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Research Article

Global Mittag–Leffler Stabilization of Fractional-Order BAM Neural Networks with Linear State Feedback Controllers

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In this paper, the global Mittag–Leffler stabilization of fractional-order BAM neural networks is investigated. First, a new lemma is proposed by using basic inequality to broaden the selection of Lyapunov function. Second, linear state feedback control strategies are designed to induce the stability of fractional-order BAM neural networks. Third, based on constructed Lyapunov function, generalized Gronwall-like inequality, and control strategies, several sufficient conditions for the global Mittag–Leffler stabilization of fractional-order BAM neural networks are established. Finally, a numerical simulation is given to demonstrate the effectiveness of our theoretical results.

1. Introduction

Bidirectional associative memory (BAM) neural networks are a type of extended unidirectional auto-associator of Hopfield neural networks. They are composed of two layers: the X-layer and the Y-layer, which can store and recall pattern pairs [1]. And they are widely applied in pattern recognition, signal processing, and associative memories. It is of great importance to investigate the dynamic stability behaviors of such networks to meet application requirements.

The fractional-order model is developed quickly because of its memory and genetic characteristics [2–5]. When the fractional order is in the interval $[0, 1]$, a new lemma for the Caputo fractional derivatives is proposed in [2]. Some properties of the Lyapunov direct method for noninteger order systems are presented in [3]. In [4], the authors extend Lyapunov direct method for noninteger order systems. In [5], the finite-time stability of fractional-order impulsive switched systems is considered. There are many research studies on fractional-order BAM neural networks [6–11]. A novel result about finite-time impulsive stability of fractional-order memristive BAM neural networks is obtained in [6]. Quasi-pinning synchronization and β-exponential pinning stabilization for a class of fractional-order BAM neural networks with time-varying delays and discontinuous neuron activations are considered in [8]. In [10], sufficient conditions for the existence, uniqueness, and global Mittag–Leffler stability for the solutions of the fractional difference model of BAM neural networks are provided. In [11], based on Cauchy–Schwartz inequality and Burkholder–Davis–Gundy inequality, some sufficient conditions are derived to ensure the uniform stability of stochastic fractional-order memristor fuzzy BAM neural networks.

The definitions of Mittag–Leffler stability and generalized Mittag–Leffler stability are proposed in [12, 13]. Subsequently, some investigations focus on Mittag–Leffler stability [14–21]. Global Mittag–Leffler stability and synchronization analysis of discrete fractional-order complex-valued neural networks with time delay are given in [14]. In [15], the global Mittag–Leffler stability of multiple equilibrium points for the impulsive fractional-order quaternion-valued neural networks is investigated by employing the Lyapunov method. In [16], sufficient conditions ensuring the existence, uniqueness, and global Mittag–Leffler stability of the solutions of the fractional-order coupled system on a network without strong connections are derived. In [17], a new criterion is proposed to ensure the Mittag–Leffler

stability for fractional-order neural networks in the quaternion field. The finite-time Mittag–Leffler stability for fractional-order quaternion-valued memristive neural networks with impulsive effect is investigated in [20]. Researchers not only study stability but also introduce control strategies to improve the stability. Different controllers have been applied to postpone Hopf bifurcation and broaden the stability domain on fractional-order systems [22–25]. Adaptive control approaches are adopted to induce Mittag–Leffler stabilization and synchronization for delayed fractional-order BAM neural networks in [26]. With linear and partial state feedback controls, global Mittag–Leffler stability of fractional-order BAM neural network is analyzed using Caputo fractional derivative and generalized Gronwall inequality [27]. In [28], state feedback stabilizing control and output feedback stabilizing control are designed for the stabilization of fractional-order memristive neural networks. However, the influence of whole state feedback controllers on stability of fractional-order BAM neural networks is not considered.

Motivated by the above discussion, we investigate the global Mittag–Leffler stability of fractional-order BAM neural networks with linear feedback controllers, including single and whole state feedback controllers. The main contributions include the following: First, a novel lemma is proposed using basic inequality to broaden the choice of the Lyapunov function. Second, linear state feedback controllers are designed to stabilize the systems. Third, some sufficient conditions for global Mittag–Leffler stability are given by using the fractional Lyapunov method and introducing feedback controllers. Finally, numerical simulations are performed to show the effect of different state feedback controllers on the selected system.

The paper is organized as follows. The preliminaries and the model descriptions are given in Section 2. Some sufficient conditions for global Mittag–Leffler stability of fractional-order BAM neural networks with two types of feedback controls are given in Section 3. In Section 4, a numerical simulation using the Adams-type forecast correction method is presented to illustrate the effectiveness of the theoretical results. Conclusions are given in Section 5.

2. Preliminaries and Model Description

In this section, some relevant definitions and lemmas about fractional calculus are introduced and the fractional-order BAM neural networks are described. Caputo fractional derivative is adopted.

Definition 1 (see [27]). The Caputo fractional derivative of order α for a function $f(t) \in C^n([0,+\infty), R)$ is defined by

$$ {}_t^C D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\mathrm{d}s, \tag{1} $$

where $n-1<\alpha<n$, $n \in N_+$, $\Gamma(\alpha)$ is Euler's gamma function, and $\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1}\exp(-s)\mathrm{d}t$.

Particularly, when $0<\alpha<1$,

$$ {}_t^C D_0^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\mathrm{d}s. \tag{2} $$

Definition 2 (see [27]). One-parameter Mittag–Leffler function is defined as

$$ E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{3} $$

and two-parameter Mittag–Leffler function is defined as

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{4} $$

where the real part $\mathrm{Re}(\alpha)$ of the complex number α is $\mathrm{Re}(\alpha)>0$, z and β are both complex numbers, and $\Gamma(\cdot)$ is Euler's gamma function. Obviously, $E_\alpha(z) = E_{\alpha,1}(z)$, $E_{0,1}(z) = (1/(1-z))$, and $E_{1,1}(z) = \exp(z)$.

In this paper, we consider the fractional-order BAM neural networks given by the following fractional differential equations:

$$ \begin{cases} {}_t^C D_0^\alpha x_i(t) = -a_i x_i(t) + \sum_{j=1}^m b_{ij} f_j\big(y_j(t)\big) + u_i(t), & t \ge 0, \\ {}_t^C D_0^\alpha y_j(t) = -c_j y_j(t) + \sum_{i=1}^n d_{ji} g_i\big(x_i(t)\big) + v_j(t), & t \ge 0, \end{cases} \tag{5} $$

where ${}_t^C D_0^\alpha$ is the Caputo derivative of order $0<\alpha<1$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$, x_i and y_j are the neural states, a_i and $c_j>0$ are the self-inhibitions, b_{ij} and d_{ji} are the synaptic connection strengths, f_j and g_i are the activation functions satisfying $f_j(0)=0$ and $g_i(0)=0$, and $u_i(t)$ and $v_j(t)$ denote the external inputs.

To ensure the existence and uniqueness of solution of system (5), the following assumption is given.

Assumption 1. The neuron activation functions f_j and g_i satisfy Lipschitz condition with the Lipschitz constants $F_j>0$ and $G_i>0$, i.e., for $i \in \{1,2,\ldots,n\}$ and $j \in \{1,2,\ldots,m\}$, $\forall \theta_1, \theta_2 \in R$,

$$ \begin{aligned} \left|f_j(\theta_1) - f_j(\theta_2)\right| &\le F_j\left|\theta_1-\theta_2\right|, \\ \left|g_i(\theta_1) - g_i(\theta_2)\right| &\le G_i\left|\theta_1-\theta_2\right|. \end{aligned} \tag{6} $$

Now, we give the definition of the globally Mittag–Leffler stability for system (5) and some relevant lemmas.

Definition 3 (see [12]) (global Mittag–Leffler stability). Under the condition of $u_i(t)=0$ and $v_j(t)=0$ $(i=1,2,\ldots,n, j=1,2,\ldots,m)$, the zero solution of system (5) is globally Mittag–Leffler stable if there exist two positive constants $\eta_1, \eta_2>0$ such that for any trajectories $x(t)$ and $y(t)$ of system (5) with initial values x_0 and y_0, satisfying

$$\|x(t)\|_p + \|y(t)\|_p \le \{(M_1(x_0) + M_2(y_0))E_\alpha(-\eta_1 t^\alpha)\}^{\eta_2}, \quad t \ge 0, \tag{7}$$

where $\|\cdot\|_p$ is p-norm, $M_1(x_0) \ge 0$, $M_2(y_0) \ge 0$, $M_1(0) = 0$, $M_2(0) = 0$, and $E_\alpha(\cdot)$ is one-parameter Mittag–Leffler function.

Remark 1. If the equilibrium point is not at the origin, it can be shifted to the origin by coordinate transformation, so we just considered the case of zero equilibrium point.

Definition 4 (see [28]) (global Mittag–Leffler stabilization). System (5) is globally Mittag–Leffler stabilizable if there exists a suitable feedback control, such that closed-loop system (5) is globally Mittag–Leffler stable.

Lemma 1 (see [29]). *If $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n$ is a vector, where $x_i(t)$ are continuous and differentiable functions, for all $i = 1, 2, \ldots, n$, and $P \in R^{n\times n}$ is a positive definite matrix, then for general quadratic form function $x^T(t)Px(t)$, we have*

$$ {}_t^C D_0^\alpha x^T(t)Px(t) \le 2x^T(t)P\,{}_t^C D_0^\alpha x(t), \quad \forall \alpha \in (0,1). \tag{8}$$

Lemma 2 (see [27]) (generalized Gronwall-like inequality). *Let $F(t)$ be a continuous function on $[0, +\infty)$, if there exist constants $\varrho_1 > 0$ and $\varrho_2 \ge 0$ such that*

$$ {}_t^C D_0^\alpha F(t) \le -\varrho_1 F(t) + \varrho_2, \quad t \ge 0. \tag{9}$$

Then,

$$F(t) \le F(0)E_\alpha(-\varrho_1 t^\alpha) + \varrho_2 t^\alpha E_{\alpha,\alpha+1}(-\varrho_1 t^\alpha), \quad t \ge 0, \tag{10}$$

where $0 < \alpha < 1$ and $E_\alpha(\cdot)$ and $E_{\alpha,\alpha+1}(\cdot)$ are one-parameter Mittag–Leffler function and two-parameter Mittag–Leffler function, respectively.

Next, we propose a new lemma by applying basic inequality $a + b \ge 2\sqrt{ab}$, $a, b \ge 0$ to broaden the choice of the Lyapunov function for analyzing the stability.

Lemma 3. *For state vectors $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n$ and $y(t) = (y_1(t), y_2(t), \ldots, y_m(t))^T \in R^m$, if there exist $\eta_3 > 0$ and two positive definite matrices $N^1_{n\times n}$ and $N^2_{m\times m}$, and all eigenvalues of N^1 and N^2 are greater than or equal to 1, the following inequality is satisfied:*

$$\sum_{i=1}^n x_i^2(t) + \sum_{j=1}^m y_j^2(t) \le \left[x(0)^T N^1 x(0) + y(0)^T N^2 y(0)\right] E_\alpha(-2\eta_3 t^\alpha). \tag{11}$$

Then,

$$\left(\sum_{i=1}^n x_i^2(t)\right)^{1/2} + \left(\sum_{j=1}^m y_j^2(t)\right)^{1/2} \le \left\{\left[2x(0)^T N^1 x(0) + 2y(0)^T N^2 y(0)\right] E_\alpha(-2\eta_3 t^\alpha)\right\}^{1/2}, \tag{12}$$

where $0 < \alpha < 1$, $E_\alpha(\cdot)$ is one-parameter Mittag–Leffler function and $x(0)$ and $y(0)$ are the initial value.

Proof. By inequality (11) and basic inequality $a + b \ge 2\sqrt{ab}$, we have

$$\begin{aligned} 2\left(\sum_{i=1}^n x_i^2(t)\right)^{1/2} \times \left(\sum_{j=1}^m y_j^2(t)\right)^{1/2} &\le \sum_{i=1}^n x_i^2(t) + \sum_{j=1}^m y_j^2(t) \\ &\le \left[x(0)^T N^1 x(0) + y(0)^T N^2 y(0)\right] E_\alpha(-2\eta_3 t^\alpha). \end{aligned} \tag{13}$$

Combining inequalities (11) and (13), we obtain

$$\begin{aligned} &\left[\left(\sum_{i=1}^n x_i^2(t)\right)^{1/2} + \left(\sum_{j=1}^m y_j^2(t)\right)^{1/2}\right]^2 \\ &\quad = \sum_{i=1}^n x_i^2(t) + \sum_{j=1}^m y_j^2(t) + 2\left(\sum_{i=1}^n x_i^2(t)\right)^{1/2} \times \left(\sum_{j=1}^m y_j^2(t)\right)^{1/2} \\ &\quad \le \left[2x(0)^T N^1 x(0) + 2y(0)^T N^2 y(0)\right] E_\alpha(-2\eta_3 t^\alpha). \end{aligned} \tag{14}$$

Then,

$$\begin{aligned} &\left(\sum_{i=1}^n x_i^2(t)\right)^{1/2} + \left(\sum_{j=1}^m y_j^2(t)\right)^{1/2} \\ &\quad \le \left\{\left[2x(0)^T N^1 x(0) + 2y(0)^T N^2 y(0)\right] E_\alpha(-2\eta_3 t^\alpha)\right\}^{1/2}. \end{aligned} \tag{15}$$

3. Main Results

In this section, linear state feedback controls are designed and some sufficient conditions are derived to ensure global Mittag–Leffler stability of fractional-order BAM neural networks.

3.1. A Single Linear State Feedback Control. The external inputs $u_i(t)$ and $v_j(t)$ in system (5), which only depend on a single linear state feedback control, are designed as follows:

$$\begin{aligned} u_i(t) &= h_i x_i(t), \\ v_j(t) &= k_j y_j(t). \end{aligned} \tag{16}$$

Theorem 1. *If*

$$H_1 = \begin{pmatrix} A & \dfrac{C}{2} \\ \dfrac{C^T}{2} & B \end{pmatrix}, \tag{17}$$

is a $(n+m)\times(n+m)$ negative definite matrix, where $A_{n\times n} = diag\{-a_i + h_i\}$ and $B_{m\times m} = diag\{-c_j + k_j\}$ are negative definite matrices and $C = (|b_{ij}|F_j + |d_{ji}|G_i)_{n\times m}$, then system (5) is globally Mittag–Leffler stable under designed control law (16).

Proof. Since H_1 is a negative definite matrix, there exist $0 < l_{1i}, \overline{l_{1j}} < 1$ such that $\overline{H}_1$ is a negative semidefinite matrix, where $\overline{A} = \mathrm{diag}\{(-a_i + h_i)l_{1i}\}$, $\overline{B} = \mathrm{diag}\{(-c_j + k_j)\overline{l_{1j}}\}$, and

$$\overline{H}_1 = \begin{pmatrix} \overline{A} & \dfrac{C}{2} \\ \dfrac{C^T}{2} & \overline{B} \end{pmatrix}. \tag{18}$$

Consider a Lyapunov function as follows:

$$V(t) = \sum_{i=1}^{n} x_i^2(t) + \sum_{j=1}^{m} y_j^2(t). \tag{19}$$

Using Lemma 1, we have

$$\begin{aligned}
&{}_t^C D_0^\alpha V(t) \\
&\le 2\sum_{i=1}^{n} x_i(t)\left(-a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j\big(y_j(t)\big) + h_i x_i(t)\right) \\
&\quad + 2\sum_{j=1}^{m} y_j(t)\left(-c_j y_j(t) + \sum_{i=1}^{n} d_{ji} g_i(x_i(t)) + k_j y_j(t)\right) \\
&\le 2\sum_{i=1}^{n}\left(-a_i x_i^2(t) + \sum_{j=1}^{m}\left|b_{ij}\right|\left|f_j\big(y_j(t)\big)\right|\left|x_i(t)\right| + h_i x_i^2(t)\right) \\
&\quad + 2\sum_{j=1}^{m}\left(-c_j y_j^2(t) + \sum_{i=1}^{n}\left|d_{ji}\right|\left|g_i(x_i(t))\right|\left|y_j(t)\right| + k_j y_j^2(t)\right) \\
&\le 2\sum_{i=1}^{n}\left((-a_i + h_i)x_i^2(t) + \sum_{j=1}^{m}\left|b_{ij}\right|\left|F_j\right|\left|y_j(t)\right|\left|x_i(t)\right|\right) \\
&\quad + 2\sum_{j=1}^{m}\left(\left(-c_j + k_j\right)y_j^2(t) + \sum_{i=1}^{n}\left|d_{ji}\right|\left|G_i\right|\left|x_i(t)\right|\left|y_j(t)\right|\right) \\
&= 2\sum_{i=1}^{n}(-a_i + h_i)x_i^2(t) + 2\sum_{j=1}^{m}\left(-c_j + k_j\right)y_j^2(t) \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left|b_{ij}\right|\left|F_j\right| + \left|d_{ji}\right|\left|G_i\right|\right)\left|x_i(t)\right|\left|y_j(t)\right| \\
&= 2\sum_{i=1}^{n}(-a_i + h_i)l_{1i}x_i^2(t) + 2\sum_{j=1}^{m}\left(-c_j + k_j\right)\overline{l_{1j}}y_j^2(t) \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left|b_{ij}\right|\left|F_j\right| + \left|d_{ji}\right|\left|G_i\right|\right)\left|x_i(t)\right|\left|y_j(t)\right| \\
&\quad + 2\sum_{i=1}^{n}(-a_i + h_i)(1 - l_{1i})x_i^2(t) + 2\sum_{j=1}^{m}\left(-c_j + k_j\right)\left(1 - \overline{l_{1j}}\right)y_j^2(t) \\
&= 2(|x(t)|, |y(t)|)^T\overline{H}_1(|x(t)|, |y(t)|) \\
&\quad + 2\sum_{i=1}^{n}(-a_i + h_i)(1 - l_{1i})x_i^2(t) + 2\sum_{j=1}^{m}\left(-c_j + k_j\right)\left(1 - \overline{l_{1j}}\right)y_j^2(t) \\
&\le 2\sum_{i=1}^{n}(-a_i + h_i)(1 - l_{1i})x_i^2(t) + 2\sum_{j=1}^{m}\left(-c_j + k_j\right)\left(1 - \overline{l_{1j}}\right)y_j^2(t).
\end{aligned} \tag{20}$$

Let

$$\gamma_1 = \min_{1\le i\le n, 1\le j\le m}\left\{(a_i - h_i)(1 - l_{1i}), (c_j - k_j)\left(1 - \overline{l_{1j}}\right)\right\}. \tag{21}$$

According to the conditions of Theorem 1, obviously $\gamma_1 > 0$. Combining (20) and (21), we obtain

$$ {}_t^C D_0^\alpha V(t) \le -2\gamma_1 V(t). \tag{22}$$

Using Lemma 2, we obtain

$$V(t) \le V(0) E_\alpha(-2\gamma_1 t^\alpha). \tag{23}$$

That is,

$$\sum_{i=1}^n x_i^2(t) + \sum_{j=1}^m y_j^2(t) \le \left[\sum_{i=1}^n x_i^2(0) + \sum_{j=1}^m y_j^2(0)\right] E_\alpha(-2\gamma_1 t^\alpha). \tag{24}$$

According to Lemma 3, we have

$$\left(\sum_{i=1}^n x_i^2(t)\right)^{1/2} + \left(\sum_{j=1}^m y_j^2(t)\right)^{1/2} \le \left\{\left[\sum_{i=1}^n 2x_i^2(0) + \sum_{j=1}^m 2y_j^2(0)\right] E_\alpha(-2\gamma_1 t^\alpha)\right\}^{1/2}. \tag{25}$$

By Definitions 3 and 4, system (5) is globally Mittag–Leffler stable under designed control law (16).

3.2. The Whole Linear State Feedback Control. Meanwhile, we consider the following the external inputs $u_i(t)$ and $v_j(t)$ in system (5) that depend on the whole linear state feedback control:

$$\begin{aligned} u_i(t) &= \sum_{i=1}^n h_i x_i(t), \\ v_j(t) &= \sum_{j=1}^m k_j y_j(t). \end{aligned} \tag{26}$$

In the proof of Theorem 1, it is a little strict to construct a negative definite matrix. Now, we will explore the stability of system (5) by establishing a nonzero matrix with zero diagonal elements and control law (26).

Theorem 2. *If the conditions*

$$\begin{aligned} \min_{1\le i\le n, 1\le j\le m}\Bigg\{ & a_i - \frac{nh_i + \sum_{I=1}^n h_I}{2} - \lambda_{\max}(H_2), c_j \\ & - \frac{mk_j + \sum_{J=1}^m k_J}{2} - \lambda_{\max}(H_2)\Bigg\} > 0, \end{aligned} \tag{27}$$

are satisfied and H_2 is a $(n+m)\times(n+m)$ nonzero matrix with zero diagonal elements, where $C = (|b_{ij}|F_j + |d_{ji}|G_i)_{n\times m}$ and

$$H_2 = \begin{pmatrix} 0 & \frac{C}{2} \\ \frac{C^T}{2} & 0 \end{pmatrix}. \tag{28}$$

$\lambda_{\max}\{H_2\}$ is the largest eigenvalue of matrix H_2; then, system (5) is globally Mittag–Leffler stable under designed control law (26).

Proof. Since the diagonal elements of H_2 are 0, the trace of N is 0, and obviously, $\lambda_{\max}\{H_2\} > 0$.

Consider the following Lyapunov function:

$$V(t) = \sum_{i=1}^n x_i^2(t) + \sum_{j=1}^m y_j^2(t). \tag{29}$$

By Lemma 1, we obtain

$$\begin{aligned} & {}_t^C D_0^\alpha V(t) \\ & \le 2\sum_{i=1}^n x_i(t)\left(-a_i x_i(t) + \sum_{j=1}^m b_{ij} f_j(y_j(t)) + \sum_{i=1}^n h_i x_i(t)\right) \\ & \quad + 2\sum_{j=1}^m y_j(t)\left(-c_j y_j(t) + \sum_{i=1}^n d_{ji} g_i(x_i(t)) + \sum_{j=1}^m k_j y_j(t)\right) \\ & = 2\sum_{i=1}^n\left(-a_i x_i^2(t) + \sum_{j=1}^m b_{ij} f_j(y_j(t)) x_i(t) + \sum_{I=1}^n h_I x_I(t) x_i(t)\right) \\ & \quad + 2\sum_{j=1}^m\left(-c_j y_j^2(t) + \sum_{i=1}^n d_{ji} g_i(x_i(t)) y_j(t) + \sum_{J=1}^m k_J y_J(t) y_j(t)\right) \\ & \le 2\sum_{i=1}^n\left(-a_i x_i^2(t) + \sum_{j=1}^m |b_{ij}||f_j(y_j(t))||x_i(t)| + \sum_{I=1}^n h_I x_I(t) x_i(t)\right) \\ & \quad + 2\sum_{j=1}^m\left(-c_j y_j^2(t) + \sum_{i=1}^n |d_{ji}||g_i(x_i(t))||y_j(t)| + \sum_{J=1}^m k_J y_J(t) y_j(t)\right) \\ & \le 2\sum_{i=1}^n\left(-a_i x_i^2(t) + \sum_{j=1}^m |b_{ij}|F_j|y_j(t)||x_i(t)| + \sum_{I=1}^n h_I x_I(t) x_i(t)\right) \\ & \quad + 2\sum_{j=1}^m\left(-c_j y_j^2(t) + \sum_{i=1}^n |d_{ji}|G_i|x_i(t)||y_j(t)| + \sum_{J=1}^m k_J y_J(t) y_j(t)\right), \end{aligned} \tag{30}$$

due to

$$\begin{aligned} x_i(t) x_I(t) &\le \frac{1}{2} x_I^2(t) + \frac{1}{2} x_i^2(t), \\ y_j(t) y_J(t) &\le \frac{1}{2} y_J^2(t) + \frac{1}{2} y_j^2(t). \end{aligned} \tag{31}$$

Substituting (31) into (30), we have

$$
\begin{aligned}
{}_t^C D_0^\alpha V(t) &\le 2\sum_{i=1}^{n}\left(-a_i + \frac{nh_i + \sum_{I=1}^{n} h_I}{2}\right)x_i^2(t) \\
&\quad + 2\sum_{j=1}^{m}\left(-c_j + \frac{mk_j + \sum_{J=1}^{m} k_J}{2}\right)y_j^2(t) \\
&\quad + 2\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left|b_{ij}\right|F_j + \left|d_{ji}\right|G_i\right)\left|x_i(t)\right|\left|y_j(t)\right| \\
&= 2\sum_{i=1}^{n}\left(-a_i + \frac{nh_i + \sum_{I=1}^{n} h_I}{2}\right)x_i^2(t) \\
&\quad + 2\sum_{j=1}^{m}\left(-c_j + \frac{mk_j + \sum_{J=1}^{m} k_J}{2}\right)y_j^2(t) \\
&\quad + 2\left(|x(t)|, |y(t)|\right)^T H_2\left(|x(t)|, |y(t)|\right) \\
&\le 2\sum_{i=1}^{n}\left(-a_i + \frac{nh_i + \sum_{I=1}^{n} h_I}{2}\right)x_i^2(t) \\
&\quad + 2\sum_{j=1}^{m}\left(-c_j + \frac{mk_j + \sum_{J=1}^{m} k_J}{2}\right)y_j^2(t) \\
&\quad + 2\lambda_{\max}(H_2)\left(\sum_{i=1}^{n} x_i^2(t) + \sum_{j=1}^{m} y_j^2(t)\right) \\
&= 2\sum_{i=1}^{n}\left(-a_i + \frac{nh_i + \sum_{I=1}^{n} h_I}{2} + \lambda_{\max}(H_2)\right)x_i^2(t) \\
&\quad + 2\sum_{j=1}^{m}\left(-c_j + \frac{mk_j + \sum_{J=1}^{m} k_J}{2} + \lambda_{\max}(H_2)\right)y_j^2(t) \\
&\le -2\gamma_2\left(\sum_{i=1}^{n} x_i^2(t) + \sum_{j=1}^{m} y_j^2(t)\right),
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
\gamma_2 = \min_{1\le i\le n, 1\le j\le m}\Bigg\{ & a_i - \frac{nh_i + \sum_{I=1}^{n} h_I}{2} - \lambda_{\max}(H_2), c_j \\
& - \frac{mk_j + \sum_{J=1}^{m} k_J}{2} - \lambda_{\max}(H_2)\Bigg\} > 0.
\end{aligned}
\tag{33}
$$

That is,

$$
{}_t^C D_0^\alpha V(t) \le -2\gamma_2 V(t). \tag{34}
$$

Using Lemma 2, we obtain

$$
V(t) \le V(0)E_\alpha\left(-2\gamma_2 t^\alpha\right). \tag{35}
$$

That is,

$$
\sum_{i=1}^{n} x_i^2(t) + \sum_{j=1}^{m} y_j^2(t) \le \left[\sum_{i=1}^{n} x_i^2(0) + \sum_{j=1}^{m} y_j^2(0)\right]E_\alpha\left(-2\gamma_2 t^\alpha\right). \tag{36}
$$

According to Lemma 3, we have

$$
\begin{aligned}
&\left(\sum_{i=1}^{n} x_i^2(t)\right)^{1/2} + \left(\sum_{j=1}^{m} y_j^2(t)\right)^{1/2} \\
&\quad \le \left\{\left[\sum_{i=1}^{n} 2x_i^2(0) + \sum_{j=1}^{m} 2y_j^2(0)\right]E_\alpha\left(-2\gamma_2 t^\alpha\right)\right\}^{1/2}.
\end{aligned}
\tag{37}
$$

By Definitions 3 and 4, system (5) is globally Mittag–Leffler stable under designed control law (26).

Remark 2. In the proof of Theorems 1 and 2, by constructing a negative definite matrix or a nonzero matrix with zero diagonal elements, we enlarge the inequality and have ${}_t^C D_0^\alpha V(t) \le -2\gamma_i V(t)$. Then, based on Lemma 2, the condition of Lemma 3 is satisfied $\sum_{i=1}^{n} x_i^2(t) + \sum_{j=1}^{m} y_j^2(t) \le [\sum_{i=1}^{n} x_i^2(0) + \sum_{j=1}^{m} y_j^2(0)]E_\alpha(-2\gamma_i t^\alpha)$, where N^1 and N^2 of Lemma 3 are set to be unit matrices. Finally, using Lemma 3 and definition of Mittag–Leffler stability, we determine that system (5) is globally Mittag–Leffler stable under different control laws.

4. Numerical Simulation

In this section, a numerical example is given to show the effectiveness of our proposed theoretical results by Adama–Bashforth–Moulton predictor-corrector algorithm [30].

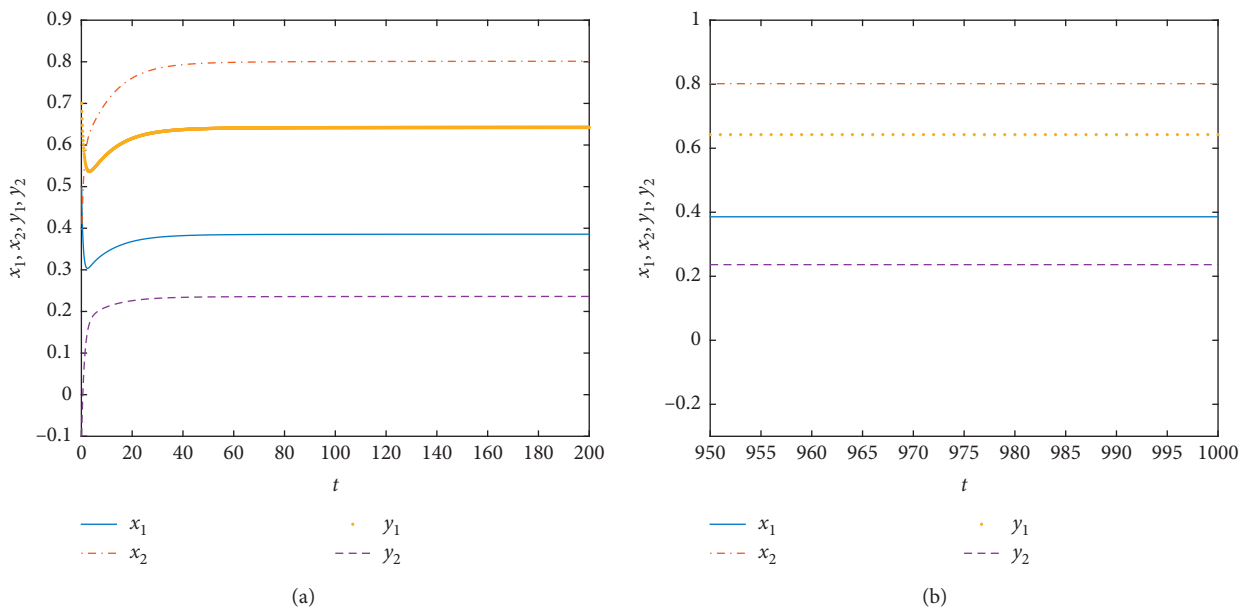
Consider the following four-dimensional fractional-order BAM neural networks:

$$
\begin{cases}
{}_t^C D_0^\alpha x_1(t) = -1.2x_1(t) + 0.5f_1(y_1(t)) + 0.6f_2(y_2(t)) + u_1(t), \\
{}_t^C D_0^\alpha x_2(t) = -0.7x_2(t) + 0.8f_1(y_1(t)) + 0.2f_2(y_2(t)) + u_2(t), \\
{}_t^C D_0^\alpha y_1(t) = -0.85y_1(t) + 0.4g_1(x_1(t)) + 0.6g_2(x_2(t)) + v_1(t), \\
{}_t^C D_0^\alpha y_2(t) = -y_2(t) + 0.1g_1(x_1(t)) + 0.3g_2(x_2(t)) + v_2(t),
\end{cases}
\tag{38}
$$

with the initial condition $(x_1(0), x_2(0), y_1(0), y_2(0))^T = (0.5, 0.4, 0.7, -0.1)^T$, where $\alpha = 0.96$, $f_j(y_j(t)) = (1/2)(|y_j(t) + 1| - |y_j(t) - 1|)$, and $g_i(x_i(t)) = \tan h(x_i(t))$, $i, j = 1, 2$.

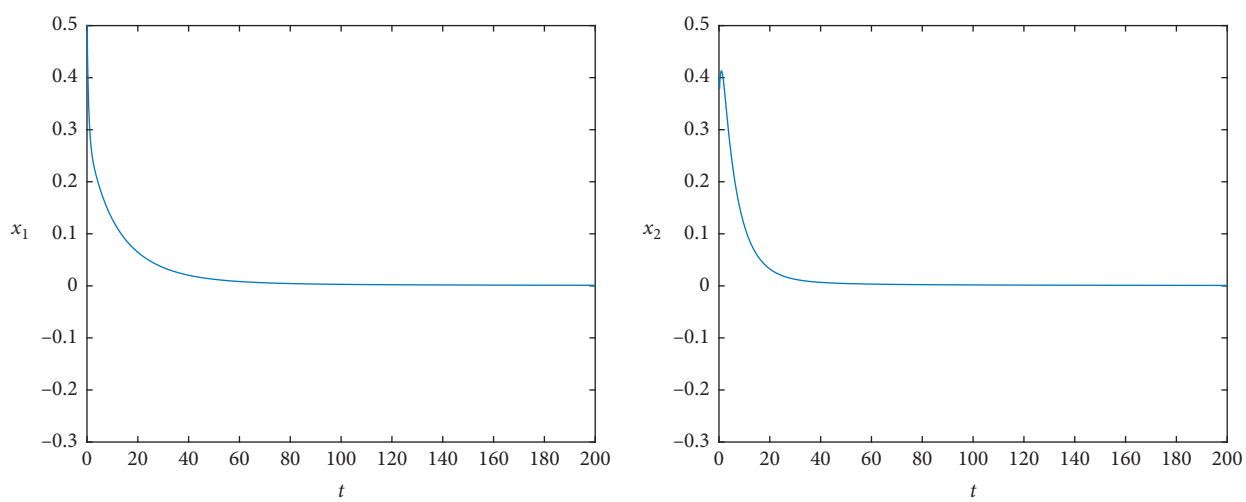
As depicted in Figure 1, the state trajectories of system (38) without external controllers $u_i(t)$ and $v_j(t)$ cannot converge to the origin.

Based on selected activation functions f_j and g_i, we have Lipschitz constants $F_j = G_i = 1$. For single state feedback controller (16), setting $u_1(t) = -0.1x_1(t)$, $u_2(t) = -0.2x_2(t)$, $v_1(t) = -0.1y_1(t)$, and $v_2(t) = -0.2y_2(t)$, we obtain a negative definite matrix H_1:



(a) (b)

Figure 1: State trajectories of system (38) without external control.



(a) (b)

Figure 2: Continued.

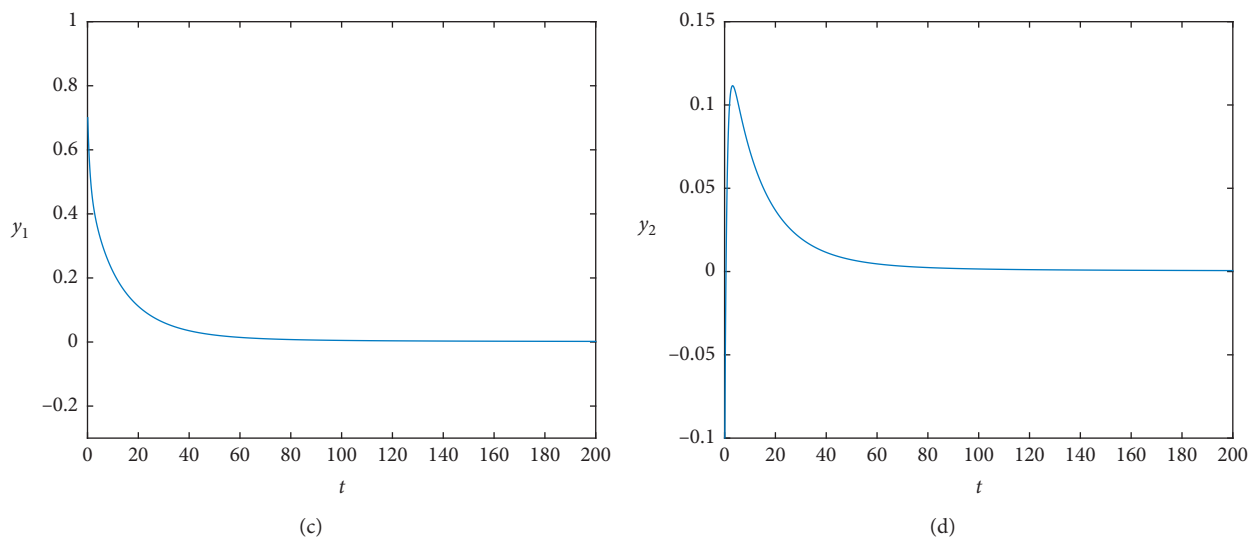


Figure 2: State trajectories of system (38) under the single state feedback laws $u_1(t) = -0.1x_1(t)$, $u_2(t) = -0.2x_2(t)$, $v_1(t) = -0.1y_1(t)$, and $v_2(t) = -0.2y_2(t)$.

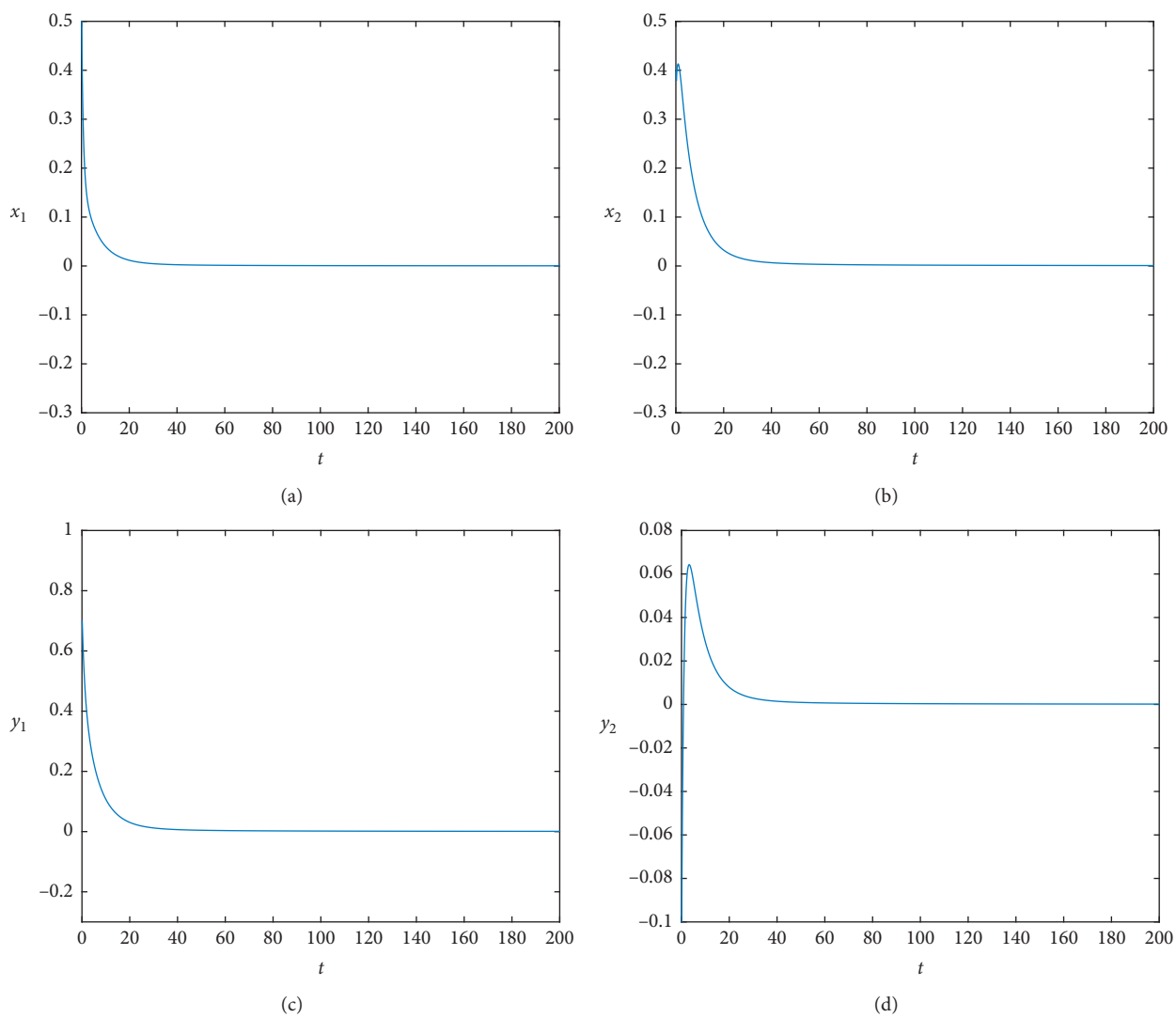


Figure 3: State trajectories of system (38) with the whole linear state feedback control laws $u_1(t) = u_2(t) = -0.2x_1(t) - 0.2x_2(t)$ and $v_1(t) = v_2(t) = -0.1y_1(t) - 0.1y_2(t)$.

$$H_1 = \begin{pmatrix} -1.3 & 0 & 0.45 & 0.35 \\ 0 & -0.9 & 0.7 & 0.25 \\ 0.45 & 0.7 & -0.95 & 0 \\ 0.35 & 0.25 & 0 & -1.2 \end{pmatrix}. \tag{39}$$

Therefore, it follows from Theorem 1 that system (38) can achieve global Mittag–Leffler stability under the designed single linear state feedback control laws $u_1(t) = -0.1x_1(t)$, $u_2(t) = -0.2x_2(t)$, $v_1(t) = -0.1y_1(t)$, and $v_2(t) = -0.2y_2(t)$. Figure 2 shows state trajectories of system (38) with the designed single linear state feedback control law. As shown in Figure 2, state trajectories of system (38) converge to the origin.

By Theorem 2, we obtain a nonzero matrix with zero diagonal elements H_2 and calculate the maximal eigenvalue of matrix $\lambda_{\max}(H_2) = 0.9258$:

$$H_2 = \begin{pmatrix} 0 & 0 & 0.45 & 0.35 \\ 0 & 0 & 0.7 & 0.25 \\ 0.45 & 0.7 & 0 & 0 \\ 0.35 & 0.25 & 0 & 0 \end{pmatrix}. \tag{40}$$

For whole linear state feedback control law (26), let $u_1(t) = u_2(t) = -0.2x_1(t) - 0.2x_2(t)$ and$v_1(t) = v_2(t) = -0.1y_1(t) - 0.1y_2(t)$, the conditions of Theorem 2 are satisfied as follows:

$$\begin{cases} 1.2 - \dfrac{2\times(-0.2)+(-0.2)+(-0.2)}{2} - \lambda_{\max}(H_2) > 0, \\ 0.7 - \dfrac{2\times(-0.2)+(-0.2)+(-0.2)}{2} - \lambda_{\max}(H_2) > 0, \\ 0.85 - \dfrac{2\times(-0.1)+(-0.1)+(-0.1)}{2} - \lambda_{\max}(H_2) > 0, \\ 1 - \dfrac{2\times(-0.1)+(-0.1)+(-0.1)}{2} - \lambda_{\max}(H_2) > 0. \end{cases} \tag{41}$$

Therefore, system (38) is globally Mittag–Leffler stable under the whole state feedback control law. Figure 3 shows state trajectories of system (38) with the designed whole linear state feedback control law.

5. Conclusion

In this paper, the global Mittag–Leffler stabilization problem for a class of fractional-order BAM neural networks is analyzed. Linear state feedback control is designed to induce the stability of fractional-order BAM neural networks, including a single linear state feedback control and the whole linear state feedback control. A novel lemma is proposed to broaden the selection of Lyapunov function. Some sufficient conditions of global Mittag–Leffler stability are given by using fractional Lyapunov stability theory, generalized Gronwall-like inequality, and the designed controls. In addition, a numerical example is given to show the influence of different state feedback controllers on the selected system. In the future research, we will try to investigate the effect of external disturbances on the stability of fractional-order BAM neural networks under linear state feedback controls. And we will explore in depth the stability of incommensurating fractional-order BAM neural networks with time delay.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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