 = 3.5$.

The parameters are selected to cover different shapes of the distribution. It can be seen from Figure 1 that

   (i) For cases 1 and 2, the surface plot of the absolutely continuous part of the joint probability density function is decreasing.

   (ii) For cases 3 to 8, the surface plot of the absolutely continuous part of the joint probability density function is increasing till it reaches the mode; then, it is decreasing.

The following algorithm is to generate (X_1, X_2) from BIK distribution.
Step 1: generate U_1, U_2, and U_3 from uniform $(0, 1)$.
Step 2: compute $T_1 = (1 - U_1^{1/\beta_1})^{1/\alpha}$, $T_2 = (1 - U_2^{1/\beta_2})^{1/\alpha}$, and $T_3 = (1 - U_3^{1/\beta_3})^{1/\alpha}$.
Step 3: define $Z_1 = 1/T_1 - 1$, $Z_2 = 1/T_2 - 1$, and $Z_3 = 1/T_3 - 1$.
Step 4: obtain $X_1 = \max(Z_1, Z_3)$, $X_2 = \max(Z_2, Z_3)$.

ALGORITHM 1: An Algorithm to Generate BIK Distribution.
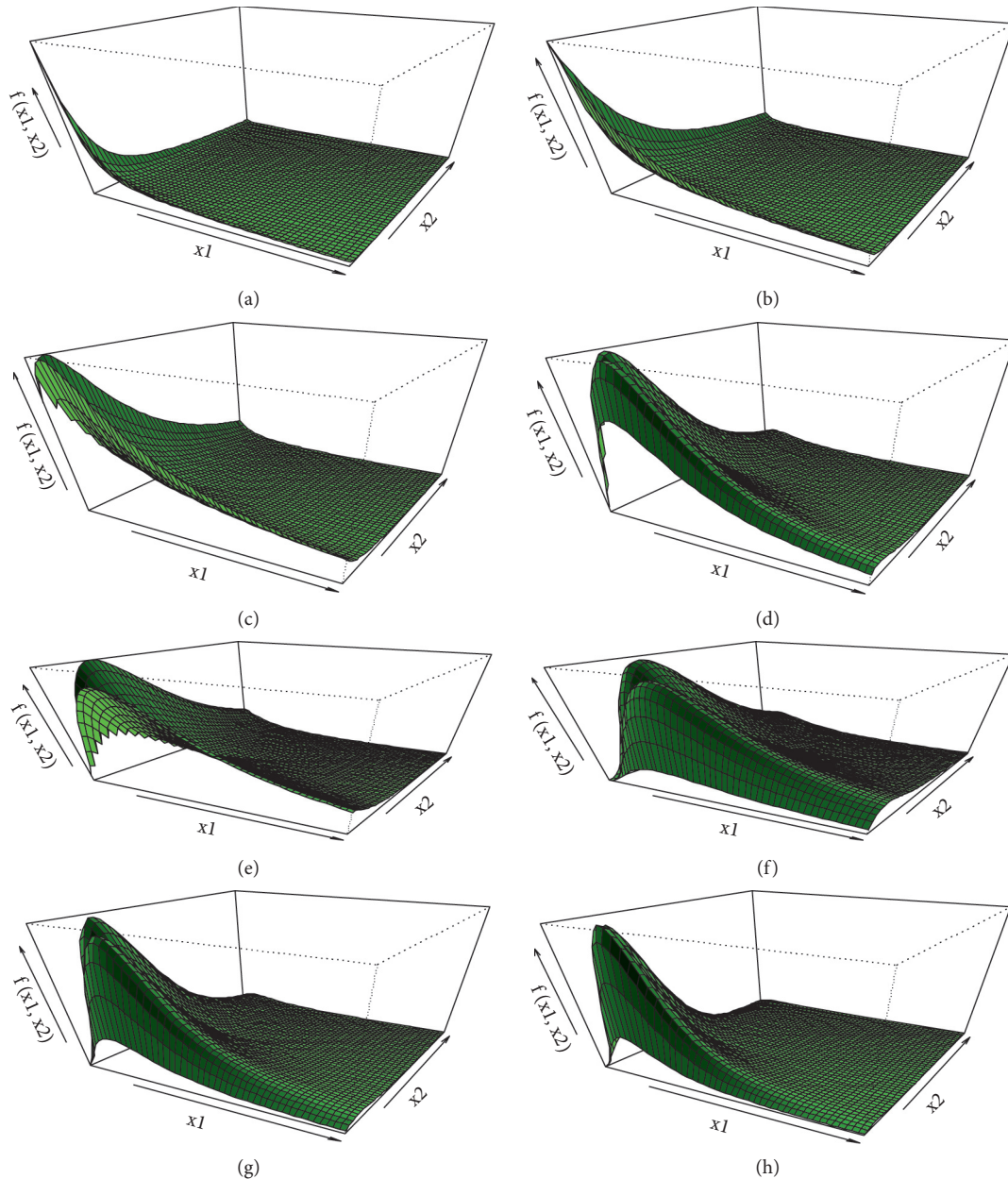
FIGURE 1: Different shapes of the absolutely continuous part of BIK pdf. (a) $\alpha = 0.2$, $\beta_1 = 0.2$, $\beta_2 = 0.2$, $\beta_3 = 0.2$; (b) $\alpha = 0.5$, $\beta_1 = 0.5$, $\beta_2 = 0.5$, $\beta_3 = 0.5$; (c) $\alpha = 0.8$, $\beta_1 = 0.8$, $\beta_2 = 0.8$, $\beta_3 = 0.8$; (d) $\alpha = 2$, $\beta_1 = 2$, $\beta_2 = 2$, $\beta_3 = 2$; (e) $\alpha = 1.5$, $\beta_1 = 1.2$, $\beta_2 = 1.3$, $\beta_3 = 2.5$; (f) $\alpha = 2$, $\beta_1 = 2.5$, $\beta_2 = 3$, $\beta_3 = 3.5$; (g) $\alpha = 2.5$, $\beta_1 = 2.5$, $\beta_2 = 3$, $\beta_3 = 2.5$; (h) $\alpha = 3.5$, $\beta_1 = 4$, $\beta_2 = 3.5$, $\beta_3 = 3.5$.

(1) Maximum Likelihood Approach. The maximum likelihood estimates of the model parameters are obtained by maximizing the log likelihood function given by equation (15). Monte Carlo simulation is performed using R package with 1000 replications and three different sample sizes $n = 30, 50$, and 70 and eight different populations.

Absolute bias (ABias), mean square error (MSE), confidence width (CW), and coverage probability (CP) are obtained

Table 1: The results of MLEs and Bayesian estimates.

			MLE				Bayesian			
			α	β_1	β_2	β_3	α	β_1	β_2	β_3
$\alpha = 0.2, \beta_1 = 0.2, \beta_2 = 0.2, \beta_3 = 0.2$	$n = 30$	ABias	0.020	0.015	0.015	0.010	0.016	0.012	0.008	0.012
		MSE	0.003	0.004	0.004	0.004	0.003	0.004	0.004	0.003
		CW (CL)	0.214	0.248	0.248	0.248	0.213	0.243	0.239	0.230
		CP	0.950	0.938	0.957	0.948	0.943	0.932	0.949	0.951
	$n = 50$	ABias	0.011	0.009	0.008	0.007	0.010	0.009	0.004	0.009
		MSE	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
		CW (CL)	0.176	0.176	0.176	0.176	0.162	0.188	0.185	0.179
		CP	0.954	0.940	0.938	0.951	0.956	0.943	0.949	0.942
	$n = 70$	ABias	0.010	0.006	0.006	0.006	0.008	0.006	0.004	0.007
		MSE	0.001	0.002	0.002	0.001	0.001	0.002	0.002	0.001
		CW (CL)	0.124	0.176	0.176	0.124	0.136	0.157	0.156	0.143
		CP	0.952	0.957	0.946	0.951	0.958	0.945	0.949	0.948
$\alpha = 0.5, \beta_1 = 0.5, \beta_2 = 0.5, \beta_3 = 0.5$	$n = 30$	ABias	0.032	0.048	0.046	0.032	0.028	0.041	0.027	0.038
		MSE	0.011	0.034	0.034	0.026	0.011	0.031	0.030	0.025
		CW (CL)	0.392	0.702	0.702	0.620	0.394	0.664	0.659	0.587
		CP	0.947	0.945	0.956	0.951	0.942	0.930	0.945	0.951
	$n = 50$	ABias	0.018	0.028	0.026	0.022	0.020	0.027	0.026	0.025
		MSE	0.006	0.018	0.018	0.014	0.006	0.017	0.018	0.014
		CW (CL)	0.304	0.512	0.512	0.464	0.299	0.501	0.504	0.448
		CP	0.948	0.944	0.945	0.957	0.954	0.949	0.946	0.937
	$n = 70$	ABias	0.017	0.019	0.021	0.019	0.013	0.021	0.017	0.019
		MSE	0.004	0.012	0.012	0.010	0.004	0.012	0.012	0.010
		CW (CL)	0.248	0.430	0.430	0.392	0.250	0.417	0.418	0.375
		CP	0.945	0.958	0.948	0.950	0.951	0.929	0.951	0.946
$\alpha = 0.8, \beta_1 = 0.8, \beta_2 = 0.8, \beta_3 = 0.8$	$n = 30$	ABias	0.045	0.088	0.086	0.066	0.037	0.072	0.073	0.060
		MSE	0.022	0.101	0.099	0.076	0.022	0.090	0.091	0.068
		CW (CL)	0.554	1.196	1.188	1.052	0.557	1.129	1.133	0.979
		CP	0.941	0.951	0.954	0.946	0.940	0.939	0.950	0.951
	$n = 50$	ABias	0.026	0.051	0.049	0.040	0.026	0.059	0.049	0.040
		MSE	0.013	0.052	0.051	0.040	0.013	0.052	0.050	0.038
		CW (CL)	0.430	0.868	0.868	0.764	0.425	0.854	0.850	0.744
		CP	0.950	0.944	0.941	0.956	0.956	0.948	0.948	0.940
	$n = 70$	ABias	0.024	0.036	0.039	0.035	0.019	0.037	0.030	0.030
		MSE	0.009	0.034	0.036	0.028	0.009	0.034	0.033	0.026
		CW (CL)	0.350	0.712	0.722	0.644	0.358	0.703	0.703	0.619
		CP	0.944	0.956	0.949	0.949	0.958	0.940	0.952	0.946
$\alpha = 1.5, \beta_1 = 1.2, \beta_2 = 1.3, \beta_3 = 2.5$	$n = 30$	ABias	0.068	0.142	0.158	0.307	0.023	0.120	0.162	0.137
		MSE	0.062	0.391	0.442	0.913	0.059	0.355	0.447	0.664
		CW (CL)	0.936	2.388	2.532	3.548	0.924	2.207	2.417	3.063
		CP	0.950	0.940	0.950	0.950	0.940	0.950	0.930	0.960
	$n = 50$	ABias	0.046	0.099	0.111	0.191	0.018	0.076	0.103	0.091
		MSE	0.035	0.203	0.231	0.444	0.034	0.189	0.219	0.369
		CW (CL)	0.712	1.722	1.834	2.504	0.703	1.633	1.738	2.043
		CP	0.960	0.950	0.950	0.960	0.960	0.920	0.950	0.970
	$n = 70$	ABias	0.039	0.068	0.082	0.157	0.013	0.068	0.051	0.090
		MSE	0.026	0.134	0.153	0.301	0.023	0.131	0.139	0.258
		CW (CL)	0.608	1.408	1.498	1.627	0.586	1.364	1.416	1.941
		CP	0.930	0.940	0.940	0.960	0.910	0.940	0.950	0.960

TABLE 1: Continued.

			MLE				Bayesian		
		α	β_1	β_2	β_3	α	β_1	β_2	β_3
$\alpha = 2, \beta_1 = 2, \beta_2 = 2, \beta_3 = 2$	$n = 30$								
		ABias	0.094	0.296	0.295	0.237	0.060	0.264	0.247	0.147
		MSE	0.099	0.890	0.892	0.680	0.094	0.811	0.790	0.519
		CW (CL)	1.176	3.510	3.518	3.096	1.166	3.312	3.237	2.723
		CP	0.943	0.949	0.950	0.952	0.949	0.931	0.948	0.952
	$n = 50$	ABias	0.055	0.169	0.166	0.136	0.040	0.185	0.169	0.101
		MSE	0.055	0.427	0.428	0.326	0.055	0.432	0.418	0.290
		CW (CL)	1.176	3.510	3.518	3.096	0.898	2.443	2.388	2.050
		CP	0.953	0.946	0.941	0.961	0.952	0.946	0.945	0.952
	$n = 70$	ABias	0.051	0.123	0.119	0.051	0.031	0.129	0.101	0.063
		MSE	0.040	0.281	0.285	0.226	0.038	0.277	0.263	0.195
		CW (CL)	0.754	2.022	2.030	1.804	0.753	1.979	1.948	1.695
		CP	0.942	0.953	0.945	0.942	0.961	0.942	0.949	0.942
$\alpha = 2, \beta_1 = 2.5, \beta_2 = 3, \beta_3 = 3.5$	$n = 30$	ABias	0.086	0.383	0.493	0.507	0.037	0.268	0.426	0.178
		MSE	0.089	1.832	2.583	2.476	0.219	1.497	2.411	0.908
		CW (CL)	1.122	5.088	5.996	5.840	0.843	4.543	5.563	3.387
		CP	0.950	0.950	0.950	0.950	0.942	0.950	0.880	0.938
	$n = 50$	ABias	0.053	0.226	0.279	0.297	0.019	0.197	0.304	0.095
		MSE	0.051	0.870	1.194	1.132	0.049	0.815	1.197	0.865
		CW (CL)	0.858	3.548	4.141	4.006	0.842	3.376	3.996	3.541
		CP	0.950	0.940	0.940	0.970	0.930	0.910	0.930	0.950
	$n = 70$	ABias	0.049	0.173	0.236	0.244	0.014	0.166	0.149	0.091
		MSE	0.036	0.574	0.800	0.766	0.033	0.543	0.688	0.607
		CW (CL)	0.712	2.894	3.382	3.294	0.697	2.770	3.116	2.997
		CP	0.940	0.940	0.950	0.960	0.950	0.930	0.930	0.960
$\alpha = 2.5, \beta_1 = 2.5, \beta_2 = 3, \beta_3 = 2.5$	$n = 30$	ABias	0.111	0.390	0.489	0.326	0.067	0.374	0.418	0.168
		MSE	0.140	1.535	2.191	1.234	0.134	1.435	1.938	0.891
		CW (CL)	1.402	4.610	5.476	4.164	1.395	4.375	5.016	3.545
		CP	0.943	0.950	0.954	0.947	0.946	0.903	0.915	0.955
	$n = 50$	ABias	0.068	0.232	0.279	0.183	0.041	0.236	0.262	0.108
		MSE	0.079	0.737	1.017	0.577	0.078	0.725	0.977	0.496
		CW (CL)	1.066	3.240	3.798	2.892	1.074	3.167	3.657	2.697
		CP	0.950	0.940	0.949	0.961	0.949	0.917	0.886	0.952
	$n = 70$	ABias	0.060	0.164	0.214	0.160	0.028	0.166	0.166	0.069
		MSE	0.057	0.479	0.669	0.399	0.037	0.472	0.622	0.333
		CW (CL)	0.902	2.636	3.094	2.394	0.903	2.581	2.972	2.217
		CP	0.948	0.950	0.950	0.943	0.966	0.887	0.883	0.950
$\alpha = 3.5, \beta_1 = 4, \beta_2 = 3.5, \beta_3 = 3.5$	$n = 30$	ABias	0.152	0.717	0.607	0.533	0.068	0.663	0.536	0.115
		MSE	0.257	4.574	3.552	2.888	0.237	4.168	3.127	1.682
		CW (CL)	1.896	7.898	6.994	6.322	1.873	7.403	6.356	4.974
		CP	0.938	0.949	0.949	0.954	0.836	0.928	0.955	0.956
	$n = 50$	ABias	0.091	0.414	0.352	0.289	0.053	0.438	0.357	0.133
		MSE	0.143	2.075	1.637	1.277	0.141	2.080	1.606	1.038
		CW (CL)	1.440	5.410	4.822	4.282	1.448	5.311	4.641	3.905
		CP	0.952	0.947	0.934	0.966	0.761	0.919	0.932	0.963
	$n = 70$	ABias	0.082	0.309	0.268	0.259	0.031	0.283	0.232	0.060
		MSE	0.103	1.352	1.066	0.883	0.099	1.297	1.012	0.701
		CW (CL)	1.214	4.394	3.908	3.542	1.222	4.275	3.773	3.240
		CP	0.940	0.951	0.945	0.953	0.696	0.928	0.957	0.953

and presented in Table 1. The numerical steps and the corresponding equations are explained in detail in Appendix B.1.

(2) Bayesian Approach. Using Bayesian approach under square error loss function, the Bayesian estimator is the posterior mean. However, it is hard to obtain the posterior mean theoretically as we have four parameters to estimate. One can use Markov Chain Monte Carlo (MCMC) simulation method to obtain it numerically.

The MCMC method uses simulation techniques to obtain a Markov sequence such that it has a limiting distribution. In the Bayesian approach, the limiting distribution is the posterior pdf as it includes all needed information about the parameters $\underline{\theta}$.

Here, the MCMC method can be used to set up a Markov chain of parameters $\underline{\theta}$ with distribution $P(\underline{\theta} \mid (X_1, X_2))$. The mean of the sequence can be considered as the posterior mean.

To perform MCMC, we used both R and WinBugs packages. Gamma prior is used with the same three sample sizes and eight populations used in ML approach. The R package with 1000 replications is used, and for each replication, WinBugs is used with 1000 replications to generate the sequence of Markov chain.

We used the Geweke test to examine the convergence of the generated Markov chain sequence. Geweke statistic (z_n) converges to normal distribution for large sample sizes. Hence, large absolute values of z_n are considered as a rejection for convergence. Only those converged sequences are used in the analysis. For more details about the Geweke test, see [16].

ABias, MSE, confidence length (CL), and CP are obtained and presented in Table 1. The numerical steps and the corresponding equations are explained in detail in Appendix B.1.

From Table 1, it can be seen that under different combinations of the parameters and for different sample sizes, ABias and MSE are relatively small. This indicates that both Bayesian and ML approaches work efficiently in estimating the parameters of BIK.

Comparing ML and Bayesian estimates, it is found that Bayesian estimates have less than or equal mean square error (MSE) than ML ones. This is clear from Figure 2.

Also, it can be seen that as the sample size (n) increases, the ABias, MSE, CW, and CL decrease for both ML and Bayesian as seen from Figure 3. Moreover, it can be seen that for most cases, the CP is around 0.95.

6.1.2. Real Datasets. Here, we analyze three real datasets to show the applicability of BIK in several fields like sports, engineering, and medicine.

(1) Football Data. The dataset has been obtained from Meintanis [23]; he used the bivariate MO exponential distribution (BE) to analyze the data. The data are about football (soccer) where at least one goal scored by the home team and at least one goal scored directly from a penalty kick, a foul kick, or any other direct kick (all of them together will be called as kick goal) by any team have been considered. Here, the variables are the time in minutes of the first kick goal scored by any team (X_1) and the time of the first goal of any type scored by the home team (X_2).

The bivariate dataset has the following structure: $X_1 < X_2$, $X_1 = X_2$, and $X_1 > X_2$. Since, $X_1 = X_2$ has a positive probability, we need a singular distribution to analyze this dataset. Here, we analyze the data using BIK distribution defined by equation (4). All the data points have been divided by 100. This is not going to make any difference in the statistical inference.

First, before analyzing the data, we fit inverted Kumaraswamy distribution to X_1, X_2 and max (X_1, X_2). To guess the initial values for the parameters of BIK model, the MLEs of the shape parameters (α, β) of the respective

inverted Kumaraswamy distribution for X_1, X_2 and max (X_1, X_2) are obtained. To check the model's fitness, we first need to illustrate goodness-of-fit tests.

Goodness-of-fit (GOF) tests are hypothesis tests regarding the distribution of some random variable (X) in a population. The objective of applying GOF tests is to measure how well the data agree with a given distribution as its population. For example, if we want to examine if the random variable (X) follows distribution $F_0(x)$, then the null hypothesis is

$$H_0: F(x) = F_0(x). \tag{42}$$

One approach for applying GOF tests is based on the empirical distribution function (EDF) $(F_n(x))$ which is defined as follows:

$$F_n(x) = \frac{\text{number of elements in the sample} \leq x}{\text{total sample size } (n)}. \tag{43}$$

This approach is based on defining a statistic to measure the discrepancy between $F_n(x)$ and $F_0(x)$. The most used statistics are modified Cramér–von Mises statistic (W^*) and Anderson–Darling statistic (A^*) which have the following formulas:

$$W^* = \frac{1}{12n} + \sum_{i=1}^{n} \left[F_0(x_{(i)}) - \frac{2i-1}{2n} \right]^2,$$

$$A^* = -n - 2 \sum_{i=1}^{n} \left[\frac{2i-1}{2n} \ln(F_0(x_{(i)})) \right. \tag{44}$$

$$\left. + \left(1 - \frac{2i-1}{2n} \right) \ln(1 - F_0(x_{(i)})) \right),$$

where $x_{(i)}$ is the value of the ith order statistics in the sample.

Large values of test statistics (or small corresponding p value) lead to the rejection of the null hypothesis. For more details about GOF tests, see D'Agostnio and Stephens [24].

Here, we apply GOF tests in order to see whether the fit based on univariate inverted Kumaraswamy distributions is reasonable for this dataset. We computed the modified Cramér–von Mises statistic (W^*) and Anderson–Darling statistic (A^*). The values of these statistics and the corresponding p values (in brackets) for X_1, X_2, and max (X_1, X_2) are illustrated in Table 2.

Based on the values of these statistics and the corresponding p values, the inverted Kumaraswamy distribution cannot be rejected for modeling the marginals and the maximum. In Table 3, the ML estimates and the posterior mean using gamma priors are obtained for the parameters of BIK. Also, credible interval length and confidence width are illustrated.

Now, we try to compare the performance of BIK, bivariate exponential (BE), and bivariate inverted Weibull (BIW) to fit this dataset. To select between models, several information criteria (IC) were presented; the main idea behind IC is to afford a balance between good fit and complexity of the model as follows:
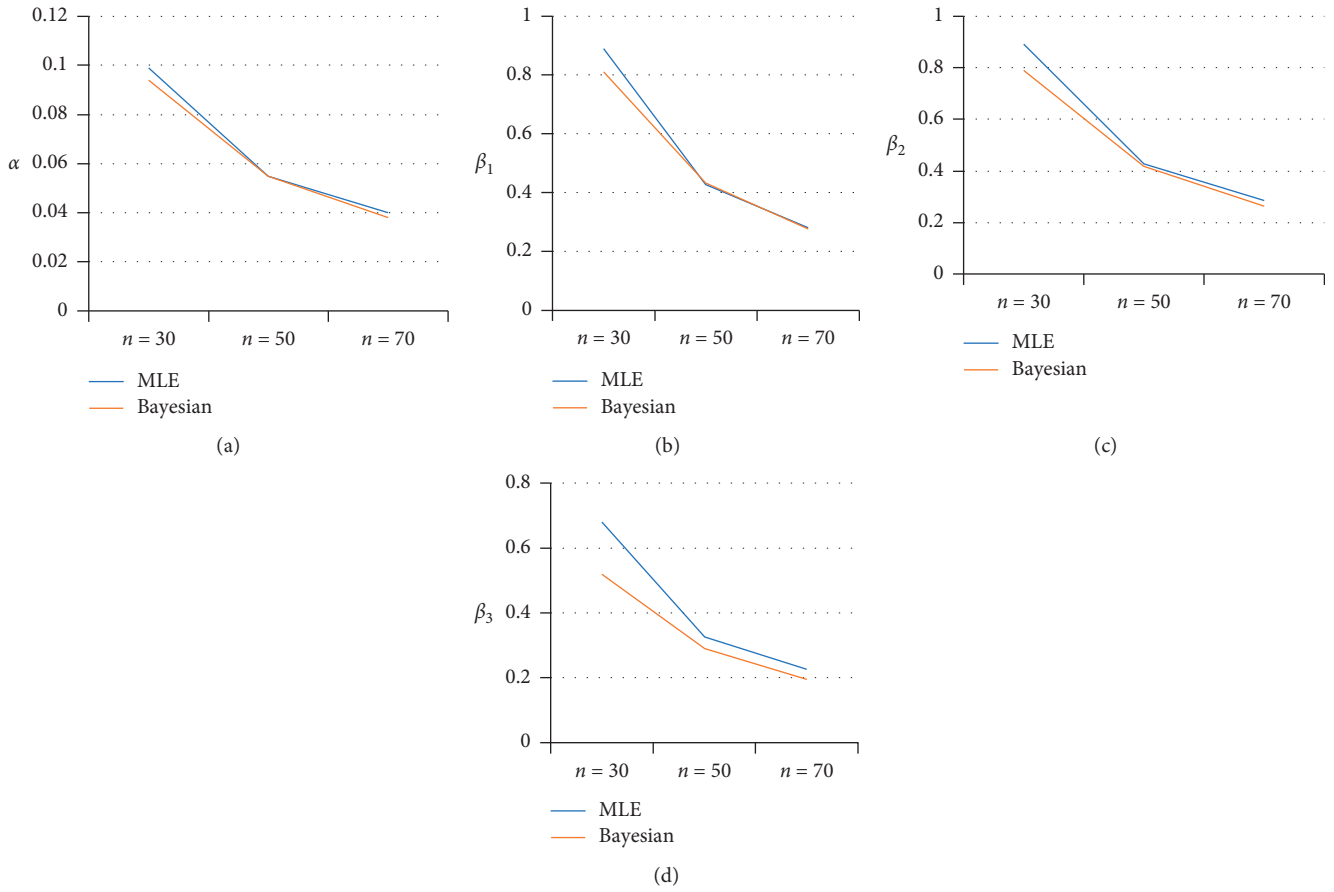
$$\text{IC} = -2 \log(L) + kp, \tag{45}$$

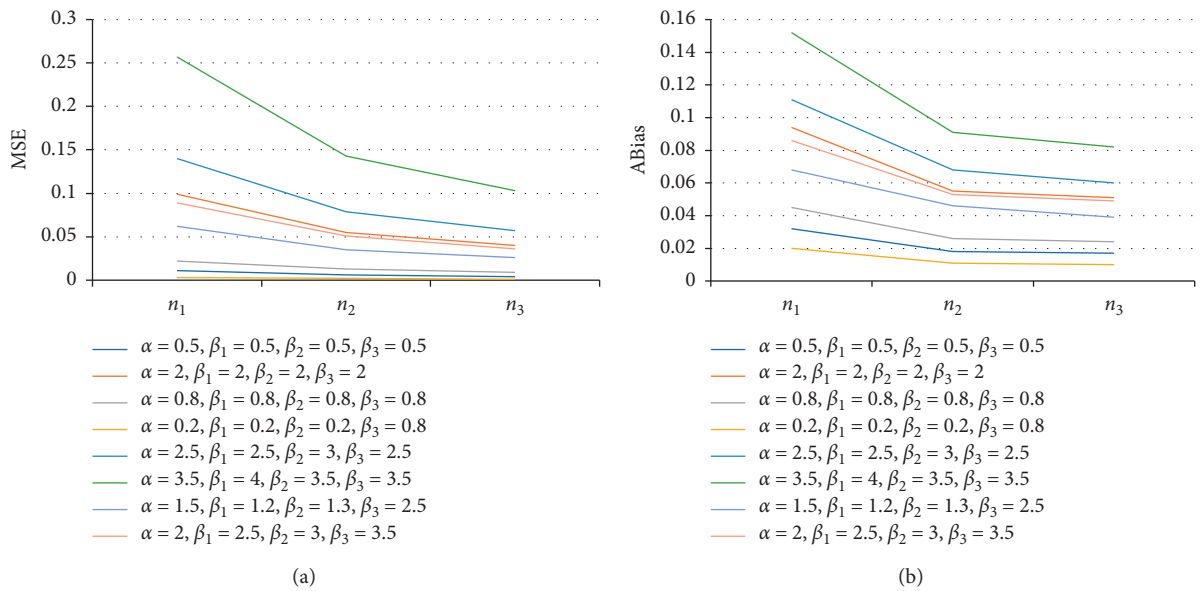Figure 2: MSE of (a) $\alpha = 2$, (b) $\beta_1 = 2$, (c) $\beta_2 = 2$, and (d) $\beta_3 = 2$.

Figure 3: (a) MSE and (b) ABias for α.

TABLE 2: Cramér–von Mises statistic (W^*) and Anderson–Darling statistic (A^*) for IK.

	X_1	X_2	$\max(X_1, X_2)$
A^*	0.966 (0.375)	0.549 (0.696)	1.395 (0.204)
W^*	0.139 (0.426)	0.083 (0.679)	0.233 (0.212)

TABLE 3: The estimates based on ML and Bayesian approaches.

Approach		α	β_1	β_2	β_3
ML	MLE	5.258	1.763	0.560	1.411
	Confidence interval width	2.824	2.026	0.886	1.374
Bayesian	Posterior mean	4.727	1.673	0.679	1.442
	Credible interval length	2.168	1.286	0.491	1.061

where k is the penalized term and p refers to number of parameters in the model.

The most commonly used IC in model selections are Akaike information criteria (AIC), Bayesian information criteria (BIC), the consistent Akaike information criteria (CAIC), and Hannan–Quinn information criteria (HQIC). Each IC has a different penalty term illustrated in the first row of Table 4.

By analyzing equation (45), we can see that the first term ($-2\log(L)$) tends to decrease when the model provides good fit. But, the second term tends to increase as the number of parameters in the model increases. The model with the lowest IC is the best (for more details about IC, see Vrieze [25]).

The log likelihood value, AIC, BIC, CAIC, and HQIC are represented in Table 4. All of the criteria suggest that BIK provides the best fit compared with BE and BIW models. This may show the importance of BIK.

(2) Motor Data. The dataset has been obtained from [26]. The data are about failure time in days for a parallel system containing two motors. The variables are time to failure for the first motor (X_1) and time to failure for the second motor (X_2). All data points have been divided by 1000. We applied GOF tests on IK, IW, and E distributions. From Table 5, based on the values of (W^*), (A^*), and the corresponding p values, only IK distribution can be used for modeling the marginals and the maximum. Hence, only BIK can be used for modeling this dataset.

ML estimates and the posterior mean using gamma priors are obtained for the parameters of BIK. Also, credible interval length and confidence width are illustrated in Table 6.

(3) Diabetic Retinopathy Data. The dataset has been obtained from [27]. The data are used by the National Eye Institute to study the effect of laser treatment on the blindness in patients with diabetic retinopathy. At the beginning of clinical trial, for each patient, one eye is randomly selected for laser treatment. The variables are time to failure for treated eye (X_1) and time to failure for untreated eye (X_2). All data points have been divided by 1000. We applied GOF on IK, IW, and E distributions. From Table 7, it can be seen that only IK distribution can be used for modeling the marginals

TABLE 4: Information criteria.

	Log (L)	AIC	BIC	CAIC	HQIC
k	—	2	$\ln(n)$	$\ln(n+1)$	$\ln(\ln(n))$
BIK	-23.53^*	55.06	61.5	61.61	57.33
BE	-44.56^{**}	95.12	99.95	100.03	96.82
BIW	-30.25^{***}	68.5	74.94	75.05	70.77

*Based on the estimates in Table 3. **Based on the estimates obtained by Meintanis [23]. ***Based on the estimates presented by Muhammed [9].

TABLE 5: (W^*) and (A^*) for IK, IW, and E.

Model		X_1	X_2	$\max(X_1, X_2)$
IK	A^*	1.133 (0.294)	0.294 (0.942)	0.488 (0.756)
	W^*	0.204 (0.261)	0.036 (0.958)	0.081 (0.692)
IW	A^*	1.491 (0.179)	0.874 (0.429)	7.165 (0.0003)
	W^*	0.267 (0.168)	0.119 (0.505)	1.509 (9.18×10^{-5})
E	A^*	3.902 (0.009)	3.363 (0.014)	94.4 (3.3×10^{-5})
	W^*	0.778 (0.007)	0.672 (0.014)	5.964 (2.2×10^{-16})

TABLE 6: The estimates based on ML and Bayesian approaches.

Approach		α	β_1	β_2	β_3
ML	MLE	16.977	12.603	13.067	0.01
	Confidence interval width	10.508	19.59	20.046	18.730
Bayesian	Posterior mean	16.950	12.900	13.210	0.034
	Credible interval length	9.020	19.152	20.720	0.594

TABLE 7: (W^*) and (A^*) for IK, IW, and E.

Model		X_1	X_2	$\max(X_1, X_2)$
IK	A^*	0.662 (0.591)	0.570 (0.676)	0.707 (0.551)
	W^*	0.092 (0.627)	0.099 (0.594)	0.090 (0.635)
IW	A^*	2.538 (0.048)	73.841 (8.4×10^{-6})	166.45 (8.4×10^{-6})
	W^*	0.369 (0.087)	11.41 (2.2×10^{-16})	18.25 (2.2×10^{-16})
E	A^*	3.886 (0.009)	214.7 (8.4×10^{-6})	271.7 (8.4×10^{-6})
	W^*	0.671 (0.015)	22.4 (2.2×10^{-16})	23.1 (2.2×10^{-16})

and the maximum. Hence, only BIK can be used for modeling this dataset.

In Table 8, ML estimates and the posterior mean using gamma priors are obtained for the parameters of BIK. Also, credible interval length and confidence width are illustrated.

From these three datasets, we can conclude that the derived BIK distribution will be of great importance.

6.2. For MIK Distribution. In this section, we present numerical results of estimation using a simulation study and a real dataset.

6.2.1. A Simulation Study. Here, we present an algorithm to generate MIK distribution. Also, we illustrate the simulation results for both ML and Bayesian approaches (Algorithm 2).

(1) Maximum Likelihood Approach. To obtain the maximum likelihood estimates, a Monte Carlo simulation is performed

TABLE 8: The estimates based on ML and Bayesian approaches.

Approach		α	β_1	β_2	β_3
ML	MLE	3.830	2.889	2.570	0.900
	Confidence interval width	1.288	2.114	1.952	1.044
Bayesian	Posterior mean	3.586	2.739	2.3796	0.907
	Credible interval length	0.930	1.326	1.133	0.700

The following algorithm is to generate (X_1, X_2, X_3) from MIK distribution.
Step 1: generate U_1, U_2, U_3, and U_4 from uniform (0, 1).
Step 2: compute $T_1 = (1 - U_1^{1/\beta_1})^{1/\alpha}$, $T_2 = (1 - U_2^{1/\beta_2})^{1/\alpha}$, $T_3 = (1 - U_3^{1/\beta_3})^{1/\alpha}$, and $T_4 = (1 - U_4^{1/\beta_4})^{1/\alpha}$.
Step 3: define $Z_1 = 1/T_1 - 1$, $Z_2 = 1/T_2 - 1$, $Z_3 = 1/T_3 - 1$, and $Z_4 = 1/T_4 - 1$.
Step 4: obtain $X_1 = \max(Z_1, Z_4)$, $X_2 = \max(Z_2, Z_4)$, and $X_3 = \max(Z_3, Z_4)$.

ALGORITHM 2: An algorithm to generate MIK distribution.

TABLE 9: The results of MLEs and Bayesian estimates for MIK.

			MLE				Bayesian			
			α	β_1	β_2	β_3	α	β_1	β_2	β_3
$\alpha = 0.9, \beta_1 = 0.9, \beta_2 = 0.9, \beta_3 = 0.9$	$n = 30$	ABias	0.046	0.063	0.050	0.061	0.045	0.037	0.020	0.032
		MSE	0.020	0.045	0.042	0.045	0.019	0.040	0.037	0.039
		CW (CL)	0.526	0.794	0.774	0.794	0.505	0.761	0.735	0.749
		CP	0.953	0.973	0.954	0.949	0.939	0.925	0.948	0.933
	$n = 50$	ABias	0.035	0.033	0.046	0.040	0.039	0.005	0.020	0.019
		MSE	0.011	0.023	0.025	0.025	0.012	0.021	0.022	0.023
		CW (CL)	0.392	0.582	0.594	0.594	0.391	0.567	0.573	0.578
		CP	0.960	0.968	0.950	0.960	0.945	0.946	0.929	0.943
	$n = 70$	ABias	0.020	0.026	0.021	0.032	0.029	0.001	0.005	0.006
		MSE	0.007	0.017	0.015	0.017	0.008	0.015	0.015	0.015
		CW (CL)	0.328	0.496	0.480	0.496	0.328	0.477	0.472	0.479
		CP	0.945	0.951	0.958	0.964	0.930	0.931	0.933	0.950
$\alpha = 1.8, \beta_1 = 0.8, \beta_2 = 1.8, \beta_3 = 0.8$	$n = 30$	ABias	0.135	0.083	0.224	0.089	0.119	0.043	0.199	0.046
		MSE	0.084	0.044	0.282	0.046	0.078	0.033	0.248	0.034
		CW (CL)	1.008	0.754	1.888	0.764	0.977	0.685	1.756	0.687
		CP	0.918	0.961	0.973	0.952	0.927	0.940	0.948	0.935
	$n = 50$	ABias	0.111	0.067	0.155	0.062	0.085	0.015	0.167	0.022
		MSE	0.051	0.025	0.148	0.025	0.044	0.018	0.148	0.018
		CW (CL)	0.774	0.568	1.380	0.568	0.749	0.511	1.335	0.516
		CP	0.939	0.962	0.950	0.958	0.935	0.956	0.925	0.938
	$n = 70$	ABias	0.086	0.057	0.155	0.049	0.072	0.011	0.120	0.015
		MSE	0.034	0.017	0.113	0.016	0.031	0.012	0.094	0.013
		CW (CL)	0.644	0.464	1.170	0.464	0.628	0.432	1.019	0.433
		CP	0.914	0.969	0.930	0.977	0.919	0.951	0.928	0.948
$\alpha = 1.8, \beta_1 = 1.8, \beta_2 = 1.8, \beta_3 = 1.8$	$n = 30$	ABias	0.112	0.210	0.173	0.219	0.090	0.195	0.169	0.167
		MSE	0.064	0.262	0.236	0.269	0.060	0.245	0.229	0.224
		CW (CL)	0.886	1.830	1.780	1.842	0.884	1.763	1.725	1.718
		CP	0.921	0.962	0.972	0.962	0.917	0.942	0.953	0.934
	$n = 50$	ABias	0.081	0.137	0.150	0.147	0.074	0.130	0.162	0.115
		MSE	0.037	0.134	0.141	0.139	0.036	0.128	0.143	0.123
		CW (CL)	0.678	1.330	1.346	1.340	0.677	1.299	1.324	1.288
		CP	0.951	0.972	0.944	0.965	0.944	0.964	0.922	0.952
	$n = 70$	ABias	0.059	0.113	0.116	0.122	0.054	0.097	0.128	0.091
		MSE	0.024	0.092	0.093	0.095	0.024	0.086	0.095	0.084
		CW (CL)	0.568	1.102	1.108	1.368	0.562	1.075	1.090	1.074
		CP	0.932	0.952	0.949	0.959	0.941	0.941	0.928	0.946

			MLE				Bayesian		
			α	β_1	β_2	β_3	α	β_1	β_2	β_3
$\alpha = 2, \beta_1 = 1.5, \beta_2 = 2, \beta_3 = 2$	$n = 30$	ABias	0.119	0.159	0.208	0.273	0.084	0.130	0.189	0.196
		MSE	0.076	0.179	0.310	0.367	0.066	0.143	0.280	0.286
		CW (CL)	0.976	1.478	2.026	2.118	0.940	1.371	1.900	1.924
		CP	0.923	0.964	0.979	0.964	0.923	0.945	0.952	0.945
	$n = 50$	ABias	0.092	0.103	0.173	0.176	0.075	0.092	0.157	0.124
		MSE	0.044	0.086	0.181	0.067	0.041	0.080	0.165	0.154
		CW (CL)	0.744	1.074	1.524	1.524	0.726	1.036	1.444	1.435
		CP	0.948	0.969	0.944	0.965	0.945	0.963	0.927	0.954
	$n = 70$	ABias	0.065	0.086	0.122	0.135	0.045	0.062	0.104	0.097
		MSE	0.029	0.055	0.115	0.119	0.027	0.053	0.105	0.104
		CW (CL)	0.620	0.894	1.240	1.256	0.606	0.856	1.188	1.189
		CP	0.927	0.950	0.957	0.967	0.933	0.945	0.950	0.950
$\alpha = 2.4, \beta_1 = 2.4, \beta_2 = 2.4, \beta_3 = 2.4$	$n = 30$	ABias	0.284	0.245	0.298	0.133	0.106	0.264	0.240	0.209
		MSE	0.494	0.456	0.508	0.098	0.093	0.463	0.441	0.410
		CW (CL)	2.520	2.466	2.538	1.108	1.110	2.424	2.366	2.340
		CP	0.967	0.973	0.967	0.934	0.943	0.948	0.954	0.950
	$n = 50$	ABias	0.184	0.204	0.198	0.095	0.070	0.156	0.200	0.132
		MSE	0.251	0.254	0.260	0.056	0.052	0.232	0.257	0.219
		CW (CL)	1.826	1.856	1.842	0.850	0.846	1.779	1.804	1.751
		CP	0.975	0.952	0.964	0.952	0.955	0.966	0.925	0.955
	$n = 70$	ABias	0.146	0.136	0.157	0.070	0.059	0.149	0.139	0.117
		MSE	0.170	0.166	0.176	0.038	0.037	0.168	0.165	0.156
		CW (CL)	1.514	1.508	1.524	0.712	0.713	1.491	1.481	1.469
		CP	0.944	0.952	0.962	0.939	0.938	0.928	0.950	0.956
$\alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.5, \beta_3 = 1.5$	$n = 30$	ABias	0.168	0.168	0.151	0.175	0.146	0.151	0.141	0.138
		MSE	0.135	0.173	0.163	0.177	0.124	0.155	0.149	0.149
		CW (CL)	1.282	1.492	1.466	1.498	1.240	1.408	1.388	1.387
		CP	0.931	0.961	0.979	0.963	0.919	0.943	0.955	0.950
	$n = 50$	ABias	0.125	0.120	0.119	0.125	0.096	0.102	0.102	0.094
		MSE	0.078	0.092	0.115	0.095	0.069	0.083	0.152	0.082
		CW (CL)	0.976	1.094	1.102	1.102	0.947	1.047	1.043	1.044
		CP	0.946	0.969	0.952	0.966	0.949	0.956	0.923	0.953
	$n = 70$	ABias	0.092	0.100	0.094	0.094	0.086	0.092	0.105	0.062
		MSE	0.051	0.064	0.062	0.062	0.050	0.060	0.063	0.054
		CW (CL)	0.812	0.910	0.902	0.902	0.799	0.876	0.878	0.862
		CP	0.913	0.942	0.950	0.971	0.927	0.949	0.921	0.966
$\alpha = 3, \beta_1 = 3, \beta_2 = 3, \beta_3 = 3$	$n = 30$	ABias	0.157	0.377	0.318	0.374	0.117	0.361	0.293	0.249
		MSE	0.141	0.837	0.795	0.833	0.131	0.796	0.716	0.658
		CW (CL)	1.336	3.268	3.180	3.264	1.328	3.150	3.025	2.975
		CP	0.936	0.966	0.974	0.969	0.944	0.932	0.961	0.956
	$n = 50$	ABias	0.121	0.242	0.283	0.258	0.096	0.214	0.281	0.174
		MSE	0.083	0.420	0.456	0.434	0.079	0.397	0.451	0.367
		CW (CL)	1.022	2.356	2.404	2.374	1.024	2.306	2.345	2.248
		CP	0.941	0.975	0.951	0.963	0.939	0.962	0.929	0.958
	$n = 70$	ABias	0.074	0.165	0.159	0.188	0.058	0.150	0.158	0.137
		MSE	0.052	0.269	0.266	0.281	0.051	0.259	0.264	0.249
		CW (CL)	0.850	1.928	1.924	1.944	0.855	1.893	1.892	1.876
		CP	0.937	0.951	0.961	0.963	0.943	0.938	0.948	0.956

TABLE 9: Continued.

|			MLE				Bayesian		
		α	β_1	β_2	β_3	α	β_1	β_2	β_3
$\alpha = 3, \beta_1 = 2.8, \beta_2 = 2.8, \beta_3 = 3.5$	$n = 30$								
	ABias	0.154	0.339	0.283	0.455	0.103	0.284	0.243	0.257
	MSE	0.139	0.700	0.633	1.221	0.126	0.621	0.582	0.896
	CW (CL)	1.330	2.998	2.916	3.948	1.314	2.834	2.755	3.513
	CP	0.937	0.964	0.977	0.970	0.864	0.938	0.847	0.914
	$n = 50$								
	ABias	0.121	0.218	0.252	0.324	0.087	0.169	0.233	0.210
	MSE	0.083	0.354	0.381	0.642	0.076	0.320	0.362	0.529
	CW (CL)	1.022	2.168	2.208	2.872	1.018	2.098	2.136	2.697
	CP	0.947	0.973	0.955	0.965	0.805	0.974	0.797	0.857
	$n = 70$								
	ABias	0.081	0.156	0.148	0.238	0.053	0.121	0.133	0.137
	MSE	0.054	0.230	0.227	0.414	0.050	0.211	0.217	0.344
	CW (CL)	0.850	1.780	1.774	2.342	0.849	1.726	1.733	2.220
	CP	0.933	0.951	0.965	0.963	0.800	0.947	0.706	0.839
$\alpha = 3, \beta_1 = 3, \beta_2 = 4, \beta_3 = 3$	$n = 30$								
	ABias	0.154	0.374	0.477	0.370	0.109	0.327	0.455	0.230
	MSE	0.136	0.827	1.558	0.822	0.120	0.726	1.441	0.617
	CW (CL)	1.312	3.250	4.520	3.244	1.284	3.048	4.307	2.914
	CP	0.937	0.963	0.973	0.971	0.941	0.951	0.970	0.963
	$n = 50$								
	ABias	0.116	0.232	0.406	0.256	0.091	0.206	0.388	0.171
	MSE	0.079	0.409	0.915	0.430	0.073	0.380	0.851	0.354
	CW (CL)	1.008	2.336	3.394	2.366	0.989	2.254	3.254	2.213
	CP	0.942	0.973	0.960	0.965	0.942	0.953	0.965	0.952
	$n = 70$								
	ABias	0.074	0.163	0.234	0.182	0.058	0.156	0.228	0.123
	MSE	0.051	0.266	0.530	0.276	0.049	0.257	0.513	0.241
	CW (CL)	0.840	1.916	2.702	1.932	0.834	1.869	2.636	1.847
	CP	0.943	0.956	0.960	0.960	0.955	0.965	0.967	0.958
$\alpha = 3.5, \beta_1 = 3.5, \beta_2 = 3.5, \beta_3 = 4$	$n = 30$								
	ABias	0.171	0.450	0.378	0.536	0.100	0.402	0.389	0.238
	MSE	0.176	1.190	1.074	1.666	0.157	1.095	1.071	1.149
	CW (CL)	1.502	3.894	3.782	4.604	1.486	3.702	3.660	4.027
	CP	0.939	0.963	0.975	0.969	0.949	0.952	0.931	0.937
	$n = 50$								
	ABias	0.132	0.281	0.330	0.371	0.080	0.253	0.268	0.166
	MSE	0.104	0.587	0.639	0.859	0.093	0.562	0.573	0.653
	CW (CL)	1.156	2.794	2.854	3.328	1.146	2.739	2.715	3.059
	CP	0.944	0.972	0.959	0.966	0.936	0.961	0.900	0.903
	$n = 70$								
	ABias	0.085	0.197	0.185	0.261	0.055	0.145	0.194	0.159
	MSE	0.068	0.381	0.372	0.545	0.065	0.351	0.378	0.465
	CW (CL)	0.968	2.292	2.280	2.708	0.967	2.230	2.258	2.583
	CP	0.940	0.952	0.963	0.970	0.950	0.959	0.871	0.864
$\alpha = 3.5, \beta_1 = 3.5, \beta_2 = 3.5, \beta_3 = 3.5$	$n = 30$								
	ABias	0.172	0.450	0.378	0.445	0.107	0.369	0.323	0.250
	MSE	0.179	1.194	1.079	1.188	0.162	1.056	0.992	0.885
	CW (CL)	1.514	3.902	3.792	3.900	1.503	3.689	3.593	3.488
	CP	0.941	0.966	0.975	0.971	0.955	0.945	0.959	0.955
	$n = 50$								
	ABias	0.133	0.284	0.331	0.308	0.100	0.263	0.308	0.199
	MSE	0.106	0.592	0.643	0.617	0.099	0.573	0.610	0.516
	CW (CL)	1.162	2.383	2.862	2.832	1.160	2.755	2.755	2.665
	CP	0.946	0.971	0.959	0.965	0.946	0.964	0.929	0.949
	$n = 70$								
	ABias	0.087	0.204	0.186	0.224	0.059	0.163	0.182	0.144
	MSE	0.069	0.387	0.375	0.398	0.066	0.362	0.374	0.346
	CW (CL)	0.968	2.302	2.286	2.312	0.978	2.252	2.259	2.220
	CP	0.937	0.954	0.963	0.963	0.941	0.951	0.956	0.956
$\alpha = 4, \beta_1 = 1.5, \beta_2 = 2, \beta_1 = 2.5$	$n = 30$								
	ABias	0.234	0.152	0.199	0.330	0.177	0.143	0.185	0.236
	MSE	0.351	0.162	0.302	0.590	0.272	0.155	0.284	0.474
	CW (CL)	1.928	1.462	2.006	2.722	1.906	1.415	1.912	2.486
	CP	0.932	0.963	0.974	0.971	0.950	0.941	0.945	0.954
	$n = 50$								
	ABias	0.178	0.100	0.167	0.231	0.141	0.092	0.155	0.167
	MSE	0.173	0.085	0.177	0.306	0.162	0.083	0.169	0.260
	CW (CL)	1.472	1.074	1.514	1.972	1.468	1.059	1.468	1.871
	CP	0.944	0.972	0.950	0.963	0.960	0.958	0.953	0.929
	$n = 70$								
	ABias	0.136	0.082	0.120	0.193	0.121	0.073	0.137	0.126
	MSE	0.117	0.059	0.113	0.209	0.116	0.057	0.120	0.177
	CW (CL)	1.234	0.894	1.234	1.626	1.241	0.883	1.236	1.559
	CP	0.929	0.950	0.962	0.971	0.943	0.939	0.946	0.935
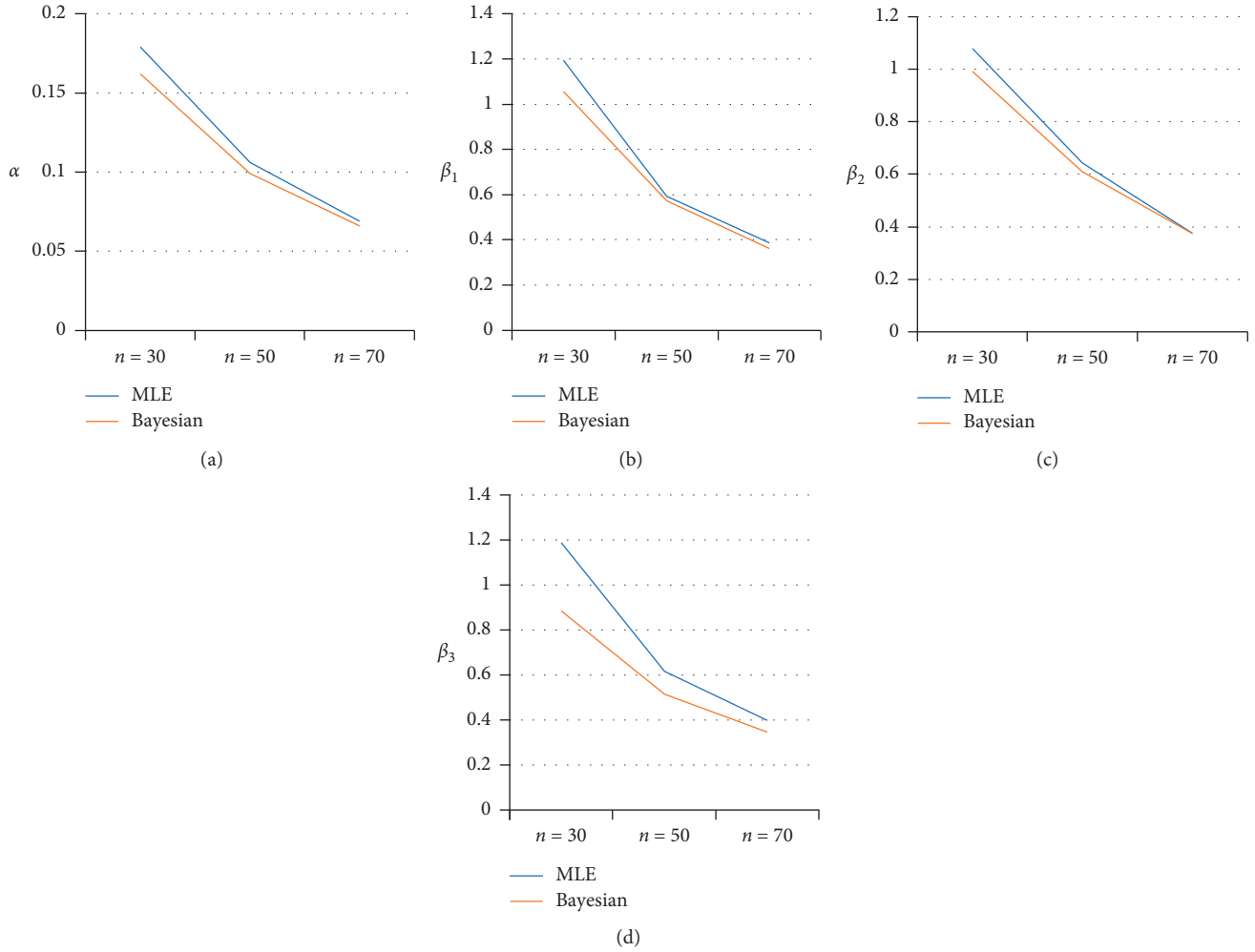
Figure 4: MSE of (a) $\alpha = 3.5$, (b) $\beta_1 = 3.5$, (c) $\beta_2 = 3.5$, and (d) $\beta_3 = 3.5$.

using R package with 1000 replications and three different sample sizes $n = 30$, 50, and 70. Different initial values of the parameters are arbitrarily chosen, varying between small and large values to cover different cases of the distribution. The parameter β_4 is assumed known for simplicity. The following twelve populations are considered.

(i) $\alpha = 0.9, \beta_1 = 0.9, \beta_2 = 0.9, \beta_3 = 0.9$.

(ii) $\alpha = 1.8, \beta_1 = 0.8, \beta_2 = 1.8, \beta_3 = 0.8$.

(iii) $\alpha = 1.8, \beta_1 = 1.8, \beta_2 = 1.8, \beta_3 = 1.8$.

(iv) $\alpha = 2, \beta_1 = 1.5, \beta_2 = 2, \beta_3 = 2$.

(v) $\alpha = 2.4, \beta_1 = 2.4, \beta_2 = 2.4, \beta_3 = 2.4$.

(vi) $\alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.5, \beta_3 = 1.5$.

(vii) $\alpha = 3, \beta_1 = 3, \beta_2 = 3, \beta_3 = 3$.

(viii) $\alpha = 3, \beta_1 = 2.8, \beta_2 = 2.8, \beta_3 = 3.5$.

(ix) $\alpha = 3, \beta_1 = 3, \beta_2 = 4, \beta_3 = 3$.

(x) $\alpha = 3.5, \beta_1 = 3.5, \beta_2 = 3.5, \beta_3 = 4$.

(xi) $\alpha = 3.5, \beta_1 = 3.5, \beta_2 = 3.5, \beta_3 = 3.5$.

(xii) $\alpha = 4, \beta_1 = 1.5, \beta_2 = 2, \beta_3 = 2.5$.

To check the behavior of the estimates, ABias, MSE, CW, and CP are computed in Table 9. The algorithm is explained in detail in Appendix B.2.

(2) Bayesian Approach. Similar to the bivariate case, MCMC simulation is used to obtain the posterior mean numerically. Absolute bias (ABias), mean square error (MSE), confidence length (CL), and coverage probability (CP) are obtained and presented in Table 9. The algorithm is explained in detail in Appendix B.2.

From Table 9, it can be seen that for different combinations of the parameters and for different sample sizes, ABias and MSE are relatively small. This indicates that both Bayesian and ML approaches work efficiently in estimating the parameters of MIK.

Comparing ML and Bayesian estimates, it is found that Bayesian estimates have less than or equal mean square error (MSE) than ML ones as seen from Figure 4. Also, for the majority of cases, Bayesian estimates have smaller ABias than ML ones.

Also, it can be seen that as the sample size (n) increases, the ABias, MSE, CW, and CL decrease for both ML and
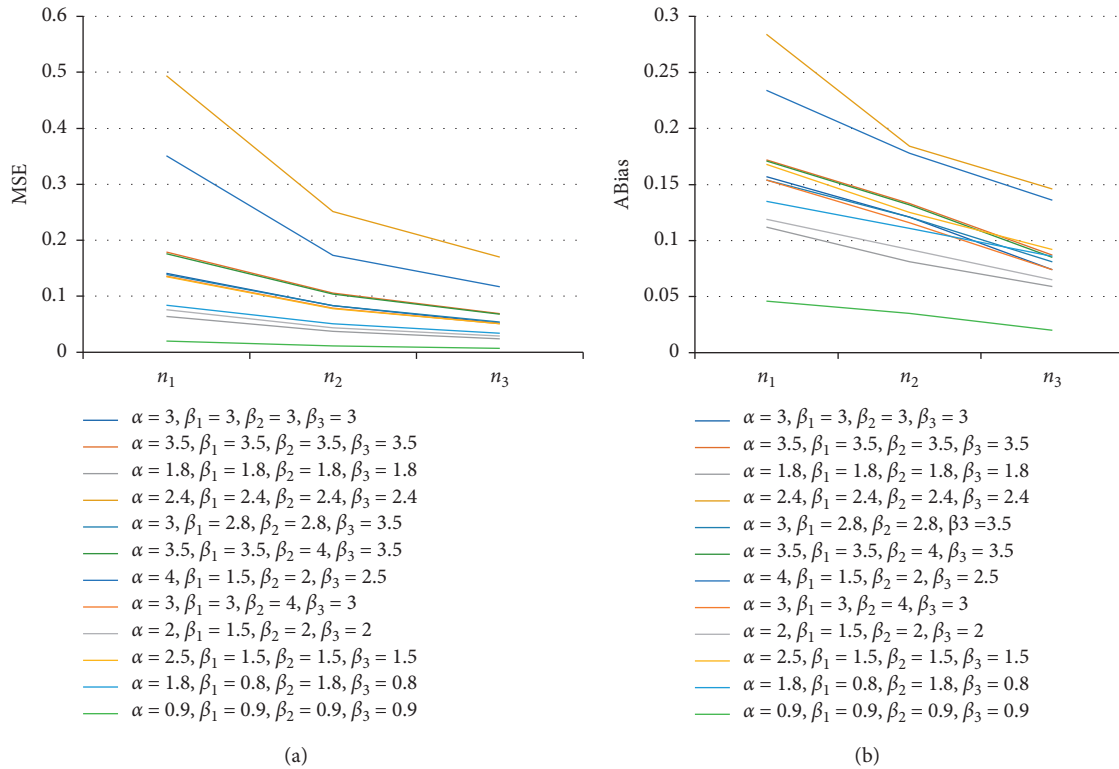
(a)

(b)

FIGURE 5: (a) MSE and (b) ABias for α.

TABLE 10: (W^*) and (A^*) for IK, IW, and E.

Model		X_1	X_2	X_3	$\max(X_1, X_2)$
IK	A^*	0.31 (0.93)	0.43 (0.82)	0.49 (0.76)	0.50 (0.75)
	W^*	0.04 (0.94)	0.07 (0.76)	0.08 (0.68)	0.08 (0.72)
IW	A^*	13.60 (7×10^{-6})	14.48 (7×10^{-6})	15.80 (7×10^{-6})	14.20 (7×10^{-6})
	W^*	2.79 (2×10^{-7})	2.99 (5×10^{-8})	3.30 (9×10^{-9})	2.91 (7×10^{-8})
E	A^*	23.95 (7×10^{-6})	24.18 (7×10^{-6})	24.75 (7×10^{-6})	24.29 (7×10^{-6})
	W^*	5.06 (2×10^{-16})	5.12 (2×10^{-16})	5.26 (2×10^{-16})	5.14 (2×10^{-16})

TABLE 11: The estimates based on ML and Bayesian approaches.

Approach		α	β_1	β_2	β_3
ML	MLE	46.949	264.199	245.852	245.519
	Confidence interval width	8.126	269.684	256.424	263.866
Bayesian	Posterior mean	48.697	270.528	226.877	257.231
	Credible interval length	4.270	69.215	61.110	64.305

Bayesian as seen from Figure 5. Moreover, it can be seen that for most cases, the CP is around 0.95.

6.2.2. A Real Dataset. Here, we analyze a real dataset to show the applicability of MIK. The dataset has been obtained from Bland and Altman [28]. It represents a set of systolic blood pressure measurement for 85 patients made by a semiautomatic blood pressure monitor; three readings were made for each patient. The variables are as follows:

X_1: first systolic blood pressure measurement.

X_2: second systolic blood pressure measurement.

X_3: third systolic blood pressure measurement.

All data points have been divided by 1000. This is not going to make any difference in the statistical inference. We applied GOF tests in order to check if the fit based on IK, IW, and E distributions is reasonable in this case. We computed the modified Cramér–von Mises statistic (W^*) and Anderson–Darling statistic (A^*). The values of these statistics and the corresponding p values (in brackets) for X_1, X_2, X_3, and $\max(X_1, X_2, X_3)$ are illustrated in Table 10.

Based on the values of these statistics and the corresponding p values, only IK distribution can be used for modeling the marginals and the maximum. Hence, only

MIK can be used for modeling these data. In Table 11, the ML estimates and the posterior mean using gamma priors are obtained for the parameters of MIK (β_4 is considered known for simplicity). Also, credible interval length and confidence width are illustrated.

7. Conclusion

In this paper, bivariate inverted Kumaraswamy (BIK) distribution is derived as a new member of bivariate Marshall–Olkin family. Its properties are also studied. Estimation is performed using both maximum likelihood (ML) and Bayesian approaches. To see the applicability of BIK distribution, three real datasets in different fields like sports, engineering, and biology have been analyzed. It is observed that the BIK model provides the best fit. Due to the wide applicability and great performance of the BIK model, a generalization to multivariate inverted Kumaraswamy (MIK) distribution is performed. To the best of our knowledge, estimation of multivariate Marshall–Olkin

family was not studied before. Here, estimation of MIK is performed using both ML and Bayesian approaches. A very convenient algorithm is proposed for both approaches. This algorithm could be applied for any multivariate Marshall–Olkin distributions. Finally, a real dataset has been analyzed to illustrate the applicability of MIK distribution, and it is observed that the MIK model provides good fit, while multivariate inverse Weibull and multivariate exponential distributions failed to fit this dataset.

Appendix

A

A.1. Proofs of Propositions 1 to 5

A.1.1. Proposition 1.
The first two cases $X_1 < X_2$ and $X_1 > X_2$ are easily obtained by taking $\delta^2 F_{X_1, X_2}(x_1, x_2)/\delta x_1 \delta x_2$. Now, to get $f_3(x)$, we use the fact that

$$\int_0^\infty \int_0^{x_2} f_1(x_1, x_2) dx_1 dx_2 + \int_0^\infty \int_0^{x_1} f_2(x_1, x_2) dx_2 dx_1 + \int_0^\infty f_3(x) dx = 1. \tag{A.1}$$

But $\int_0^\infty \int_0^{x_2} f_1(x_1, x_2) dx_1 dx_2 = \beta_2/\beta_1 + \beta_2 + \beta_3$, and $\int_0^\infty \int_0^{x_1} f_2(x_1, x_2) dx_2 dx_1 = \beta_1/\beta_1 + \beta_2 + \beta_3$.

Hence, we have $\int_0^\infty f_3(x) dx = \beta_3/\beta_1 + \beta_2 + \beta_3$.

Therefore,

$f_3(x) = \beta_3/\beta_1 + \beta_2 + \beta_3 f_{IK}(x, \beta_1 + \beta_2 + \beta_3, \alpha)$.

A.1.2. Proposition 2.
Let A be the following event:

$$A = \{U_1 < U_3\} \cap \{U_2 < U_3\}. \tag{A.2}$$

Then, $P(A) = \beta_3/\beta_1 + \beta_2 + \beta_3$ and $p(A) = \beta_1 + \beta_2/\beta_1 + \beta_2 + \beta_3$.

Therefore, $F_{X_1, X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2 \mid A) P(A) + P(X_1 \leq x_1, X_2 \leq x_2 \mid \bar{A}) P(\bar{A})$.

For $z = \min(x_1, x_2), P(X_1 \leq x_1, X_2 \leq x_2 \mid A) = [1 - (1 + z)^{-\alpha}]^{\beta_1 + \beta_2 + \beta_3}$ and $P(X_1 \leq x_1, X_2 \leq x_2 \mid \bar{A})$ can be obtained by subtraction.

It can be seen that $[1 - (1 + z)^{-\alpha}]^{\beta_1 + \beta_2 + \beta_3}$ is the singular part as its second partial derivative is zero when $x_1 \neq x_2$. Thus, $P(X_1 \leq x_1, X_2 \leq x_2 \mid \bar{A}) P(\bar{A})$ is the absolutely continuous part as its mixed partial derivative is a density function.

A.1.3. Proposition 3

(a) $F_{X_1, X_2}(x_1, x_2) = F_{IK}(x_1, \beta_1, \alpha) F_{IK}(x_2, \beta_2, \alpha) F_{IK}$ (z, β_3, α)

where $z = \min(x_1, x_2)$.

$$\lim_{x_2 \to 0} F_{X_1, X_2}(x_1, x_2) = (1 - (1 + x_1)^{-\alpha})^{\beta_1 + \beta_3},$$
$$\lim_{x_1 \to 0} F_{X_1, X_2}(x_1, x_2) = (1 - (1 + x_2)^{-\alpha})^{\beta_2 + \beta_3}. \tag{A.3}$$

(b) $P(\max\{X_1, X_2\} \leq x) = P(X_1 \leq x, X_2 \leq x) = P(U_1 \leq x, U_2 \leq x, U_3 \leq x) = (1 - (1 + x)^{-\alpha})^{\beta_1 + \beta_2 + \beta_3}$.

(c) Conditional distribution of X_1 given $X_2 \leq x_2$ is given by

$$F_{X_1 \mid X_2 \leq x_2}(x_1) = \begin{cases} \dfrac{F_{IK}(x_1, \beta_1 + \beta_3, \alpha) F_{IK}(x_2, \beta_2, \alpha)}{F_{IK}(x_2, \beta_2 + \beta_3, \alpha)} & \text{if } x_1 \leq x_2 \\[2ex] F_{IK}(x_1, \beta_1, \alpha) & \text{if } x_2 < x_1 \end{cases}, \tag{A.4}$$

that is,

$$F_{X_1 \mid X_2 \leq x_2}(x_1) = \begin{cases} (1 - (1 + x_1)^{-\alpha})^{\beta_1 + \beta_3} (1 - (1 + x_2)^{-\alpha})^{-\beta_3} & \text{if } x_1 \leq x_2 \\[1ex] (1 - (1 + x_1)^{-\alpha})^{\beta_1} & \text{if } x_2 < x_1. \end{cases} \tag{A.5}$$

A.1.4. Proposition 4.
Starting with

$$M(t_1, t_2) = E(e^{t_1 x_1 + t_2 x_2}) = \iint_{\text{all } x_1, x_2} f(x_1, x_2) e^{t_1 x_1 + t_2 x_2} dx_1 dx_2, \tag{A.6}$$

substituting for $f(x_1, x_2)$ by the corresponding formula, and then using change of variable technique and the following facts, the formula is derived.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \ (p+q)^n = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i}, \ B_x(\alpha, \beta) = \frac{x^\alpha}{\alpha} {}_2F_1(\alpha, 1-\beta; \alpha+1; x) \ \text{and}$$

$$\cdot \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} {}_2F_1(c, d; h; u) = B(\alpha, \beta) {}_3F_1(\alpha, c, d; h, \alpha+\beta; 1).$$

(A.7)

A.1.5. Proposition 5. Starting with

$$E(x_1^s x_2^r) = \iint_{\text{all } x_1, x_2} f(x_1, x_2) x_1^s x_2^r dx_1 dx_2, \quad r, s = 1, 2, 3, \dots,$$

(A.8)

substituting for $f(x_1, x_2)$ by corresponding formula, and then applying change of variable technique and the following facts, the formula is derived.

$$(p+q)^n = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i}, \ B_x(\alpha, \beta) = \frac{x^\alpha}{\alpha} {}_2F_2(\alpha, 1-\beta; \alpha+1; x) \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} {}_2F_1(c, d; h; u) du = B(\alpha, \beta) {}_3F_2(\alpha, c, d; h, \alpha+\beta; 1).$$

(A.9)

A.2. Maximum Likelihood Estimators for BIK

$$\frac{\partial \log L}{\partial \beta_1} = \frac{n_1}{\beta_1 + \beta_3} + \frac{n_2}{\beta_1} + \sum_{i \in I_1 \cup I_2} \log\left(1 - (1 + x_{1i})^{-\alpha}\right) + \sum_{i \in I_3} \log\left(1 - (1 + x_i)^{-\alpha}\right) = 0,$$

$$\frac{\partial \log L}{\partial \beta_2} = \frac{n_1}{\beta_2} + \frac{n_2}{\beta_2 + \beta_3} + \sum_{i \in I_1 \cup I_2} \log\left(1 - (1 + x_{2i})^{-\alpha}\right) + \sum_{i \in I_3} \log\left(1 - (1 + x_i)^{-\alpha}\right) = 0,$$

$$\frac{\partial \log L}{\partial \beta_3} = \frac{n_1}{\beta_1 + \beta_3} + \frac{n_2}{\beta_2 + \beta_3} + \frac{n_1}{\beta_3} + \sum_{i \in I_1} \log\left(1 - (1 + x_{1i})^{-\alpha}\right) + \sum_{i \in I_2} \log\left(1 - (1 + x_{2i})^{-\alpha}\right) + \sum_{i \in I_3} \log\left(1 - (1 + x_i)^{-\alpha}\right) = 0,$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{2n_1 + 2n_2 + n_3}{\alpha} - \left[\sum_{i \in I_1 \cup I_2} \log(1 + x_{1i}) + \sum_{i \in I_1 \cup I_2} \log(1 + x_{2i}) + \sum_{i \in I_3} \log(1 + x_i)\right]$$

(A.10)

$$+ (\beta_1 + \beta_3 - 1) \sum_{i \in I_1} \frac{(1 + x_{1i})^{-\alpha} \log(1 + x_{1i})}{1 - (1 + x_{1i})^{-\alpha}} + (\beta_2 - 1) \sum_{i \in I_1} \frac{(1 + x_{2i})^{-\alpha} \log(1 + x_{2i})}{1 - (1 + x_{2i})^{-\alpha}}$$

$$+ (\beta_1 - 1) \sum_{i \in I_2} \frac{(1 + x_{1i})^{-\alpha} \log(1 + x_{1i})}{1 - (1 + x_{1i})^{-\alpha}} + (\beta_2 + \beta_3 - 1) \sum_{i \in I_2} \frac{(1 + x_{2i})^{-\alpha} \log(1 + x_{2i})}{1 - (1 + x_{2i})^{-\alpha}}$$

$$+ (\beta_1 + \beta_2 + \beta_3 - 1) \sum_{i \in I_3} \frac{(1 + x_i)^{-\alpha} \log(1 + x_i)}{1 - (1 + x_i)^{-\alpha}} = 0.$$

The MLEs $\widehat{\underline{\theta}} = (\widehat{\alpha}, \widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3)$ of $\underline{\theta} = (\alpha, \beta_1, \beta_2, \beta_3)$ are obtained by solving the above nonlinear system of equations. The solutions are numerically obtained in Section 6.

The asymptotic variance covariance matrix is given by

$$I(\alpha, \beta_1, \beta_2, \beta_3)\big|_{\widehat{\alpha}, \widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3} = \begin{bmatrix} I_{11} & 0 & I_{13} & I_{14} \\ 0 & I_{22} & I_{23} & I_{24} \\ I_{13} & I_{23} & I_{33} & I_{34} \\ I_{14} & I_{24} & I_{34} & I_{44} \end{bmatrix},$$

(A.11)

where

$$I_{11} = \frac{-\partial^2 \log L}{\partial \beta_1^2}\bigg|_{\widehat{\beta}_1, \widehat{\beta}_2} = \frac{n_1}{(\widehat{\beta}_1 + \widehat{\beta}_2)^2} + \frac{n_2}{\widehat{\beta}_1^2},$$

$$I_{13} = \frac{-\partial^2 \log L}{\partial \beta_1 \partial \beta_3}\bigg|_{\widehat{\beta}_1, \widehat{\beta}_3} = \frac{n_1}{(\widehat{\beta}_1 + \widehat{\beta}_3)^2},$$

$$I_{14} = \frac{-\partial^2 \log L}{\partial \beta_1 \partial \alpha}\bigg|_{\widehat{\alpha}} = -\sum_{i \in I_1} \frac{(1 + x_{1i})^{-\widehat{\alpha}} \log(1 + x_{1i})}{1 - (1 + x_{1i})^{-\widehat{\alpha}}} - \sum_{i \in I_2} \frac{(1 + x_{1i})^{-\widehat{\alpha}} \log(1 + x_{1i})}{1 - (1 + x_{1i})^{-\widehat{\alpha}}} - \sum_{i \in I_3} \frac{(1 + x_i)^{-\widehat{\alpha}} \log(1 + x_i)}{1 - (1 + x_i)^{-\widehat{\alpha}}},$$

$$I_{22} = \frac{-\partial^2 \log L}{\partial \beta_2^2}\bigg|_{\widehat{\beta}_2, \widehat{\beta}_3} = \frac{n_1}{\widehat{\beta}_2^2} + \frac{n_2}{(\widehat{\beta}_2 + \widehat{\beta}_3)^2},$$

$$I_{23} = \frac{-\partial^2 \log L}{\partial \beta_2 \partial \beta_3}\bigg|_{\widehat{\beta}_2, \widehat{\beta}_3} = \frac{n_2}{(\widehat{\beta}_2 + \widehat{\beta}_3)^2}, \qquad \text{(A.12)}$$

$$I_{24} = \frac{-\partial^2 \log L}{\partial \beta_2 \partial \alpha}\bigg|_{\widehat{\alpha}} = -\sum_{i \in I_1} \frac{(1 + x_{2i})^{-\widehat{\alpha}} \log(1 + x_{2i})}{1 - (1 + x_{2i})^{-\widehat{\alpha}}} - \sum_{i \in I_2} \frac{(1 + x_{2i})^{-\widehat{\alpha}} \log(1 + x_{2i})}{1 - (1 + x_{2i})^{-\widehat{\alpha}}} - \sum_{i \in I_3} \frac{(1 + x_i)^{-\widehat{\alpha}} \log(1 + x_i)}{1 - (1 + x_i)^{-\widehat{\alpha}}},$$

$$I_{33} = \frac{-\partial^2 \log L}{\partial \beta_3^2}\bigg|_{\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3} = \frac{n_3}{\widehat{\beta}_3^2} + \frac{n_1}{(\widehat{\beta}_1 + \widehat{\beta}_3)^2} + \frac{n_2}{(\widehat{\beta}_2 + \widehat{\beta}_3)^2},$$

$$I_{34} = \frac{-\partial^2 \log L}{\partial \beta_3 \partial \alpha}\bigg|_{\widehat{\alpha}} = -\sum_{i \in I_1} \frac{(1 + x_{1i})^{-\widehat{\alpha}} \log(1 + x_{1i})}{1 - (1 + x_{1i})^{-\widehat{\alpha}}} - \sum_{i \in I_2} \frac{(1 + x_{2i})^{-\widehat{\alpha}} \log(1 + x_{2i})}{1 - (1 + x_{2i})^{-\widehat{\alpha}}} - \sum_{i \in I_3} \frac{(1 + x_i)^{-\widehat{\alpha}} \log(1 + x_i)}{1 - (1 + x_i)^{-\widehat{\alpha}}},$$

$$I_{44} = \frac{-\partial^2 \log L}{\partial \alpha^2}\bigg|_{\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3, \widehat{\alpha}} = \sum_{l=1}^{6} H_l,$$

where

$$H_1 = \frac{2n_1 + 2n_2 + n_3}{\widehat{\alpha}^2},$$

$$H_2 = (\widehat{\beta}_1 + \widehat{\beta}_3 - 1) \sum_{i \in I_1} \frac{(1 + x_{1i})^{-\widehat{\alpha}} [\log(1 + x_{1i})]^2}{\left(1 - (1 + x_{1i})^{-\widehat{\alpha}}\right)^2},$$

$$H_3 = (\widehat{\beta}_2 - 1) \sum_{i \in I_1} \frac{(1 + x_{2i})^{-\widehat{\alpha}} [\log(1 + x_{2i})]^2}{\left(1 - (1 + x_{2i})^{-\widehat{\alpha}}\right)^2},$$

$$H_4 = (\widehat{\beta}_1 - 1) \sum_{i \in I_2} \frac{(1 + x_{1i})^{-\widehat{\alpha}} [\log(1 + x_{1i})]^2}{\left(1 - (1 + x_{1i})^{-\widehat{\alpha}}\right)^2},$$

$$H_5 = (\widehat{\beta}_2 + \widehat{\beta}_3 - 1) \sum_{i \in I_2} \frac{(1 + x_{2i})^{-\widehat{\alpha}} [\log(1 + x_{2i})]^2}{\left(1 - (1 + x_{2i})^{-\widehat{\alpha}}\right)^2},$$

$$H_6 = (\widehat{\beta}_1 + \widehat{\beta}_2 + \widehat{\beta}_3 - 1) \sum_{i \in I_3} \frac{(1 + x_i)^{-\widehat{\alpha}} [\log(1 + x_i)]^2}{\left(1 - (1 + x_i)^{-\widehat{\alpha}}\right)^2}.$$

$$\text{(A.13)}$$

Using the asymptotic distribution of the MLEs, the confidence intervals can be obtained as

$$\widehat{\underline{\theta}} \pm z_{\gamma/2} \sqrt{\widehat{\text{var}}(\widehat{\underline{\theta}})}, \qquad \text{(A.14)}$$

where $\widehat{\underline{\theta}} = (\widehat{\alpha}, \widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3)$, $\widehat{\text{var}}(\widehat{\underline{\theta}})$ is the estimated variance, and $z_{\gamma/2}$ is the upper $\gamma^{th}/2$ percentile of the standard normal table.

B

B.1. Algorithms for BIK

B.1.1. Monte Carlo Simulation Algorithm for BIK. The following algorithm is used to perform Monte Carlo simulation study using R package and get ML estimates for BIK distribution.

Step 1: n independent samples (X_1, X_2) from BIK distribution are generated as follows:

(a) Generate U_1, U_2, and U_3 from uniform (0, 1).

(b) Compute $T_1 = (1 - U_1^{1/\beta_1})^{1/\alpha}$, $T_2 = (1 - U_2^{1/\beta_2})^{1/\alpha}$, and $T_3 = (1 - U_3^{1/\beta_3})^{1/\alpha}$.

(c) Define $Z_1 = 1/T_1 - 1, Z_2 = 1/T_2 - 1$, and $Z_3 = 1/T_3 - 1$.

(d) Obtain $X_1 = \max(Z_1, Z_3)$, $X_2 = \max(Z_2, Z_3)$.

Step 2: the maximum likelihood estimates (MLEs) and the corresponding variance (Var) are obtained and stored.

Step 3: lower (L) and upper (U) bounds for 95% confidence interval (CI) are calculated (i.e., $L(U) = \text{MLE} - (+)1.96 * \sqrt{\text{Var}}$) and stored.

Step 4: the three previous steps are repeated 1000 times.

Step 5: for converged datasets (this is obtained from R using optim function), the average of MLE, variance, and CI is obtained as follows:

$$
\begin{aligned}
\text{average of MLE (AMLE)} &= \frac{\sum_{i=1}^{\text{number of converged datasets}} \text{MLE}_i}{\text{number of converged datasets}}, \\
\text{average of Var (AVar)} &= \frac{\sum_{i=1}^{\text{number of converged data sets}} \text{Var}_i}{\text{number of converged datasets}}, \\
\text{average of CI (ACI)} &= \frac{\sum_{i=1}^{\text{number of converged datasets}} \text{CI}_i}{\text{number of converged datasets}}.
\end{aligned}
\tag{B.1}
$$

Step 6: absolute bias (ABias), mean square error (MSE), and confidence width (CW) are obtained as follows:

Step 7: the coverage probability (CP) is obtained as follows:

$$
\begin{aligned}
\text{ABias} &= |\text{AMLE} - \text{intial value}|, \\
\text{MSE} &= \text{AVar} + \text{Bias}^2, \\
\text{CW} &= \text{upper limit of ACI} - \text{lower limit of ACI}.
\end{aligned}
\tag{B.2}
$$

$$
\text{CP} = \frac{\text{number of times the intial value falls inside the converged samples' CI}}{\text{number of converged samples}}.
\tag{B.3}
$$

B.1.2. MCMC Simulation Algorithm for BIK. The following algorithm is used to perform Markov Chain Monte Carlo (MCMC) simulation study using R and WinBugs packages and get Bayesian estimates for BIK distribution.

Step 1: using R package: n independent samples (X_1, X_2) from BIK distribution are generated using the same procedure in Appendix B.1.1.

Step 2: the generated dataset is sent to WinBugs package and gamma priors are defined.

Step 3: WinBugs is used with 1000 replications to generate the sequence of Markov chain.

Step 4: WinBugs provides posterior mean (PM), standard deviation (SD), and credible interval (CI).

Step 5: results are sent back to R package and stored.

Step 6: the five previous steps are repeated 1000 times.

Step 7: for converged datasets (i.e., Geweke test results <1.96, it is obtained using coda function in R package), the average of posterior mean (PM), variance (Var), and credible interval (CI) is obtained as follows:

$$
\begin{aligned}
\text{average of posterior means (APM)} &= \frac{\sum_{i=1}^{\text{number of converged datasets}} \text{PM}_i}{\text{number of converged datasets}}, \\
\text{average of Var (AVar)} &= \frac{\sum_{i=1}^{\text{number of converged datasets}} (\text{SD}_i)^2}{\text{number of converged datasets}}, \\
\text{average of CI (ACI)} &= \frac{\sum_{i=1}^{\text{number of converged datasets}} \text{CI}_i}{\text{number of converged datasets}}.
\end{aligned}
\tag{B.4}
$$

Step 8: absolute bias (ABias), mean square error (MSE), and credible length (CL) are obtained as follows:

$$\text{ABias} = |\text{APM} - \text{intial value}|. \qquad \text{(B.5)}$$

MSE and CL are the same as in Appendix B.1.1.

Step 9: the coverage probability (CP) is the same as step 7 in Appendix B.1.1.

B.2. Algorithms for MIK

B.2.1. Monte Carlo Simulation Algorithm for MIK. The following algorithm is used to perform Monte Carlo simulation study using R package and get ML estimates for MIK distribution.

Step 1: n independent samples (X_1, X_2, X_3) from MIK distribution are generated as follows:

(a) Generate $U_1, U_2, U_3,$ and U_4 from uniform $(0, 1)$.

(b) Compute $T_1 = (1 - U_1^{1/\beta_1})^{1/\alpha}$, $T_2 = (1 - U_2^{1/\beta_2})^{1/\alpha}$, $T_3 = (1 - U_3^{1/\beta_3})^{1/\alpha}$, and $T_4 = (1 - U_4^{1/\beta_4})^{1/\alpha}$.

(c) Define $Z_1 = 1/T_1 - 1, Z_2 = 1/T_2 - 1, \quad Z_3 = 1/T_3 - 1,$ and $Z_4 = 1/T_4 - 1$.

(d) Obtain $X_1 = \max(Z_1, Z_4)$, $X_2 = \max(Z_2, Z_4)$, and $X_3 = \max(Z_3, Z_4)$

Steps 2 till 7 are the same as in Appendix B.1.1.

B.2.2. MCMC Simulation Algorithm for MIK. The following algorithm is used to perform Markov Chain Monte Carlo (MCMC) simulation study using R and WinBugs packages and get Bayesian estimates for MIK distribution.

Step 1: using R package: n independent samples (X_1, X_2, X_3) from MIK distribution are generated using the same procedure in Appendix B.2.1.

Steps 2 till 9 are the same as in Appendix B.1.2.

Data Availability

The datasets used in the example application are available at [23, 26–28].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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