Research Article

Consensus Control of Nonlinear Multiagent Systems with Incremental Quadratic Constraints and Time Delays

Dachao Wang [iD], **Fang Song** [iD], **Wei Zhang** [iD], **and Zhi Hu** [iD]

Laboratory of Intelligent Control and Robotics, Shanghai University of Engineering Science, Shanghai 201620, China

Correspondence should be addressed to Fang Song; songfang2005@163.com

This paper considers the problem of consensus control for a class of nonlinear multiagent systems with incremental quadratic constraints and time delays. Each agent exchanges state information through a strongly connected communication topology. Based on the information obtained from neighboring agents, a distributed consensus protocol is designed. A delay-independent consensus condition is formed for the protocol to solve the consensus problem by employing Lyapunov–Krasovskii functional method. In order to deal with the nonlinear terms in matrix inequalities, an iterative algorithm is proposed by using the Schur complement lemma and the cone complementary linearization method. The nonlinearities under consideration are more general than many other nonlinearities considered in related literature studies since the incremental quadratic constraints include many other known nonlinearities as some special cases. Finally, we give a numerical example to illustrate the effectiveness of the proposed consensus control protocol.

1. Introduction

The consensus control problem of multiagent systems (MAS) has drawn lots of attention from researchers of systems and control community over the past few decades [1]. This is partly because the consensus problem has wide applications in different areas such as optimization, formation control, sensor network, dynamic agents of network, and cooperative surveillance.

In the past few years, researchers have made considerable progresses in the consensus control area and many aspects of consensus control are fully investigated. Many important results have been analyzed and published about the consensus control for linear, second-order, and high-order MAS, as seen in [2–7]. Finite-time consensus problem also attracts much attention from researchers [8, 9], and they mainly investigated how MAS can reach consensus within a certain time. In [10], the authors worked on the consensus problem for linear MAS under a time-invariant communication topology. In order to save energy in real-word mechanical systems, many different methods, such as event-trigger-based control [11, 12], intermittent control [13], and

sampled-data-based control [14], are invented to help solving consensus problem [15]. Instead of focusing on node consensus, authors in [16] studied nonnegative edge consensus due to physical considerations. As an important topic, group consensus also received much attention and many important works are published [17, 18].

It is worth mentioning that most of the existing papers are focusing on linear MAS, but physical systems always contain complicated nonlinearities. Therefore, consensus problem for nonlinear MAS deserves more attention. However, there exists no universal approach to solve the consensus problem for nonlinear MAS, and researchers are mostly concentrating on some types of systems with special nonlinearities (see, e.g., [19–21]). Researchers in [19] provided a linear matrix inequality-based method to design adaptive consensus control protocols for nonlinear MAS with Lipschitz nonlinearities by introducing adaptive coupling weights. Further, this method was extended to solve the consensus problem for nonlinear MAS whose nonlinearities satisfy one-side Lipschitz constraints in [20].

Time delays often appear in systems control and may bring instability to systems. Therefore, it is essential to

consider time delays in system control. Liu et al. [22] studied the consensus problem for nonlinear Lipschitz time-delay systems with input saturation. In [23], by applying a reduction method, the authors investigated the consensus problem for Lipschitz nonlinear MAS with input delays. Time delays also appear in communication between agents and pose a grave threat on consensus control, so many authors considered the consensus problem with communication delay (see, e.g., [24, 25]).

On the other hand, the incremental quadratic constraints (δQC) were first proposed in [26]. As stated in [27], δQC can be described by a set of multiplier matrices which we call incremental multiplier matrices (δ MMs) in a particular form, and many other types of nonlinearities, such as Lipschitz, one-side Lipschitz, and incremental sector bound nonlinearities, can be rewritten in a unified form of δQC. Many references consider the control and observation problem for systems with nonlinearities satisfying δQC [26–32]. In [28], the authors proposed a secure chaotic communication scheme of chaotic systems which satisfy δQC. In [31], full-order and reduced-order observers for discrete-time systems whose nonlinearities satisfy δQC are designed. Adaptive state observers are designed for incremental quadratically nonlinear systems in [32]. However, to the best of our knowledge, we can hardly find any papers that investigate the consensus control for nonlinear MAS whose nonlinearities satisfy δQC.

In this paper, we aim to focus on investigating the consensus problem for nonlinear MAS whose nonlinearities satisfy δQC with time delays. Compared with the current literature, there are about two main contributions in this work. Firstly, we design a distributed consensus protocol for incremental quadratically nonlinear MAS, and the considered nonlinearities include many common nonlinearities as some special cases. In other words, our results generalize and unify quite a few consensus protocol-design problems for many nonlinear systems. Secondly, this paper considers the sabotage of time delays, and a delay-independent consensus condition is established for the proposed consensus protocol by employing Lyapunov–Krasovskii functional method. Moreover, an iterative algorithm was proposed to handle nonlinear terms in matrix inequalities by applying cone complement linearization (CCL) method. We transform the nonconvex feasibility problem to some sequential optimization problem. Consequently, it can solve the proposed sufficient conditions via the linear matrix inequalities (LMI) method. Furthermore, the gain matrix and coupling weight can be computed through the proposed iterative algorithm.

The rest of this paper is organized as follows. Preliminaries are given in Section 2. The consensus problem for incremental quadratically nonlinear systems with time delays is investigated in Section 3. In Section 4, we provide a numerical example to illustrate the effectiveness of the developed results. Finally, Section 5 draws the conclusions.

In this paper, we use $x \in \mathbb{R}^n$ representing a vector of n real elements. $\mathbb{R}^{m \times n}$ denotes the set of all m by n real matrices. $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}$, i.e., given $x, y \in \mathbb{R}^n$,

then $\langle \cdot, \cdot \rangle = x^T y$. $\| \cdot \|$ denotes the Euclidean norm. $E > 0$ means E is a symmetric positive definite matrix and $E \geq 0$ denotes symmetric positive semidefinite matrix. $E - F \geq 0$ represents that the matrix $E - F$ is symmetric semipositive definite. I_n represents an identity matrix of dimension n. I is an identity matrix of any appropriate dimension. $E \otimes F$ denotes the Kronecker product of matrices E and F. Minimize $\mathrm{Tr}(\sum_{i=1}^{N} A_i)$ represents the minimum trace for a set of matrices A_i. The notation $*$ is used to denote the blocks induced by symmetry.

2. Preliminaries

In this paper, we consider an incremental quadratically nonlinear MAS with N agents represented as follows:

$$\begin{aligned} \dot{x}_i(t) &= A x_i(t) + A_h x_i(t-h) + B\varphi(t, z_i(t)) + F u_i(t), \\ z_i(t) &= C x_i(t) + C_h x_i(t-h), \end{aligned} \tag{1}$$

with

$$x_i(t) = \eta_i(t), \quad \forall t \in [-h, 0], i = 1, 2, \ldots, N, \tag{2}$$

where $A \in \mathbb{R}^{n \times n}, A_h \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, F \in \mathbb{R}^{n \times r}$ and C and $C_h \in \mathbb{R}^{q \times n}$ are constant matrices. $x_i(t) \in \mathbb{R}^n$ is the state vector and $x_i(t-h) \in \mathbb{R}^n$ is the time-delay state. $u_i \in \mathbb{R}^r$ is the control input or protocol. The scalar $h > 0$ is the constant delay. η_i is the continuous initial condition. We put all the nonlinear time-varying terms into a vector-valued nonlinear function $\varphi(t, z) \in \mathbb{R}^m$.

Those agents exchange information via a network modeled by a communication graph $\mathscr{G}$ with $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. $\mathscr{V} = \{1, \ldots, N\}$ means the set of nodes, and $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$ represents the set of edges. In this paper, nodes denote agents while edges represent communication links. The adjacency matrix of a graph $\mathscr{G}$ is denoted by $A = [a_{ij}]_{N \times N}$, where N means the number of agents in this network and

$$a_{ij} = \begin{cases} 0, & \text{if } (v_i, v_j) \notin \mathscr{E}, \\ 0, & \forall i = j, \\ 1, & \text{if } (v_i, v_j) \in \mathscr{E}, \end{cases} \tag{3}$$

where a_{ij} denotes the connection between agents ith and jth of the network of all agents. $a_{ij} = 0$ means there is no connection between agents ith and jth. $a_{ij} = 1$ means connection between agents ith and jth exists. Given an undirected communication graph, we have $a_{ij} = a_{ji}$. The Laplacian matrix $\mathscr{L}$ for MAS is defined as follows:

$$\mathscr{L} = [\mathscr{L}_{ij}]_{N \times N}, \tag{4}$$

where

$$\mathscr{L}_{ij} = \begin{cases} \sum_{j=1, i \neq j}^{N} a_{ij}, & \forall i = j, \\ -a_{ij}, & \forall i \neq j. \end{cases} \tag{5}$$

In this paper, we consider a distributed consensus protocol based on the states feedback of neighboring agents (see [19]):

$$u_i(t) = cK \sum_{j=1}^{N} a_{ij}\big(x_i(t) - x_j(t)\big), \qquad (6)$$

where $c > 0$ is the coupling weight between neighboring agents, $K \in \mathbb{R}^{p \times n}$ is the feedback gain matrix and that will be computed later.

Lemma 1 (see [33]). *The network $\mathcal{G}$ has a spanning tree that connects any two agents in the MAS if and only if Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ has a simple zero eigenvalue and all the other eigenvalues have positive real parts.*

Lemma 2 (see [34]). *For any given symmetric matrix $R \in \mathbb{R}^{n \times n}$ with the form $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}$ where $R_{11} \in \mathbb{R}^{r \times r}$, $R_{12} \in \mathbb{R}^{r \times (n-r)}$, and $R_{22} \in \mathbb{R}^{(n-r) \times (n-r)}$, then the following three conditions are equal:*

$$\begin{aligned} &1 \colon R < 0, \\ &2 \colon R_{11} < 0, R_{22} - R_{12}^T R_{11}^{-1} R_{12} < 0, \qquad (7) \\ &3 \colon R_{22} < 0, R_{11} - R_{12} R_{22}^{-1} R_{12}^T < 0. \end{aligned}$$

Definition 1 (see [30]). A symmetric matrix $M \in \mathbb{R}^{(q+m) \times (q+m)}$ is an incremental multiplier matrix (δMM) for vector-valued function $\varphi(t, z(t))$ if it satisfies the following incremental quadratic constraints (δQC):

$$\begin{bmatrix} \delta z \\ \delta \varphi \end{bmatrix}^{\mathrm{T}} M \begin{bmatrix} \delta z \\ \delta \varphi \end{bmatrix} \geq 0, \qquad (8)$$

for all $z_1(t), z_2(t) \in \mathbb{R}^q$ and all $\varphi(t, z_1(t)), \varphi(t, z_2(t)) \in \mathbb{R}^m$, where $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$, in which $M_{11} = M_{11}^{\mathrm{T}} \in \mathbb{R}^{q \times q}$ and $M_{22} = M_{22}^{\mathrm{T}} \in \mathbb{R}^{m \times m}$, and $\delta z = z_1(t) - z_2(t)$ and $\delta \varphi = \varphi(t, z_1(t)) - \varphi(t, z_2(t))$.

In this paper, we consider a vector-valued nonlinear function φ, which can be described in the form of δQC characterized by a set of symmetric matrices. Actually, δQC include many common nonlinearities as some special cases, such as the sector constraint, slope-restricted, Lipschitz, and one-side Lipschitz nonlinearities. We refer the interested readers to [26–32] for more details. Some nonlinearities satisfying δQC will be listed.

Remark 1. If φ is globally Lipschitz nonlinearity for z (see, e.g., [31]), there is a positive constant $L_g > 0$ for any $z_1, z_2 \in \mathbb{R}^q$ such that

$$\|\delta \varphi\| \leq L_g \|\delta z\|, \qquad (9)$$

which is equivalent to (where $q = m$)

$$L_g^2 \delta z^{\mathrm{T}} \delta z - \delta \varphi^{\mathrm{T}} \delta \varphi \geq 0. \qquad (10)$$

Therefore, inequality (8) is satisfied by choosing

$$M = v \begin{bmatrix} L_g^2 I_q & 0 \\ 0 & -I_m \end{bmatrix}, \qquad (11)$$

with $v > 0$.

Remark 2. If φ satisfies quadratically inner bounded constraints with respect to z (see, e.g., [31]), there exist some scalars $\alpha_2, \alpha_3 \in \mathbb{R}$ such that (where $q = m$)

$$\|\delta \varphi\|^2 \leq \alpha_2 \|z\|^2 + \alpha_3 \langle \delta \varphi, \delta z \rangle. \qquad (12)$$

The incremental multiplier matrix can be chosen as follows:

$$M = v \begin{bmatrix} \alpha_2 I_q & \dfrac{\alpha_3}{2} I_q \\[2mm] \dfrac{\alpha_3}{2} I_q & -I_q \end{bmatrix}, \qquad (13)$$

with $v > 0$.

It is worth mentioning that as shown in [26, 27], by properly selecting the incremental multiplier matrix, most of the nonlinearities can be rewritten in the form of (8). So the nonlinearities considered in this paper are more common than other nonlinearities considered in recent literature studies.

Assumption 1. All the nonlinear agents as described by (1) communicate with each other through a strongly connected graph $\mathcal{G}$.

Assumption 2. The nonlinear vector-valued functions φ in (1) satisfies δQC as defined in Definition 1 with a known incremental multiplier matrix $M \in \mathrm{M}$.

Remark 3. For MAS described by (1), we can have the following condition from Assumption 2:

$$\begin{bmatrix} \overline{z}(t) \\ \overline{\varphi}(t, z(t)) \end{bmatrix}^{\mathrm{T}} \overline{M} \begin{bmatrix} \overline{z}(t) \\ \overline{\varphi}(t, z(t)) \end{bmatrix} \geq 0, \qquad (14)$$

where

$$\begin{aligned} \overline{z}^{\mathrm{T}}(t) &= \Big[\big(z_1(t) - z^*\big)^{\mathrm{T}}, \big(z_2(t) - z^*\big)^{\mathrm{T}}, \ldots, \big(z_3(t) - z^*\big)^{\mathrm{T}} \Big], \\ \overline{\varphi}(t, z(t)) &= \Big[\big(\varphi(t, z_1(t)) - \varphi(t, z^*)\big)^{\mathrm{T}}, \ldots, \\ &\qquad \big(\varphi(t, z_N(t)) - \varphi(t, z^*)\big)^{\mathrm{T}} \Big]^{\mathrm{T}}, \\ M &= \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}, \\ \overline{M} &= \begin{bmatrix} I_N \otimes M_{11} & I_N \otimes M_{12} \\ I_N \otimes M_{12}{}^T & I_N \otimes M_{22} \end{bmatrix}. \end{aligned}$$

$$(15)$$

3. Main Results

The consensus problem under Assumptions 1 and 2 for the nonlinear MAS (1) with a constant time delay is investigated in the following section.

Theorem 1. *Consider the MAS described in (1) satisfying Assumptions 1 and 2 with a given incremental multiplier matrix M. Suppose that there exist matrices $P > 0, Q > 0$ and scalars $\alpha > 0, \tau > 0$ such that the following matrix inequality is satisfied:*

$$\begin{bmatrix} \overline{\Phi}_{11} & A_h & B \\ * & -Q & 0 \\ * & * & 0 \end{bmatrix} + \alpha \psi \Psi^T M \Psi \psi < 0, \qquad (16)$$

where

$$\overline{\Phi}_{11} = AP + PA^T + PQP - \tau FF^T,$$
$$\Psi = \begin{bmatrix} C & C_h & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad (17)$$
$$\psi = \mathrm{diag}(P, I_N, I_N).$$

Then, the consensus control protocol given by (6) can asymptotically solve the consensus problem for MAS (1) with the feedback control gain $K = -F^T P^{-1}$ and the coupling weight $c \geq \tau/2\,\partial(L)$, where $\partial(L)$ represents the minimum nonzero eigenvalue of the graph $\mathscr{G}$.

Proof. Letting $e_i(t) = x_i(t) - (1/N)\sum_{j=1}^N x_j(t)$, it follows that $e_i(t)$ and $z_i(t)$ satisfy the following dynamics:

$$\dot{e}_i(t) = Ae_i(t) + A_h e_i(t-h) + B\varphi(t, z_i(t))$$
$$- \frac{1}{N}\sum_{j=1}^N B\varphi(t, z_i(t))$$
$$+ \sum_{j=1}^N ca_{ij} FK(e_i(t) - e_j(t)),$$
$$z_i(t) - \frac{1}{N}\sum_{j=1}^N z_j(t) = Ce_i(t) + C_h e_i(t-h),$$
$$i = 1, 2, \ldots, N. \qquad (18)$$

Firstly, we consider the following Lyapunov function candidate:

$$V(t) = \sum_{i=1}^N e_i^T(t) P^{-1} e_i(t)$$
$$+ \sum_{i=1}^N \int_{t-h}^t e_i(\alpha)^T Q e_i(\alpha) d\alpha, \qquad (19)$$

where $P > 0, Q > 0$ and $P, Q \in \mathbb{R}^{n \times n}$.

The time derivative of $V(t)$ along the trajectory of the consensus error dynamic system in (18) is given by

$$\dot{V}(t) = 2\sum_{i=1}^N e_i(t)^T P^{-1}(Ae_i(t) + A_h e_i(t-h))$$
$$+ 2\sum_{i=1}^N e_i(t)^T P^{-1} B\left(\varphi(t, z_i(t)) - \frac{1}{N}\sum_{j=1}^N \varphi(t, z_j(t))\right)$$
$$+ 2\sum_{i=1}^N e_i(t)^T P^{-1} FK \sum_{j=1}^N ca_{ij}(e_i(t) - e_j(t))$$
$$+ \sum_{i=1}^N \left(e_i(t)^T Q e_i(t) - e_i(t-h)^T Q e_i(t-h)\right). \qquad (20)$$

Noticing that $\sum_{i=1}^N e_i^T(t) = 0$, so one can have

$$\sum_{i=1}^N e_i(t)^T P^{-1} B\left(\varphi(t, z^*) - \frac{1}{N}\sum_{j=1}^N \varphi(t, z_j(t))\right) = 0. \qquad (21)$$

Then, relative terms in (20) satisfy

$$2\sum_{i=1}^N e_i(t)^T P^{-1} B\left(\varphi(t, z_i(t)) - \frac{1}{N}\sum_{j=1}^N \varphi(t, z_j(t))\right) \qquad (22)$$
$$= 2\sum_{i=1}^N e(t)^T P^{-1} B(\varphi(t, z_i(t)) - \varphi(t, z^*)).$$

Let $e(t) = [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)]^T$ and $e(t-h) = [e_1^T(t-h), e_2^T(t-h), \ldots, e_N^T(t-h)]^T$.

Then, the following inequality is derived by (20) with $K = -F^T P^{-1}$:

$$\dot{V}(t) = e^T(t)\left(I_N \otimes (P^{-1}A + A^T P^{-1})\right)e(t)$$
$$+ e^T(t)\left(I_N \otimes 2P^{-1}A_h\right)e(t-h)$$
$$- e^T(t)\left(2c\mathscr{L} \otimes P^{-1}FF^T P^{-1}\right)e(t)$$
$$+ e^T(t)\left(I_N \otimes 2P^{-1}B\right)\overline{\varphi}(t, z(t))$$
$$+ e^T(t)(I_N \otimes Q)e(t) - e^T(t-h)(I_N \otimes Q)e(t-h), \qquad (23)$$

where $\overline{\varphi}(t, z(t)) = [(\varphi(t, z_1(t)) - \varphi(t, z^*))^T, \ldots, (\varphi(t, z_N(t)) - \varphi(t, z^*))^T]^T$.

Let $\Omega^T \mathscr{L} \Omega = \Lambda$ and $\Lambda = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_N)$, where Ω is an unitary matrix satisfying $\Omega^{-1} = \Omega^T$ and λ_i denotes the positive eigenvalues of the Laplacian matrix $\mathscr{L}$. By substituting $\mathscr{L} = \Omega\Lambda\Omega^T$, $\widetilde{\xi} = I \otimes (\Omega^T \otimes I_n)\xi$, and $\xi = [ce^T(t) \ e^T(t-h) \ (\overline{\varphi}(t, z(t)))^T]^T$, one can have

$$- e^T(t)\left(2c\mathscr{L} \otimes P^{-1}FF^T P^{-1}\right)e(t)$$
$$= -\sum_{i=1}^N \widetilde{e}_i^T(t) 2c\lambda_i P^{-1}FF^T P^{-1}\widetilde{e}_i(t). \qquad (24)$$

Notice that

$$- \sum_{i=1}^{N} \widetilde{e}_i^{\mathrm{T}}(t) 2c\lambda_i P^{-1} FF^{\mathrm{T}} P^{-1} \widetilde{e}_i(t)$$

$$\leq - \widetilde{e}^{\mathrm{T}}(t) \left(I_N \otimes 2\, \partial(L) c P^{-1} FF^{\mathrm{T}} P^{-1} \right) \widetilde{e}(t). \tag{25}$$

Then, we choose the coupling weight c such that $2\,\partial(\mathscr{L})c \geq \tau$, which implicates

$$- \widetilde{e}^{\mathrm{T}}(t) \left(I_N \otimes 2\, \partial(L) c P^{-1} FF^{\mathrm{T}} P^{-1} \right) \widetilde{e}(t)$$

$$\leq - \widetilde{e}^{\mathrm{T}}(t) \left(I_N \otimes \tau P^{-1} FF^{\mathrm{T}} P^{-1} \right) \widetilde{e}(t). \tag{26}$$

So (23) can be written into a compact form as follows:

$$\dot{V}(t) = \widetilde{\xi}^{\mathrm{T}} \Phi \widetilde{\xi}, \tag{27}$$

where

$$\Phi = \begin{bmatrix} \Phi_{11} & I_N \otimes P^{-1} A_h & I_N \otimes P^{-1} B \\ * & I_N \otimes -Q & 0 \\ * & * & 0 \end{bmatrix}, \tag{28}$$

$$\Phi_{11} = I_N \otimes \left(P^{-1} A + AP^{-1} + Q - \tau P^{-1} FF^{\mathrm{T}} P^{-1} \right).$$

Secondly, we have the following equality from (1):

$$\begin{bmatrix} \overline{z}(t) \\ \overline{\varphi}(t, z(t)) \end{bmatrix} = \overline{\Psi} \xi, \tag{29}$$

where

$$\overline{\Psi} = \begin{bmatrix} I_N \otimes C & I_N \otimes C_h & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{30}$$

Substituting (29) into (14) yields

$$\xi^{\mathrm{T}} \overline{\Psi}^{\mathrm{T}} \overline{M} \overline{\Psi} \xi \geq 0. \tag{31}$$

The nonlinear $\varphi(t, z_i(t))$ satisfies Definition 1 with a symmetric matrix M, so we can define a set of incremental multiplier matrices by $\overline{M}_\Omega = \{\alpha \overline{M}; \alpha > 0\}$, where $\overline{M}_\Omega \in \mathrm{M}$. Now by substituting $\overline{M}$ with $\alpha \overline{M}$ in (31) and applying $\widetilde{\xi} = I \otimes (\Omega^T \otimes I_n) \xi$, (31) can be further expressed as follows:

$$\widetilde{\xi}^{\mathrm{T}} \overline{\Psi}^{\mathrm{T}} \alpha \overline{M} \overline{\Psi} \widetilde{\xi} \geq 0. \tag{32}$$

Adding the terms on the left side of (31) to the right side of (27) yields

$$\dot{V}(t) \leq \widetilde{\xi}^{\mathrm{T}} \left(\Phi + \overline{\Psi}^{\mathrm{T}} \alpha \overline{M} \overline{\Psi} \right) \widetilde{\xi}. \tag{33}$$

If $\dot{V}(t) < 0$, the synchronization error e will converge to the origin according to the Lyapunov stability theory. Therefore, we need to verify that

$$\Phi + \overline{\Psi}^{\mathrm{T}} \alpha \overline{M} \overline{\Psi} < 0. \tag{34}$$

By utilizing the Kronecker product, it can be validated that the inequality (34) is equivalent to

$$I_N \otimes \left(\begin{bmatrix} \Phi_{11} & P^{-1} A_h & P^{-1} B \\ * & -Q & 0 \\ * & * & 0 \end{bmatrix} + \Psi^{\mathrm{T}} \alpha M \Psi \right) < 0, \tag{35}$$

where Ψ is defined in Theorem 1.

Since $I_N > 0$, the condition (35) holds if and only if

$$\begin{bmatrix} \widetilde{\Phi}_{11} & P^{-1} A_h & P^{-1} B \\ * & -Q & 0 \\ * & * & 0 \end{bmatrix} + \Psi^T \alpha M \Psi < 0, \tag{36}$$

where

$$\widetilde{\Phi}_{11} = P^{-1} A + AP^{-1} + Q - \tau P^{-1} FF^{\mathrm{T}} P^{-1}. \tag{37}$$

Then multiply both the left and right sides of the matrix in (36) by a reversible matrix $\psi = \mathrm{diag}(P, I_N, I_N)$ for a similar transformation, and we have condition (16). If condition (16) holds, then $\dot{V}(t) < 0$. Hence, the consensus error dynamic system (18) is asymptotically stable, and the consensus error $e \longrightarrow 0$ as $t \longrightarrow \infty$. Therefore, the considered MAS can reach consensus. This completes the proof of Theorem 1. □

Remark 4. Theorem 1 offers a sufficient condition that the nonlinear time-delay MAS can reach consensus. Then, we need to find a solution to the constraint in Theorem 1 to retrieve the consensus control gain matrix K and the coupling weight c.

After partitioning M in the form of (8) and applying the Schur complement lemma to (16), the inequality (16) is equivalent to

$$\begin{bmatrix} AP + PA^T - \tau FF^T + \alpha PC^T M_{11} CP & A_h + \alpha PC^T M_{11} C_h & B + \alpha PC^T M_{12} & P \\ * & -Q + \alpha C_h^T M_{11} C_h & \alpha C_h^T M_{12} & 0 \\ * & * & \alpha M_{22} & 0 \\ * & * & * & -Q^{-1} \end{bmatrix} < 0. \tag{38}$$

It should be noted that inequality (38) is not linear with the variables P and Q so we cannot directly utilize a convex optimization algorithm to find a solution. Therefore, in order to complete the design of the consensus control law in this paper, we can transform the nonlinear constraints into corresponding convex optimization problems subject to LMI constraints. A general treatment to (38) is to employ the cone complementary linearization [20, 35–37]. On the other hand, a necessary condition for the feasibility of (38) is $M_{22} < 0$ obviously. Further, when $M_{11} > 0$, $M_{11} = 0$ or

$C^T M_{11} C < 0$, the matrix inequality (38) can be transformed to different sequential corresponding convex optimization problems subject to LMI constraints, respectively, which deserve further concrete discussions.

Based on the above discussions, we have the following corollaries to Theorem 1.

Corollary 1. *Consider the MAS described in (1) satisfying Assumptions 1 and 2 and the given incremental multiplier M satisfying $M_{22} < 0$ and $M_{11} = 0$. Suppose that there exist matrices $P > 0, W > 0, \widetilde{W} > 0, Q > 0, \widetilde{Q} > 0$ and scalars $\alpha > 0, \tau > 0$ such that the following matrix inequalities are satisfied:*

$$\begin{bmatrix} AP + PA^T - \tau FF^T & A_h & (B + \alpha PC^T M_{12}) & P \\ * & -Q & \alpha C_h^T M_{12} & 0 \\ * & * & \alpha M_{22} & 0 \\ * & * & * & -W \end{bmatrix} < 0, \tag{39}$$

$$\begin{bmatrix} \widetilde{W} & I \\ * & \widetilde{Q} \end{bmatrix} \geq 0, \tag{40}$$

$$\begin{aligned} \widetilde{W}W = I, \\ \widetilde{Q}Q = I. \end{aligned} \tag{41}$$

Then, the consensus control protocol given by (6) can asymptotically solve the consensus problem for MAS given by (1) with the feedback control gain $K = -F^T P^{-1}$ and the coupling weight $c \geq \tau/2 \partial(L)$, where $\partial(L)$ represents the minimum nonzero eigenvalue of the graph $\mathscr{G}$.

$$\begin{bmatrix} AP + PA^T - \tau FF^T & (A_h + \alpha PC^T M_{11} C_h) & (B + \alpha PC^T M_{12}) & P & \sqrt{\alpha}\, PC^T M_{11} \\ * & (-Q + \alpha C_h^T M_{11} C)_h & \alpha C_h^T M_{12} & 0 & 0 \\ * & * & \alpha M_{22} & 0 & 0 \\ * & * & * & -W & 0 \\ * & * & * & * & -M_{11} \end{bmatrix} < 0. \tag{45}$$

$$\begin{bmatrix} \widetilde{W} & I \\ * & \widetilde{Q} \end{bmatrix} \geq 0,$$

$$\begin{aligned} \widetilde{W}W = I, \\ \widetilde{Q}Q = I. \end{aligned} \tag{46}$$

Then, the consensus control protocol given by (6) can asymptotically solve the consensus problem for MAS given by (1) with the feedback control gain $K = -F^T P^{-1}$ and the coupling weight $c \geq \tau/2 \partial(L)$, where $\partial(L)$ represents the minimum nonzero eigenvalue of the graph $\mathscr{G}$.

Proof. Firstly, (38) turns into the following with $M_{11} = 0$:

$$\begin{bmatrix} AP + PA^T - \tau FF^T & A_h & (B + \alpha PC^T M_{12}) & P \\ * & -Q & \alpha C_h^T M_{12} & 0 \\ * & * & \alpha M_{22} & 0 \\ * & * & * & -Q^{-1} \end{bmatrix} < 0. \tag{42}$$

Secondly, we here introduce a new matrix variable $W > 0$ and make sure that

$$Q^{-1} \geq W. \tag{43}$$

Note that (43) is equivalent to

$$\begin{bmatrix} \widetilde{W} & I \\ * & \widetilde{Q} \end{bmatrix} \geq 0, \tag{44}$$

where $\widetilde{W} = W, \widetilde{Q} = Q^{-1}$. Then, we replace Q^{-1} with W in (42) and have (39). If (39)–(41) hold, (42) is satisfied. This completes the proof of Corollary 1. □

Similarly, we have the following two corollaries when $M_{11} > 0$ and $C^T M_{11} C < 0$, respectively. We omit proofs here.

Corollary 2. *Consider the MAS described in (1) satisfying Assumptions 1 and 2 and the given incremental multiplier M satisfying $M_{22} < 0$ and $M_{11} > 0$. Suppose that there exist matrices $P > 0, Q > 0, \widetilde{Q} > 0, W > 0, \widetilde{W} > 0$, and scalars $\alpha > 0, \tau > 0$, such that the following inequalities are satisfied:*

Corollary 3. *Consider the MAS described in (1) satisfying Assumptions 1 and 2 and the given incremental multiplier M satisfying $M_{22} < 0$ and $C^T M_{11} C < 0$. Suppose that there exist matrices $P > 0, Q > 0, W_1 > 0, W_2 > 0, \widetilde{P} > 0, \widetilde{Q} > 0, \widetilde{W}_1 > 0$,*

$\widetilde{W}_2 > 0$, *and scalars* $\alpha > 0, \tau > 0$, *such that the following inequalities are satisfied:*

$$
\begin{bmatrix}
AP + PA^{\mathrm{T}} - \tau FF^{\mathrm{T}} - \alpha W_2 & (A_h + \alpha PC^{\mathrm{T}} M_{11} C_h) & (B + \alpha PC^{\mathrm{T}} M_{12}) & P \\
* & (-Q + \alpha C_h^{\mathrm{T}} M_{11} C_h) & \alpha C_h^{\mathrm{T}} M_{12} & 0 \\
* & * & \alpha M_{22} & 0 \\
* & * & * & -W_1
\end{bmatrix} < 0.
$$

$$
\begin{bmatrix} \widetilde{W}_1 & I \\ * & \widetilde{Q} \end{bmatrix} \geq 0,
$$

$$
\begin{bmatrix} -C^{\mathrm{T}} M_{11} C & \widetilde{P} \\ * & \widetilde{W}_2 \end{bmatrix} \geq 0,
$$

$$\widetilde{W}_1 W_1 = I,$$

$$\widetilde{W}_2 W_2 = I,$$

$$\widetilde{Q} Q = I,$$

$$\widetilde{P} P = I. \tag{47}$$

Then, the consensus control protocol given by (6) can asymptotically solve the consensus problem for MAS given by (1) with the feedback control gain $K = -F^{\mathrm{T}} P^{-1}$ and the coupling weight $c \geq \tau/2 \partial(L)$, where $\partial(L)$ represents the minimum nonzero eigenvalue of the graph $\mathcal{G}$.

As mentioned in [20, 35, 36], we can deal with the original feasibility problem in Corollary 1 by solving the following optimization with variables $\widetilde{W}, W, P, Q$, and $\widetilde{Q}$:

$$
\begin{cases}
\text{min Trace} & (\widetilde{W}W + \widetilde{Q}Q) \\
\text{subject to} & \begin{bmatrix} \widetilde{W} & I \\ * & W \end{bmatrix} \geq 0, \begin{bmatrix} \widetilde{Q} & I \\ * & Q \end{bmatrix} \geq 0 \\
\text{and inequalities (39), (40).}
\end{cases} \tag{48}
$$

In (48), equalities (46) are transformed into inequalities $\begin{bmatrix} \widetilde{W} & I \\ * & W \end{bmatrix} \geq 0, \begin{bmatrix} \widetilde{Q} & I \\ * & Q \end{bmatrix} \geq 0$. It ensures that we can attain $\widetilde{W} = W^{-1}$ and $\widetilde{Q} = Q^{-1}$ with the required accuracy. The specific algorithm is given in Algorithm 1.

The original feasibility problem in Corollary 2 can be tackled by solving the following optimization subject to LMI constraints with variables $\widetilde{W}, W, P, Q$, and $\widetilde{Q}$:

$$
\begin{cases}
\text{min Trace} & (\widetilde{W}W + \widetilde{Q}Q) \\
\text{subject to} & \begin{bmatrix} \widetilde{W} & I \\ * & W \end{bmatrix} \geq 0, \begin{bmatrix} \widetilde{Q} & I \\ * & Q \end{bmatrix} \geq 0 \\
\text{and inequalities (45).}
\end{cases} \tag{49}
$$

The original feasibility problem in Corollary 3 can be tackled by solving the following optimization subject to

LMI constraints with variables $\widetilde{W}_1, W_1, \widetilde{W}_2, W_2, \widetilde{P}, P, Q$, and $\widetilde{Q}$:

$$
\begin{cases}
\text{min Trace} & (\widetilde{W}_1 W_1 + \widetilde{W}_2 W_2 + \widetilde{P}P + \widetilde{Q}Q) \\
\text{subject to} & \begin{bmatrix} \widetilde{W}_1 & I \\ * & W_1 \end{bmatrix} \geq 0, \begin{bmatrix} \widetilde{W}_2 & I \\ * & W_2 \end{bmatrix} \geq 0 \\
& \begin{bmatrix} \widetilde{Q} & I \\ * & Q \end{bmatrix} \geq 0, \begin{bmatrix} \widetilde{P} & I \\ * & P \end{bmatrix} \geq 0 \\
\text{and inequalities (47).}
\end{cases} \tag{50}
$$

Remark 5. It is worth mentioning that during the derivation process in the proof of Theorem 1, we introduced a positive scalar τ that $2\partial(\mathcal{L})c \geq \tau$, where $\partial(\mathcal{L})$ represents the minimum nonzero eigenvalue of the graph G. Hence, we need to find a positive τ to choose c. In our results, $\tau > 0$ needs to be chosen such that the inequality in Theorem 1 is satisfied. In fact, τ needs to be adjusted few times to find a feasible solution in Algorithm 1. Once the inequality in Theorem 1 is satisfied with a positive τ, then we can select the coupling weight c as long as it satisfies $2\partial(\mathcal{L})c \geq \tau$, which will be exemplified in the simulation part.

Remark 6. The inequalities in corollaries can be solved by applying CCL algorithm and LMI-tools. After using LMI-tools of MATLAB, we can find a feasible solution to the constraints in (48) with respect to variables $\widetilde{W}, W, P, Q$, and $\widetilde{Q}$. Those constraints ensure $\widetilde{Q}$ and $\widetilde{W}$ are the inverses of Q and W, respectively. We can attain the variables with the required accuracy from a freedom in

---

(1) Choose proper scalars $\tau > 0$ and $\alpha > 0$ to guarantee a feasible solution to the inequalities in (48).

(2) Find a feasible set $\widetilde{W}_0, W_0, \widetilde{Q}_0, Q_0, P_0$ so that inequalities in (48) can be satisfied. Adjust scalars τ and α and try again when feasible solutions cannot be found. Then, set $k = 0$.

(3) Solve the following minimization optimization problem with respect to variables $\widetilde{W}, W, P, Q,$ and $\widetilde{Q}$.
   $\min \text{Trace}(W_k\widetilde{W} + W\widetilde{W}_k + \widetilde{Q}_kQ + \widetilde{Q}Q_k)$, subject to inequalities in (48). Find the new optimization solution $\widetilde{W}, W, P, Q, \widetilde{Q}$, then set $\widetilde{W}_{k+1} = \widetilde{W}, W_{k+1} = W, \widetilde{Q}_{k+1} = \widetilde{Q}, Q_{k+1} = Q, P_{k+1} = P$.

(4) If the inequality (42) is satisfied with the matrices Q, P obtained in Step 3, then we can choose coupling weight c according to $2\partial(\mathscr{L})c \geq \tau$ and obtain the control gain matrix K by $K = -F^{\mathrm{T}}P^{-1}$. Stop when inequality (42) is still not feasible after a certain number of iterations. Then, set $k = k + 1$ and go to Step 3.

ALGORITHM 1: The constraints in Corollary 1 can be tackled by solving (48) through the following iterative algorithm.

minimization. After verifying conditions in Corollary 1 are satisfied, we can design the gain matrix via $K = -F^{\mathrm{T}}P^{-1}$ and the coupling weight c to complete the consensus control law design. The interested readers refer to [35–37] for more details about CCL algorithm. By making appropriate modifications to Algorithm 1, we can easily derive algorithms for Corollary 2 and Corollary 3.

Remark 7. δQC include many common nonlinearities as some special cases. In the sense, δQC can be employed for more general scenarios. Therefore, the results in this paper have wider application scenarios compared with the existing references ([20, 21], etc.).

4. Simulation Example

In this section, we provide a numerical simulation to illustrate the effectiveness of the proposed consensus control protocol. For simplicity, we only simulate Corollary 1. Consider a network of mobile agents. Each agent with nonlinear dynamics remains the same as in (1) with the following parameters (from [38]):

$$
\begin{aligned}
A &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \\
A_h &= \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \\
B &= \begin{bmatrix} 0.2 & 0.7 \\ 0.3 & 0.4 \end{bmatrix}, \\
F &= \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}, \\
C &= \begin{bmatrix} 0.6 & 0.8 \end{bmatrix}, \\
C_h &= \begin{bmatrix} 0.5 & 0.9 \end{bmatrix}, \\
\varphi(t, z) &= 0.5 \begin{bmatrix} 0.5(1 + \sin t) & 0.5(1 - \sin t) \end{bmatrix}^{\mathrm{T}} * z,
\end{aligned} \tag{51}
$$

where φ is a nonlinear function since t is changeable. z contains the time-delay term $C_hx(t - h)$, and $h = 4$ in this simulation. $\varphi(t, z)$ and z satisfy δQC with an incremental multiplier matrix [38]:

$$
M = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & -1 & 0 \\ 0.5 & 0 & -1 \end{bmatrix}. \tag{52}
$$

Those agents exchange information under a connection graph as shown in Figure 1. Thus, we can obtain the following Laplacian matrix, the communication matrix between agents:

$$
\mathscr{L} = \begin{bmatrix}
2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 3 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & -1 & 4 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 \\
-1 & 0 & -1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 2
\end{bmatrix}. \tag{53}
$$

The minimum nonzero eigenvalue of the graph $G\partial(\mathscr{L}) = 0.29$. Let $\tau = 2.3$ and $\alpha = 0.2$. By applying Algorithm 1 with the LMI toolbox of MATLAB, we can find a feasible solution to (42) with

$$
\begin{aligned}
P &= \begin{bmatrix} 2.9926 & -0.6004 \\ -0.6004 & 5.2398 \end{bmatrix}, \\
Q &= \begin{bmatrix} 13.2397 & 0.0000 \\ 0.0000 & 13.2397 \end{bmatrix}.
\end{aligned} \tag{54}
$$

We choose $c = 4$ here for $2\partial(\mathscr{L})c \geq \tau$. The feedback control gain matrix $K = -F^{\mathrm{T}}P^{-1} = \begin{bmatrix} -0.3812 & -0.2345 \end{bmatrix}$. Now we have completed the consensus control protocol design.
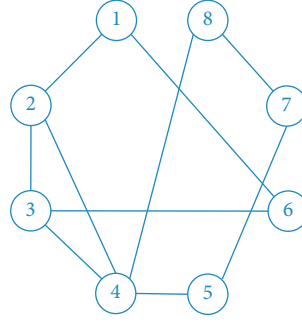
FIGURE 1: Communication topology of MAS in (1).

(a)

(b)
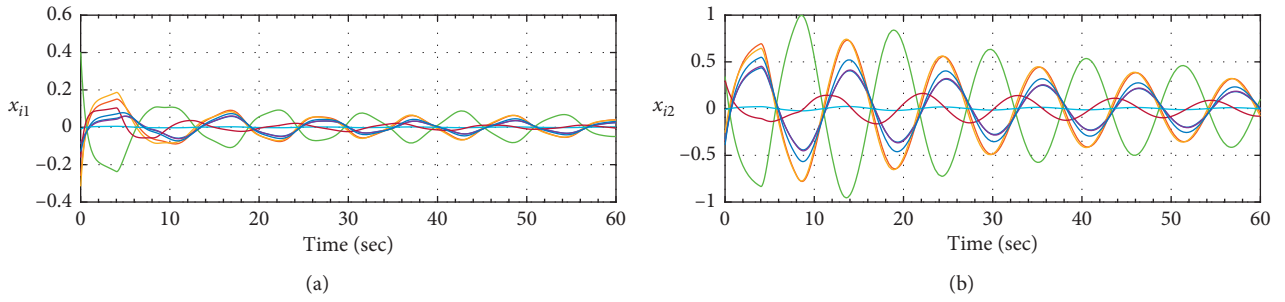
FIGURE 2: The state for agents 1, 2, ..., 8 without consensus control.
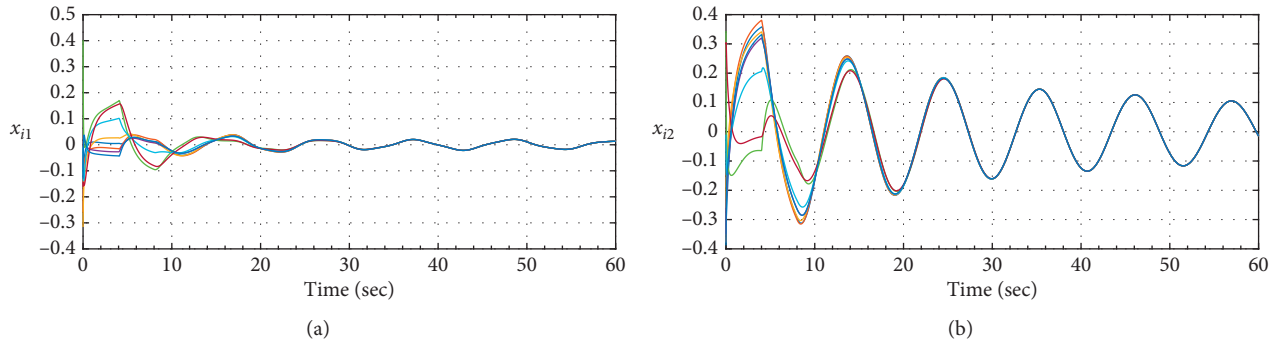

(a)

(b)

FIGURE 3: The state for agents 1, 2, ..., 8 under consensus control.
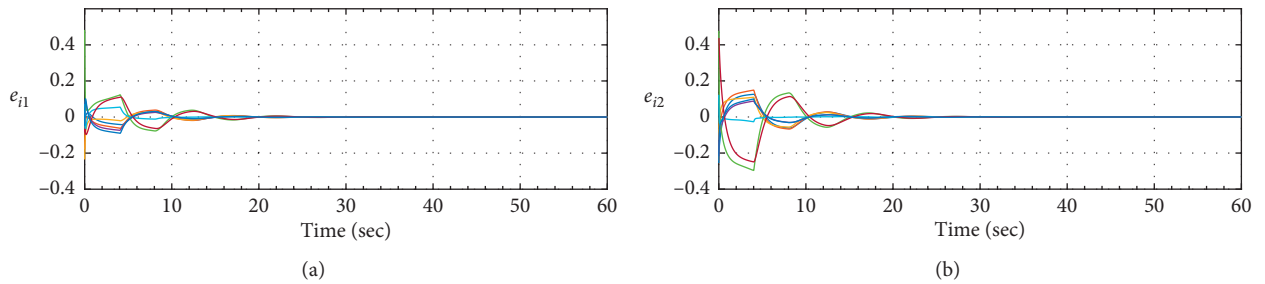

(a)

(b)

FIGURE 4: The consensus error for agents 1, 2, ..., 8 under consensus control.

As we can see in Figure 2, the nonlinear time-delay MAS did not reach consensus without the consensus control law. The state trajectories are shown in Figure 3. We can see that all agents reach consensus after applying the designed consensus law. All the corresponding states of each agent change synchronously over time. In Figure 4, the state errors of each agent converged to the origin for about 20 seconds. This demonstrated the effectiveness

of the proposed consensus control law design method in this paper.

5. Conclusion

The consensus control problem has been investigated for the nonlinear time-delay MAS whose nonlinearities satisfy δQC under an undirected communication graph. A distributed consensus control protocol-design criterion was proposed for nonlinear MAS. We have considered a more general class of nonlinearities which satisfy δQC. By utilizing graph theory and Lyapunov theory, we analyzed the consensus error dynamic stability of the proposed control law. Then, sufficient conditions have been deduced as the form of matrix inequalities to solve the consensus problem for the considered MAS. In order to cope with the nonlinear terms, we transformed the nonlinear constraints into some optimization problems subject to LMI constraints and proposed iterative algorithms by resorting to the Schur complement lemma and the CCL method. Then, the gain matrix and the coupling weight in the protocol can be easily computed to complete our consensus control law design. A numerical example is provided to illustrate the effectiveness of the developed results.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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