Research Article

Two-Step Extended Sampling Method for the Inverse Acoustic Source Problem

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Recently, a new method, called the extended sampling method (ESM), was proposed for the inverse scattering problems. Similar to the classical linear sampling method (LSM), the ESM is simple to implement and fast. Compared to the LSM which uses full-aperture scattering data, the ESM only uses the scattering data of one incident wave. In this paper, we generalize the ESM for the inverse acoustic source problems. We show that the indicator function of ESM, which is defined using the approximated solutions of some linear ill-posed integral equations, is small when the support of the source is contained in the sampling disc and is large when the source is outside. This behavior is similar to the ESM for the inverse scattering problem. Numerical examples are presented to show the effectiveness of the method.

1. Introduction

Recently, a new method, called the extended sampling method (ESM), was proposed for the inverse scattering problems using a single incident wave at a fixed frequency [1]. ESM is based on an idea similar to that of the linear sampling method, which has been studied extensively in the literature [2, 3]. Although the ESM can only reconstruct the location and approximate the size of the target, it uses much less data than the classical linear sampling method, which needs full-aperture data (far-field pattern of all incident directions and all observation directions). This property makes the ESM suitable for the inverse source problems.

Inverse source problems (ISP) have many applications such as pollution source detection, optical tomography, and sound source localization [4–8]. It is known that the inverse source problem using a single frequency far-field data does not have a unique solution [6, 9]. In this paper, we consider the inverse problem to determine the location and size of an acoustic source at a fixed frequency. The ESM is generalized for the inverse acoustic source problems. We show that the indicator function of the proposed method, which is defined

using the approximated solutions of some linear ill-posed integral equations, is small when the support of the source is contained in the sampling disc and is large when the source is outside. The behavior of the indicator function is similar to the ESM for the inverse scattering problem. Numerical examples are presented to validate the effectiveness of the method.

The ESM was first proposed for the inverse scattering problem to reconstruct the obstacles [1]. To treat the inverse source problem, we introduced an artificial intermediate scattering problem, whose far-field pattern is the same as the original source problem. This enables us to generalize the ESM for the inverse source problems with little change in the numerical algorithm.

The rest of this paper is arranged as follows. In Section 2, we introduce the acoustic source problems and give a brief presentation of the ESM for the inverse scattering problems. Section 3 contains the generalization of the ESM for the inverse source problem and its mathematical justification. In particular, we show that the indicator function of the ESM is small when the support of the source is contained in the sampling disc and is large when the source is outside. Section 4

contains several validating numerical examples in two dimensions. We make some conclusions in Section 5.

2. Preliminaries

In this section, we describe the acoustic source problem and the associated inverse problem of interests. Then, we briefly introduce the ESM for the inverse acoustic scattering problem.

2.1. Acoustic Source Problem. Let $D \subset \mathbb{R}^2$ be a bounded domain and $k > 0$ be the wave number. Let $f(x) \in L^2(D)$ represent the source with compact support D . The acoustic source problem is to find a function $u \in H_{\text{loc}}^1(\mathbb{R}^2)$ satisfying the Helmholtz equation

$$\Delta u + k^2 u = f(x), \quad \text{in } \mathbb{R}^2, \quad (1)$$

and the Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} r^{1/2} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \quad r = |x|. \quad (2)$$

The above problem has a solution u given by

$$u(x) = \int_D \Phi(x, y) f(y) dy, \quad x \in \mathbb{R}^2, \quad (3)$$

where

$$\Phi(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|), \quad (4)$$

is the fundamental solution of the Helmholtz equation. Here, $H_0^{(1)}$ is the Hankel function of the first kind and order 0 given by

$$H_0^{(1)} = J_0 + iY_0, \quad (5)$$

where the real part J_0 is the Bessel function of order 0 and the imaginary part Y_0 is the Neumann function of order 0 (see page 73 of [3]).

Since the solution u to (1) is radiating, it has the following asymptotic behavior (see, e.g., Theorem 2.6 in [3]):

$$u(x) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \frac{e^{ik|x|}}{|x|^{1/2}} u_\infty(\hat{x}) + O(|x|^{-(3/2)}), \quad r = |x| \rightarrow \infty, \quad (6)$$

where $\hat{x} = x/|x| \in \mathbb{S} := \{\hat{x} \in \mathbb{R}^2 \mid |\hat{x}| = 1\}$ and $u_\infty(\hat{x})$ is called the far-field pattern of u . From the asymptotic behavior of the Hankel function $H_0^{(1)}$ and (3), we have that

$$u_\infty(\hat{x}) = \int_D e^{-ik\hat{x} \cdot y} f(y) dy, \quad \hat{x} \in \mathbb{S}. \quad (7)$$

The *inverse source problem* (ISP) considered in this paper is to find the location and size of D from the far-field pattern $u_\infty(\hat{x})$.

2.2. Extended Sampling Method. The ESM was first proposed in [1] to obtain the location and size of an obstacle using the scattering data of one incident plane wave. We first give a quick introduction for the scattering problem of a sound-soft obstacle. Note that the method works for other types of obstacles.

We still use $D \subset \mathbb{R}^2$ to denote a bounded domain occupied by the sound-soft obstacle. Let $u^i(x) := e^{ikx \cdot d}$, $x, d \in \mathbb{R}^2$ be the incident plane wave, where $|d| = 1$ is the direction. The direct exterior scattering problem is to find the scattered field u^s such that

$$\Delta u^s + k^2 u^s = 0 \text{ in } \mathbb{R}^2 \setminus \overline{D},$$

$$u^s = -u^i \text{ on } \partial D, \quad (8)$$

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0.$$

Since the scattered field u^s is radiating, it has an asymptotic expansion

$$u^s(x) = \frac{e^{i(\pi/4)}}{\sqrt{8k\pi}} \frac{e^{ikr}}{\sqrt{r}} \left\{ u_\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right) \right\}, \quad \text{as } r := |x| \rightarrow \infty, \quad (9)$$

uniformly in all directions $\hat{x} = x/|x|$. The function $u_\infty(\hat{x})$ is the far-field pattern of u^s due to the incident field u^i .

When the obstacle is a disc centered at the origin with radius R , denoted by B , the far-field pattern can be written down exactly (see, e.g., Chp. 3 of [3]):

$$u_\infty^B(\hat{x}; d) = -e^{-i(\pi/4)} \sqrt{\frac{2}{\pi k}} \left(\frac{J_0(kR)}{H_0^{(1)}(kR)} + 2 \sum_{n=1}^{\infty} \frac{J_n(kR)}{H_n^{(1)}(kR)} \cos(n\theta) \right), \quad \hat{x} \in \mathbb{S}, \quad (10)$$

where J_n is the Bessel function, $H_n^{(1)}$ is the Hankel function of the first kind of order n , and $\theta = \angle(\hat{x}, d)$ is the angle between $\hat{x}$ and d . If the disc is shifted to z , i.e.,

$$B_z := \{x + z; x \in B, z \in \mathbb{R}^2\}, \quad (11)$$

the far-field pattern is simply

$$u_\infty^{B_z}(\hat{x}; d) = e^{ikz \cdot (d - \hat{x})} u_\infty^B(\hat{x}, d), \quad \hat{x} \in \mathbb{S}. \quad (12)$$

The inverse scattering problem is to find the location and size of D from the far-field pattern $u_\infty(\hat{x})$ due to a single incident plane wave. For a point $z \in \mathbb{R}^2$, the ESM defines a far-field operator $\mathcal{F}_z: L^2(\mathbb{S}) \rightarrow L^2(\mathbb{S})$ using the far-field pattern of the disc B_z due to the incident plane waves of all directions $d \in \mathbb{S}$:

$$(\mathcal{F}_z g)(\hat{x}) = \int_{\mathbb{S}} u_\infty^{B_z}(\hat{x}, d) g(d) ds(d), \quad \hat{x} \in \mathbb{S}, \quad (13)$$

where $s(\cdot)$ indicates the surface integral.

Let $u^s(x)$ and $u_\infty(\hat{x})$ be the scattered field and far-field pattern of the scatterer D due to one incident wave, respectively. Using the far-field operator $\mathcal{F}_z$, we set up a far-field equation

$$(\mathcal{F}_z g)(\hat{x}) = u_\infty(\hat{x}), \quad \hat{x} \in \mathbb{S}. \quad (14)$$

Suppose that Ω is a domain having D inside. One can generate a set of sampling points S for Ω . For $z \in S$, the following result is proved in [1]. Let B_z be a sound-soft disc

centered at z with radius R and assume that k^2 is not a Dirichlet eigenvalue for B_z . For the far-field equation (14), if $D \subset B_z$, for a given $\varepsilon > 0$, there exists a function $g_z^\varepsilon \in L^2(\mathbb{S})$ such that

$$\left\| \int_{\mathbb{S}} u_\infty^{B_z}(\hat{x}, d) g_z^\varepsilon(d) ds(d) - u_\infty(\hat{x}) \right\|_{L^2(\mathbb{S})} < \varepsilon, \quad (15)$$

and the Herglotz wave function

$$v_{g_z^\varepsilon}(x) := \int_{\mathbb{S}} e^{ikx \cdot d} g_z^\varepsilon(d) ds(d), \quad x \in B_z, \quad (16)$$

converges to the solution $w \in H^1(B_z)$ of the Helmholtz equation with $w = -u^s$ on ∂B_z as $\varepsilon \rightarrow 0$. Otherwise, if $D \cap B_z = \emptyset$, every $g_z^\varepsilon \in L^2(\mathbb{S})$ that satisfies (15) for a given $\varepsilon > 0$ is such that

$$\lim_{\varepsilon \rightarrow 0} \|v_{g_z^\varepsilon}\|_{H^1(B_z)} = \infty. \quad (17)$$

Remark 1. The value k^2 is called a Dirichlet eigenvalue for a bounded domain B if there exists at least one nontrivial solution u to the following problem:

$$\begin{aligned} \Delta u + k^2 u &= 0, \text{ in } B, \\ u &= 0, \text{ on } \partial B. \end{aligned} \quad (18)$$

Using (12), $u_\infty^{B_z}(\cdot, d)$ can be computed easily. The far-field pattern $u_\infty(\hat{x})$ is the measured data due to an incident plane wave. For each sampling point $z \in S$, (14) can be solved by some regularization scheme, say Tikhonov regularization. From the above discussion, the approximate solution $\|g_z^\alpha\|_{L^2(\mathbb{S})}$ should be relatively large when D is not inside B_z and relatively small when D is inside B_z . Then, an approximation of the location and support of D can be obtained by plotting $\|g_z^\alpha\|_{L^2(\mathbb{S})}$ for all sampling points $z \in S$.

Remark 2. Equation (14) is similar to that of the linear sampling method by Colton and Kirsch [2]. The far-field equation of the linear sampling method is

$$\int_{\mathbb{S}} u_\infty(\hat{x}; d) g_z(d) ds(d) = \Phi_\infty(\hat{x}; z), \quad \hat{x} \in \mathbb{S}, \quad (19)$$

where $u_\infty(\hat{x}; d)$, $\hat{x}, d \in \mathbb{S}$ is the full-aperture far-field pattern of the obstacle D due to the incident plane wave $u^i = \exp(ikx \cdot d)$. For a sampling point $z \in \Omega$ which contains D , using some appropriate regularization scheme, one obtains an approximate solution g_z^α . In general, the norm $\|g_z^\alpha\|$ is smaller for $z \in D$ and larger for $z \notin D$. Since $u_\infty(\hat{x}; d)$ is the kernel in (19), the linear sampling method cannot directly process the far-field pattern due to a single incident plane wave.

3. Two-Step ESM for the ISP

Now we generalize the ESM for the inverse source problems. Let $u_\infty(\hat{x})$ be the far-field pattern for the inverse source problem defined in (7). Since the solution of (1) is radiating, we can use the same methodology as for the inverse scattering problems. The far-field equation for the inverse source problem keeps the same as (9), i.e.,

$$\int_{\mathbb{S}} u_\infty^{B_z}(\hat{x}, d) g_z(d) ds(d) = u_\infty(\hat{x}), \quad \hat{x} \in \mathbb{S}. \quad (20)$$

The only difference is that, in (20), the far-field pattern is for the acoustic source problem defined in Section 2.1.

The following theorem holds for the inverse source problem.

Theorem 1. *Let B_z be a sound-soft disc centered at z with radius R . Assume that the radius R is large enough such that $\overline{D} \subset B_z$ for some $z \in \mathbb{R}^2$. Furthermore, assume that k^2 is not a Dirichlet eigenvalue for B_z . Let $u_\infty(\hat{x})$ be the far-field pattern for the acoustic source problem defined in (7). Then, the following properties hold for the far-field equation (14):*

Case 1. If $\overline{D} \subset B_z$, for a given $\varepsilon > 0$, there exists a function $g_z^\varepsilon \in L^2(\mathbb{S})$ such that

$$\left\| \int_{\mathbb{S}} u_\infty^{B_z}(\hat{x}, d) g_z^\varepsilon(d) ds(d) - u_\infty(\hat{x}) \right\|_{L^2(\mathbb{S})} < \varepsilon, \quad (21)$$

and the Herglotz wave function

$$v_{g_z^\varepsilon}(x) := \int_{\mathbb{S}} e^{ikx \cdot d} g_z^\varepsilon(d) ds(d), \quad x \in B_z, \quad (22)$$

converges to the solution $w \in H^1(B_z)$ of the Helmholtz equation with $w = u$ on ∂B_z as $\varepsilon \rightarrow 0$.

Case 2. If $\overline{D} \cap \overline{B_z} = \emptyset$, every $g_z^\varepsilon \in L^2(\mathbb{S})$ that satisfies (15) for a given $\varepsilon > 0$ is such that

$$\lim_{\varepsilon \rightarrow 0} \|v_{g_z^\varepsilon}\|_{H^1(B_z)} = \infty. \quad (23)$$

Proof. It is easy to see that, under the conditions of Cases 1 and 2, one can find a domain D_0 such that $D \subset D_0$. Define

$$\begin{aligned} d_1 &= \max_{x \in \partial D, y \in \partial D_0} |x - y|, \\ d_2 &= \min_{x \in \partial D, y \in \partial D_0} |x - y|. \end{aligned} \quad (24)$$

Since $\overline{D} \subset B_z$, there exist a positive number σ small enough such that $0 < d_2 < d_1 < \sigma$. Consequently, we can further require that $D_0 \subset B_z$ for some $z \in \mathbb{R}^2$.

Due to the well-posedness of the acoustic source problem, there exists a radiating solution u for (1). We define an auxiliary scattering problem for a sound-soft obstacle of finding the scattered field u^s such that

$$\begin{aligned} \Delta u_0^s + k^2 u_0^s &= 0 \text{ in } \mathbb{R}^2 \setminus \overline{D_0}, \\ u_0^s &= u|_{\partial D_0} \text{ on } \partial D_0, \end{aligned} \quad (25)$$

$$\lim_{r \rightarrow \infty} \sqrt{r} \frac{\partial u_0^s}{\partial r} - iku_0^s = 0.$$

Then, there exists a radiating solution u^s to the above equation. Furthermore, u^s coincides with u in $\mathbb{R}^2 \setminus \overline{D_0}$. By introducing D_0 and the related scattering problem, one actually deals with an inverse scattering problem instead of the inverse source problem. The rest of the proof is then

based on the idea of Theorem 3.1 of [1]. For completeness, we present the details as follows.

Recall that the Herglotz wave operator $\mathcal{H}: L^2(\mathbb{S}) \longrightarrow H^{1/2}(\partial B_z)$ is defined as

$$(\mathcal{H}g)(x) := \int_{\mathbb{S}} e^{ikx \cdot d} g(d) ds(d), \quad x \in \partial B_z. \quad (26)$$

It is well known that $\mathcal{H}$ is injective and has a dense range provided k^2 is not a Dirichlet eigenvalue for the negative Laplacian for B_z [3].

Define the near-to-far-field operator by

$$\mathcal{N}: H^{1/2}(\partial B_z) \longrightarrow L^2(\mathbb{S}), \quad (27)$$

which maps the boundary value of a radiating solution $u \in H_{\text{loc}}^1(\mathbb{R}^2 \setminus \overline{B}_z)$ to its far-field pattern u_{∞} . Then, the operator $\mathcal{N}$ is bounded and injective and has a dense range [3].

Consider the case of $\overline{D} \subset B_z$. For any ε , there exists g_z^{ε} such that

$$\|\mathcal{H}g_z^{\varepsilon} + u^s\|_{L^2(\partial B_z)} \leq \frac{\varepsilon}{\|\mathcal{N}\|}. \quad (28)$$

Then,

$$\mathcal{N}(-\mathcal{H}g_z^{\varepsilon}) = \int_{\mathbb{S}} u_{\infty}^{B_z}(\cdot, d) g_z^{\varepsilon}(d) ds(d). \quad (29)$$

Since $D_0 \subset B_z$, we also have

$$\mathcal{N}(U^s) = u_{\infty}. \quad (30)$$

Subtracting (29) from (30), and using (28), we have g_z^{ε} such that

$$\begin{aligned} \left\| \int_{\mathbb{S}} u_{\infty}^{B_z}(\hat{x}, d) g_z^{\varepsilon}(d) ds(d) - u_{\infty}(\hat{x}) \right\|_{L^2(\mathbb{S})} &= \|\mathcal{N}(-\mathcal{H}g_z^{\varepsilon}) - \mathcal{N}(U^s)\|_{L^2(\mathbb{S})} \\ &\leq \|\mathcal{N}\| \cdot \|\mathcal{H}g_z^{\varepsilon} + u^s\|_{L^2(\mathbb{S})} \\ &\leq \varepsilon, \end{aligned} \quad (31)$$

for any ε . Taking $\varepsilon \longrightarrow 0$ in (28), the Herglotz wave function

$$v_{g_z^{\varepsilon}}(x) := \int_{\mathbb{S}} e^{ikx \cdot d} g_z^{\varepsilon}(d) ds(d), \quad x \in B_z, \quad (32)$$

converges to the unique solution $w \in H^1(B_z)$ of the following problem

$$\begin{aligned} \Delta w + k^2 w &= 0, \quad \text{in } B_z, \\ w &= -u^s, \quad \text{on } \partial B_z, \end{aligned} \quad (33)$$

which proves Case 1.

For the case of $\overline{D} \cap \overline{B}_z = \emptyset$, there exists $g_z^{\varepsilon} \in L^2(\mathbb{S})$ such that

$$\left\| \int_{\mathbb{S}} u_{\infty}^{B_z}(\hat{x}, d) g_z^{\varepsilon}(d) ds(d) - u_{\infty}(\hat{x}) \right\|_{L^2(\mathbb{S})} < \varepsilon, \quad (34)$$

for every $\varepsilon > 0$. Assume to the contrary that there exists a sequence $v_{g_z^{\varepsilon_n}}$ such that $\|v_{g_z^{\varepsilon_n}}\|_{H^1(B_z)}$ remains bounded as $\varepsilon_n \longrightarrow 0, n \longrightarrow \infty$. Without loss of generality, we assume that $v_{g_z^{\varepsilon_n}}$ converges to $v_{g_z} \in H^1(B_z)$ weakly as $n \longrightarrow \infty$, where

$$v_{g_z}(x) = \int_{\mathbb{S}} e^{ikx \cdot d} g_z(d) ds(d), \quad x \in B_z. \quad (35)$$

Let $v^s \in H_{\text{loc}}^1(\mathbb{R}^2 \setminus \overline{B}_z)$ be the unique solution of the following problem:

$$\begin{aligned} \Delta v^s + k^2 v^s &= 0 \text{ in } \mathbb{R}^2 \setminus \overline{B}_z, \\ v^s &= -v_{g_z} \text{ on } \partial B_z, \end{aligned} \quad (36)$$

$$\lim_{r \longrightarrow \infty} \sqrt{r} \left(\frac{\partial v^s}{\partial r} - ikv^s \right) = 0.$$

Its far-field pattern is

$$v_{\infty} = \int_{\mathbb{S}} u_{\infty}^{B_z}(x, d) g_z(d) ds(d). \quad (37)$$

From (34), as $\varepsilon \longrightarrow 0$, we have

$$\int_{\mathbb{S}} u_{\infty}^{B_z}(x, d) g_z(d) ds(d) = U_{\infty}. \quad (38)$$

Consequently,

$$v_{\infty} = u_{\infty}. \quad (39)$$

Then by Rellich's lemma (see Lemma 2.12 in [3]), the scattered fields coincide in $\mathbb{R}^2 \setminus (\overline{B}_z \cup \overline{D}_0)$ and

$$\tilde{u}^s := v^s = u^s, \quad \text{in } \mathbb{R}^2 \setminus (\overline{B}_z \cup \overline{D}_0). \quad (40)$$

We have that v^s has an extension into $\mathbb{R}^2 \setminus B_z$ and u^s has an extension into $\mathbb{R}^2 \setminus D_0$. Since $D_0 \cap B_z = \emptyset$, $\tilde{u}^s$ can be extended from $\mathbb{R}^2 \setminus (\overline{B}_z \cup \overline{D}_0)$ into all of $\mathbb{R}^2$; that is, $\tilde{u}^s$ is an entire solution to the Helmholtz equation. Since $\tilde{u}^s$ also satisfies the radiation condition, it must vanish identically in all of $\mathbb{R}^2$. This leads to a contradiction since $\tilde{u}^s$ does not vanish identically. $\square$

Remark 3. The case of $D \cap B_z \neq \emptyset$ and $D \not\subset B_z$ is not covered by the above theorem. In fact, the behavior of the regularized solution g_z can be analyzed if one knows D . However, this is irrelevant for the inverse problem since D is always unknown.

The above theorem provides a guidance to determine the location of the source. However, the choice of R is rather arbitrary. To find both the location and rough size of the source, we propose a two-step ESM as follows.

Two-Step ESM for the ISP

(A) Find the location of D .

- (i) Assume that a domain Ω containing D is known a priori.
- (ii) Choose a large enough radius R for B_z and generate a set S of sampling points for Ω .

FIGURE 1: Sample meshes for two test domains.

FIGURE 2: Reconstructions for $f_1 = 5$. (a) $D_1, k = 1$. (b) $D_1, k = 6$. (c) $D_2, k = 1$. (d) $D_2, k = 6$.

FIGURE 3: Reconstructions for $f_2 = x^2 + y^2$. (a) D_1 , $k = 1$. (b) D_1 , $k = 6$. (c) D_2 , $k = 1$. (d) D_2 , $k = 6$.

- (iii) Compute $u_{\infty}^{B_z}(\hat{x}, d)$ for all $\hat{x} \in \mathbb{S}$ and $d \in \mathbb{S}$.
- (iv) For each sampling point $z \in S$,
 - (a) set up a discrete version of (20);
 - (b) use the Tikhonov regularization to compute an approximate g_z^α to (20) (α is the regularization parameter).
- (v) Take the global minimum point $z^* \in S$ for $g_{zL^2}^\alpha$ as the location for D .

(B) Find the size of D .

- (i) Generate a decreasing sequence of R_j , $j = 1, 2, \dots, J$ such that $R > R_1 > \dots > R_J$.
- (ii) For $j = 1, 2, \dots, J$,
 - (a) generate S_j with the distance between sampling points being roughly R_j ;
 - (b) carry out Step A using S_j and a sampling disc with radius R_j ;
 - (c) find the minimum point $z_j^* \in S_j$ and set $D_j := B_{z_j^*}^{R_j}$. If $z_j \notin D_{j-1}$, stop.
- (iii) Choose z_{j-1} and $B_{z_{j-1}}^{R_{j-1}}$ to be the location and approximate support of D , respectively.

Remark 4. The requirement that k^2 is not a Dirichlet eigenvalue of B_z is not essential. One can avoid this by choosing a slightly different R .

4. Numerical Examples

In this section, we present some numerical examples to demonstrate the effectiveness of the proposed method. The synthetic data of the direct source problems are computed using the integral formula (7). Two wavenumbers $k = 1$ and $k = 6$ are selected for tests. We first generate a triangular mesh $\mathcal{T}$ for D with mesh size $h \approx 0.01$ (see Figure 1).

The far-field pattern is approximated by

$$u^\infty(\hat{x}) \approx \sum_{T \in \mathcal{T}} e^{-ik\hat{x} \cdot y_T} f(y_T) |T|, \quad (41)$$

where $T \in \mathcal{T}$ is a triangle, y_T is the center of T , and $|T|$ is the area of T . The interval $[0, 2\pi]$ is uniformly divided into 40 intervals, and let $\theta_j = 2j\pi/40$, $j = 0, \dots, 39$. For all examples, assume that $u^\infty(\hat{x}_j)$, $\hat{x}_j = (\cos\theta_j, \sin\theta_j)^T$ are the measured data. Note that the quadrature rule leads to about 3% error for the far-field pattern, which can be viewed as the noise.

Let the sampling domain be given by $\Omega := [-8, 8] \times [-8, 8]$. The interval $[-8, 8]$ is equally divided into 160 intervals and we end up with 161×161 sampling points uniformly distributed in Ω . We denote by

FIGURE 4: Reconstructions for $f_3(x, y) = \exp(ikr)$. (a) $D_1, k = 1$. (b) $D_1, k = 6$. (c) $D_2, k = 1$. (d) $D_2, k = 6$.

S the set of all sampling points. For each point $z \in S$, compute

$$\begin{aligned} u_{\infty}^{B_z}(\hat{x}_i; d_j), \quad \hat{x}_i &= (\cos \theta_i, \sin \theta_i), \\ d_j &= (\cos \theta_j, \sin \theta_j)^T, \quad i, j = 0, 1, \dots, 39, \end{aligned} \quad (42)$$

using (12). Simply applying the trapezoidal rule to (20), one gets the following linear system:

$$Ug = u, \quad (43)$$

where

$$\begin{aligned} U_{ij} &= \frac{\pi}{20} u_{\infty}^{B_z}(\hat{x}_i; d_j), \quad i, j = 0, 1, \dots, 39, \\ g &= (g_z(d_1), \dots, g_z(d_{39}))^T, \quad j = 0, 1, \dots, 39, \\ u &= u_{\infty}(d_j), \quad j = 0, 1, \dots, 39. \end{aligned} \quad (44)$$

Then, the Tikhonov regularization is used to solve an approximated solution g_z^α of (43) with a fixed parameter $\alpha = 10^{-5}$ for all $z \in S$.

Three functions are used

$$f_1(x, y) = 5, \quad (x, y) \in D,$$

$$f_2(x, y) = x^2 + y^2, \quad (x, y) \in D,$$

$$f_3(x, y, k) = \exp(ikr), \quad r = \sqrt{x^2 + y^2}, \quad (x, y) \in D, \quad (45)$$

where $i = \sqrt{-1}$. We consider two domains (supports of the source functions) D_1 and D_2 (see Figure 1), the unit square with vertices

$$(-1, -3), (0, -3), (0, -2), (-1, -2), \quad (46)$$

and an L-shaped domain with vertices

$$(2, 2), (3, 2), (3, 2.25), (2.25, 2.25), (2.25, 3), (2, 2). \quad (47)$$

We first consider the constant function $f_1 = 5$. In Figure 2, we plot the contours of the indicator function $I_z = |g_z^\alpha|$ in the sampling domain. It is clear that the values of I_z is small in the neighborhood of the source. The boundary of the exact support is the solid line. The circle centered at the minimum value of I_z (dotted line) is the output of the two-step ESM, which correctly provides the location and rough size of the source for both wave numbers $k_1 = 1$ and $k_2 = 6$.

Figure 3 is the results for $f_2 = x^2 + y^2$; that is, the source is a function of position.

Finally, in Figure 4, the results for $f_3(x, y, k) = \exp(ikr)$ are shown. The reconstructions again correctly provide the location and rough size of the source, and the dependence of f_3 on k does not change the results significantly.

5. Conclusions

A two-step ESM is proposed for the inverse source problem. The ESM was originally developed for the inverse scattering problem. The new setup of the far-field equation makes it possible to treat the inverse source problem. The behavior of the solutions is theoretically justified. Numerical examples show that the two-step ESM can be used to find the location and rough size of the support of the acoustic source effectively.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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