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Research Article

Existence of Positive Weak Solutions for Quasi-Linear Kirchhoff Elliptic Systems via Sub-Supersolutions Concept

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By using the sub- and supersolutions concept (Schmitt, 2007), we prove in this paper the existence of positive solutions of quasi-linear Kirchhoff elliptic systems in bounded smooth domains. This work is an extension of the recent work of Boulaaras et al., 2020.

1. Introduction

The scope of nonlinear partial differential equations is quite wide. One of the main advances in the development of nonlinear PDEs has been the study of wave propagation, then comes the equations related to chemical and biological phenomena, and later, the equations related to solid mechanics, fluid dynamics, acoustics, nonlinear optics, plasma physics, quantum field theory, and engineering.

Studying these equations is a daunting task because there are no general methods for solving them. Each problem requires an appropriate approach depending on the type of linearity ([1–10]).

The p-Laplacian operator is a model of quasi-linear elliptic operators which makes it possible to model physical phenomena such as the flow of non-Newtonian aids, reaction flow systems, nonlinear elasticity, the extraction of petroleum, astronomy, through porous media, and glaciology. Several authors in this field obtained many results of existence (see, for example, [1, 3, 5, 11, 12]).

In this work, we consider the following quasi-linear elliptic system:

$$\begin{cases} -A\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = \lambda u^\alpha v^\gamma, & \text{in } \Omega, \\ -B\left(\int_\Omega |\nabla v|^2 \mathrm{d}x\right)\Delta v = \lambda u^\delta v^\beta, & \text{in } \Omega, \\ v = u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N\ (N \geq 3)$ is a bounded domain and its boundary $\partial\Omega$. Also, A and B are two continuous functions on $\mathbb{R}^+$, and the parameters α, β, δ, and γ satisfy the following conditions:

$$\begin{cases} 0 \leq \alpha < 1, \\ 0 \leq \beta < 1, \\ \delta, \gamma > 0, \\ \theta = (1-\alpha)(1-\beta) - \gamma\delta > 0 \text{ for each } \lambda > 0. \end{cases} \quad (2)$$

Within previous studies [13–15], some nonlocal elliptical problems of the Kirchhoff type of the following model were extensively studied:

$$\begin{cases} M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\triangle u = h(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \quad (3)$$

where Ω is a bounded open domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $h(x,u)$ the right hand side is defined for some exceptional functions similar to those in [13–16]. In addition, M is a defined and continuous function on $\mathbb{R}_+$ with values in $\mathbb{R}_+^*$. In recent years, various Kirchhoff or $p(x)$-Kirchhoff-type problems have been widely studied by many authors due to their theoretical and practical importance. Such problems are often referred to as nonlocal due to the presence of a full term on Ω or in $\mathbb{R}^n$. It is well known that this problem is analogous to the stationary problem of a model introduced by Kirchhoff [17].

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0. \tag{4}$$

More specifically, Kirchhoff proposed this model as an extension of the wave equation of the Alembert classic by considering the effects of variations in the length of the strings during vibration. The parameters of the above equation have the following meanings: E is Young's modulus of the material, ρ is the mass density, L is the length of the chain, h is the section area, and P_0 is the initial tension.

In recent work in [18], we have discussed the existence of the weak positive solution for the following Kirchhoff elliptic systems:

$$\begin{cases} -A\left(\|\nabla u\|_{L^2(\Omega)}\right)\triangle u=\lambda_1 u^\alpha+\mu_1' v^\beta, & \text{in }\Omega,\\ -B\left(\|\nabla u\|_{L^2(\Omega)}\right)\triangle v=\lambda_2' u^c+\mu_2' v^d, & \text{in }\Omega,\\ u=v=0, & \text{on }\partial\Omega, \end{cases} \tag{5}$$

where λ_1, μ_1', λ_2', and μ_2' are positive parameters, $\alpha+c<1$, and $\beta+d<1$.

Motivated by the recent work in [13, 14, 18, 19] and by using the sub- and supersolution method which is defined in [20], existence of positive solutions of quasi-linear Kirchhoff elliptic systems is shown in bounded smooth domains.

The paper outline is as follows: some assumptions and definitions related to problem (1) are given in Section 2. Finally, our main result is given in Section 3.

2. Preliminaries and Assumptions

We assume the following hypothesis:

($H1$): we assume that $M\colon \mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a nonincreasing and continuous function which satisfies

$$\lim_{t\longrightarrow 0^+} M(t)=m_0, \tag{6}$$

where $m_0>0$, and there exists $a_i, b_i>0, i=1,2$ such that

$$a_1\le A(t)\le a_2,\ b_1\le B(t)\le b_2 \quad \text{for all t}\in\mathbb{R}^+. \tag{7}$$

($H2$): and

$$\begin{aligned}&\alpha,\beta\in C(\overline{\Omega}),\\ &\alpha(x)\ge\alpha_0>0,\beta(x)\ge\beta_0>0\end{aligned} \tag{8}$$

for all $x\in\Omega$.

($H3$): f, g, h, and τ are C^1 on $(0,+\infty)$ and increasing functions, where

$$\begin{cases}\lim_{t\longrightarrow+\infty} f(t)=+\infty, \lim_{t\longrightarrow+\infty} g(t)=+\infty,\\ \lim_{t\longrightarrow+\infty} h(t)=+\infty=\lim_{t\longrightarrow+\infty}\tau(t)=+\infty.\end{cases} \tag{9}$$

($H4$): $\exists\gamma>0$ such that

$$\begin{aligned}&\lim_{t\longrightarrow+\infty}\frac{h(t)f\left(k\left[g(t)^\gamma\right]\right)}{t}=0, \quad \text{for all } k>0,\\ &\lim_{t\longrightarrow+\infty}\frac{\tau\left(kt^\gamma\right)}{t^{\gamma-1}}=0, \quad \text{for all } k>0.\end{aligned} \tag{10}$$

Lemma 1 (see [14]). *Under assumption ($H1$),we suppose further that function $H(t):=tM(t^2)$ is increasing on $\mathbb{R}$.*

We assume that u and v are couple nonnegative functions, where

$$\begin{cases} -M\left(\int_\Omega|\nabla u|^2 dx\right)\Delta u\ge -M\left(\int_\Omega|\nabla v|^2 dx\right)\Delta v, & \text{in }\Omega,\\ u=v=0, \quad \text{on }\partial\Omega, \end{cases} \tag{11}$$

and then $u\ge v$ a.e. in Ω.

Lemma 2 (see [1]). *If M verifies the conditions of Lemma 1, then for each $f\in L^2(\Omega)$, there exists a unique solution $u\in H_0^1(\Omega)$ to the M-linear problem:*

$$-M\left(\int_\Omega|\nabla u|^2 dx\right)\Delta u=f(x)\text{ in }\Omega\text{ and }u=0\text{ in }\partial\Omega. \tag{12}$$

Lemma 3 (see [1]). *Let w solve $\Delta w=g$ in Ω. If $g\in C(\Omega)$, then $w\in C^{1,\alpha}(\Omega)$ for any $\alpha\in(0,1)$, so particularly, w is continuous in Ω.*

Definition 1. Let $(u,v)\in(H_0^1(\Omega)\cap L^\infty(\Omega)\times H_0^1(\Omega)\cap L^\infty(\Omega))$, and (u,v) is said a weak solution of (1) if it satisfies

$$\begin{aligned}&A\left(\int_\Omega|\nabla u|^2 dx\right)\int_\Omega\nabla u\nabla\phi dx=\lambda\int_\Omega u^\alpha v^\gamma\phi dx, \quad \text{in }\Omega,\\ &B\left(\int_\Omega|\nabla v|^2 dx\right)\int_\Omega\nabla v\nabla\psi dx=\lambda\int_\Omega u^\delta v^\beta\,\psi dx, \quad \text{in }\Omega,\end{aligned} \tag{13}$$

for all $(\phi,\psi)\in(H_0^1(\Omega)\times H_0^1(\Omega))$.

Definition 2. We call the following nonnegative functions $(\underline{u},\underline{v})$, respectively; $(\overline{u},\overline{v})$ in $(H_0^1(\Omega)\cap L^\infty(\Omega)\times H_0^1(\Omega)\cap L^\infty(\Omega))$ are a weak subsolution (respectively, upersolution) of (1) if they verify $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})=(0,0)$ in $\partial\Omega$:

$$\begin{aligned}
&A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\int_\Omega \nabla \underline{u}\,\nabla\phi dx \le \lambda\int_\Omega \underline{u}^\alpha\,\underline{v}^\gamma\phi dx \text{ in } \Omega,\\
&B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\int_\Omega \nabla \underline{v}\,\nabla\psi dx \le \lambda\int_\Omega \underline{u}^\delta\,\underline{v}^\beta\,\psi dx \text{ in } \Omega,\\
&A\left(\int_\Omega |\nabla \overline{u}|^2 dx\right)\int_\Omega \nabla \overline{u}\nabla\phi dx \ge \lambda\int_\Omega \overline{u}^\alpha\overline{v}^\gamma\phi dx \text{ in } \Omega,\\
&B\left(\int_\Omega |\nabla \overline{v}|^2 dx\right)\int_\Omega \nabla \overline{v}\nabla\psi dx \ge \lambda\int_\Omega \overline{u}^\delta\overline{v}^\beta\,\psi dx \text{ in } \Omega,
\end{aligned} \tag{14}$$

for all $(\phi,\psi) \in (H_0^1(\Omega)\times H_0^1(\Omega))$.

Before proving our main result, we need to prove the existence of weak supersolution and subsolution in the following section.

3. Weak Existence Results

3.1. Existence of Weak Supersolution. The existence of a positive weak supersolution for system (1) is established such that each component belongs to $C^{0,\rho}(\overline{\Omega})$, for $\rho \in (0,1)$.

Lemma 4. *Suppose that $(H1)$ holds, $0 \le \alpha,\beta < 1$, $\delta,\gamma > 0$, and $\theta = (1-\alpha)(1-\beta) - \gamma\delta > 0$. Then, system (1) possesses a positive weak supersolution*

$$(\overline{u},\overline{v}) \in L^2\left(0,T,C^{0,\rho_1}(\overline{\Omega})\right)\times L^2\left(0,T,C^{0,\rho_2}(\overline{\Omega})\right), \tag{15}$$

for $\rho_i \in [0,1]$, $i = 1,2$ and $\lambda > 0$.

Proof. Let $e_i \in C^{0,\rho_i}(\overline{\Omega})$, for $i = 1,2$, $\rho_i > 0$, be the solution of the following problem:

$$\begin{cases} -\triangle e_i = 1, & \text{in } \Omega,\\ e_i = 0, & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Then, by the strong maximum principle, we get $e_i > 0$ in Ω, $i = 1,2$.

We define

$$(\overline{u},\overline{v}) = (C_1e_1, C_2e_2), \tag{17}$$

where C_1 and C_2 are positive constants which we will fix them later.

Let $(\phi,\psi) \subset (H_0^1(\Omega)\times H_0^1(\Omega))$, with $(\phi,\psi) \ge 0$.

Then, we obtain

$$\begin{aligned}
A\left(\int_\Omega |\nabla\overline{u}|^2 dx\right)\int_\Omega \nabla\overline{u}\nabla\phi dx &= A\left(\int_\Omega |\nabla\overline{u}|^2 dx\right)C_1\int_\Omega \nabla e_1\nabla\phi dx\\
&= A\left(\int_\Omega |\nabla\overline{u}|^2 dx\right)C_1\int_\Omega \phi dx\\
&\ge a_1C_1\int_\Omega \phi dx,
\end{aligned} \tag{18}$$

and similarly,

$$\begin{aligned}
B\left(\int_\Omega |\nabla\overline{v}|^2 dx\right)\int_\Omega \nabla\overline{v}\nabla\psi dx &= B\left(\int_\Omega |\nabla\overline{v}|^2 dx\right)C_2\int_\Omega \psi dx\\
&\ge b_1C_2\int_\Omega \psi dx.
\end{aligned} \tag{19}$$

If

$$\begin{aligned}
&l = \|e_1\|_\infty, \quad L = \|e_2\|_\infty,\\
&0 \le \alpha < 1, \quad 0 \le \beta < 1,\\
&\lambda > 0, \quad \theta > 0,
\end{aligned} \tag{20}$$

and $(H1)$ holds, it is easy to prove that there exist positive constants C_1 and C_2 such that

$$\begin{aligned}
a_1C_1^{1-\alpha} &= \lambda C_2^\gamma l^\alpha L^\gamma,\\
b_1C_2^{1-\beta} &= \lambda C_1^\gamma l^\delta L^\beta.
\end{aligned} \tag{21}$$

Thus, from (21), we obtain for all $x \in \Omega$

$$\begin{aligned}
\lambda\overline{u}^\alpha\overline{v}^\gamma &\le \lambda C_1^\alpha C_2^\gamma l^\alpha L^\gamma \le a_1C_1,\\
\lambda\overline{u}^\delta\overline{v}^\beta &\le \lambda C_1^\delta C_2^\beta l^\delta L^\beta \le b_1C_2.
\end{aligned} \tag{22}$$

Therefore, by using (18), (19), and (22), we conclude that

$$\begin{aligned}
&A\left(\int_\Omega |\nabla\overline{u}|^2 dx\right)\int_\Omega \nabla\overline{u}\nabla\phi dx \ge \lambda\int_\Omega \overline{u}^\alpha\overline{v}^\gamma\phi dx, \quad \text{in } \Omega,\\
&B\left(\int_\Omega |\nabla\overline{v}|^2 dx\right)\int_\Omega \nabla\overline{v}\nabla\psi dx \ge \lambda\int_\Omega \overline{u}^\delta\overline{v}^\beta\psi dx, \quad \text{in } \Omega.
\end{aligned} \tag{23}$$

Hence, $(\overline{u},\overline{v}) \in C^{0,\rho_1}(\overline{\Omega})\times C^{0,\rho_2}(\overline{\Omega})$ is a positive weak supersolution of system (1). □

3.2. Existence of Weak Subsolution. Existence of a positive weak subsolution for system (1) is proved such that each component belongs to $C^0(\overline{\Omega})$.

Lemma 5. *We assume that $(H1)$ holds:*

$$\begin{aligned}
&0 \le \alpha,\beta < 1, \delta,\gamma > 0,\\
&\theta = (1-\alpha)(1-\beta) - \gamma\delta > 0.
\end{aligned} \tag{24}$$

Therefore, system (1) possesses a positive weak subsolution $(\underline{u},\underline{v}) \in C^0(\overline{\Omega})\times C^0(\overline{\Omega})$, for all $\lambda > 0$.

Proof. We assume that λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet condition with ϕ_1 which is its corresponding eigenfunction and $\phi_1$belongs to $C^{0,\rho_1}(\overline{\Omega})\times C^{0,\mu_1}(\overline{\Omega})$, $\phi_1 > 0$ in Ω and $|\nabla\phi_1| \ge \sigma_1$ on $\partial\Omega$, for some positive constants σ_1,μ_1, and ρ_1.

We define

$$(\underline{u},\underline{v}) = \left(c\phi_1^2, c^k\phi_1^2\right) \tag{25}$$

which belongs to $(C^0(\overline{\Omega})\cap C^1(\overline{\Omega}))\times(C^0(\overline{\Omega})\cap C^1(\overline{\Omega}))$, with $c > 0$ to be fixed later, and

$$\frac{\delta}{1-\beta} < k < \frac{1-\alpha}{\gamma} \tag{26}$$

because $\theta > 0$, $1-\alpha > 0$, and $1-\beta > 0$. Then, for all $(\phi, \psi) \in (H_0^1(\Omega) \times H_0^1(\Omega))$, with ϕ, $\psi \geq 0$, we have

$$\begin{aligned} A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\int_\Omega \nabla \underline{u}\, \nabla\phi dx &= 2cA\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\int_\Omega \phi_1 \nabla\phi_1 \nabla\phi, \\ &= 2cA\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\int_\Omega \left[\lambda_1\phi_1^2 - |\nabla\phi_1|^2\right]\phi dx. \end{aligned} \tag{27}$$

Similarly,

$$\begin{aligned} B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\int_\Omega \nabla \underline{v}\, \nabla\psi dx &= 2c^k B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\int_\Omega \\ &\quad \cdot \left[\lambda_1\phi_1^2 - |\nabla\phi_1|^2\right]\psi dx. \end{aligned} \tag{28}$$

Since $\phi_1 = 0$ and $|\nabla\phi_1| \geq \sigma_1$ on $\partial\Omega$, there exists $\eta > 0$ such that, for every $x \in \Omega_\eta = \{x \in \Omega\colon d(x, \partial\Omega) \leq \eta\}$, we have

$$\begin{aligned} \left[\lambda_1\phi_1^2 - |\nabla\phi_1|^2\right] &\leq 0, \\ \left[\lambda_1\psi_1^2 - |\nabla\psi_1|^2\right] &\leq 0. \end{aligned} \tag{29}$$

Then, for each $\lambda > 0$, we get

$$A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\int_{\Omega_\eta} \nabla \underline{u}\, \nabla\phi dx \leq 0 \leq \lambda \int_{\Omega_\eta} \underline{u}^\alpha\, \underline{u}^\gamma \phi dx, \tag{30}$$

for all $\phi \in H_0^1(\Omega)$, $\phi \geq 0$, and

$$B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\int_{\Omega_\eta} \nabla \underline{v}\, \nabla\psi dx \leq 0 \leq \lambda \int_{\Omega_\eta} \underline{u}^\delta\, \underline{u}^\beta \psi dx, \tag{31}$$

for all $\psi \in H_0^1(\Omega)$ and $\psi \geq 0$.

Now, as $\phi_1 > 0$ in Ω and ϕ_1 is continuous, then there exists $\mu > 0$ such that $\phi_1(x) \geq \mu > 0$ for all $x \in \Omega\backslash\overline{\Omega}_\eta$. Therefore, from (26), we obtain $a_0 > 0$ such that the following inequalities hold:

$$2b_2\lambda_1 c^{k(1-\beta)-\delta}\phi_1^{2-2\beta}(x) \leq \lambda\mu^{2\delta} \leq \lambda\phi_1^{2\delta}(x), \quad \forall x \in \frac{\Omega}{\overline{\Omega}_\eta}, \tag{32}$$

$$2a_2\lambda_1 c^{1-\alpha-k\gamma}\phi_1^{2-2\alpha}(x) \leq \lambda\mu^{2\gamma} \leq \lambda\phi_1^{2\gamma}(x), \quad \forall x \in \frac{\Omega}{\overline{\Omega}_\eta}, \tag{33}$$

for each $c \in (0, a_0)$.

Then,

$$\begin{aligned} &2cA\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\left[\lambda_1\phi_1^2 - |\nabla\phi_1|^2\right]\phi \\ &\quad \leq 2a_2 c\lambda_1\phi_1^2 \\ &\quad = 2a_2\lambda_1 c^{1-\alpha-k\gamma}\phi_1^{2-2\alpha}\left[c^{k\gamma}c^\alpha\phi_1^{2\alpha}\right]. \end{aligned} \tag{34}$$

By (33), we have

$$\begin{aligned} 2cA\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\left[\lambda_1\phi_1^2 - |\nabla\phi_1|^2\right] &\leq \lambda\phi_1^{2\gamma}c^{k\gamma}c^\alpha\phi_1^{2\alpha} \\ &= \lambda\, \underline{u}^\alpha\, \underline{v}^\gamma. \end{aligned} \tag{35}$$

And similarly, from (32), we have

$$\begin{aligned} &2c^k B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\left[\lambda_1\phi_1^2 - |\nabla\phi_1|^2\right] \\ &\quad \leq \lambda\, \underline{u}^\delta\, \underline{v}^\beta \end{aligned} \tag{36}$$

in $\Omega/\overline{\Omega}_\eta$ and each $c \in (0, a_0)$.

Therefore,

$$A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\int_{\Omega/\overline{\Omega}_\eta} \nabla \underline{u}\, \nabla\phi dx \leq \lambda \int_{\Omega/\overline{\Omega}_\eta} \underline{u}^\alpha\, \underline{v}^\gamma \phi dx, \tag{37}$$

$$B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\int_{\Omega/\overline{\Omega}_\eta} \nabla \underline{v}\, \nabla\psi dx \leq \lambda \int_{\Omega/\overline{\Omega}_\eta} \overset{\delta\beta}{\underline{u}\underline{v}}\psi dx. \tag{38}$$

Hence, from (30), (31), (37), and (38), it follows that

$$\begin{cases} A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\left[\int_{\Omega_\eta} \nabla \underline{u}\, \nabla\phi dx + \int_{\Omega/\overline{\Omega}_\eta} \nabla \underline{u}\, \nabla\phi dx\right] \\ = A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\int_\Omega \nabla \underline{u}\, \nabla\phi dx \leq \int_\Omega \underline{u}^\alpha \underline{v}^\gamma \phi dx, \end{cases} \tag{39}$$

$$\begin{aligned} &B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\left[\int_{\Omega_\eta} \nabla \underline{v}\, \nabla\psi dx + \int_{\Omega/\overline{\Omega}_\eta} \nabla \underline{v}\, \nabla\psi dx\right] \\ &\quad = B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\int_\Omega \nabla \underline{v}\, \nabla\psi dx \leq \lambda \int_\Omega \underline{u}^\delta\, \underline{u}^\beta \psi dx. \end{aligned} \tag{40}$$

Then, by (39) and (40), $(\underline{u}, \underline{v})$ is a positive weak subsolution of system (1), for each $c \in (0, a_0)$. □

4. Main Result

In this section, we give the result of the existence of the positive weak solution to quasi-linear elliptic system (1) by using the sub- and supersolution method which has been already used for some classical elliptic equations by known authors (see [1, 4, 11, 19, 21]).

Theorem 1. *Suppose that* $(H1)$ *holds,* $0 \leq \alpha, \beta < 1, \delta, \gamma > 0$, *and* $\theta = (1-\alpha)(1-\beta) - \gamma\delta > 0$ *as well as under the results of Lemma 4 and 5. Then, system (1) possesses a weak solution* $(u, v) \in (H_0^1(\Omega) \times H_0^1(\Omega))$, *where each component is positive and belongs to* $C^{0,\rho}(\overline{\Omega}) \cap C^{1,\mu}(\Omega)$ *for some* $\rho \in [0, 1]$, $\mu > 0$, *and each* $\lambda > 0$.

Proof 3. In order to obtain a weak solution of problem (1), we shall use the arguments by Azzouz and Bensedik [13]. For this purpose, we define a sequence $\{(u_n, v_n)\} \subset (H_0^1(\Omega) \times H_0^1(\Omega))$ as follows: $u_0 := \overline{u}$, $v_0 = \overline{v}$, and (u_n, v_n) is the unique solution of the system

$$\begin{cases} -A\left(\int_\Omega |\nabla u_n|^2 dx\right)\triangle u_n = \lambda u_{n-1}^{\alpha} v_{n-1}^{\gamma}, & \text{in } \Omega, \\ -B\left(\int_\Omega |\nabla v_n|^2 dx\right)\triangle v_n = \lambda u_{n-1}^{\delta} v_{n-1}^{\beta}, & \text{in } \Omega, \\ u_n = v_n = 0, & \text{on } \partial\Omega. \end{cases} \quad (41)$$

Problem (41) is (A, B)–linear in the sense that if

$$\left(u_{n-1}, v_{n-1}\right) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right) \quad (42)$$

is given, the right-hand sides of (41) are independent of u_n, v_n.

Set

$$A(t) = tA\left(t^2\right), \quad B(t) = tB\left(t^2\right). \quad (43)$$

Then, since

$$A(\mathbb{R}) = \mathbb{R},\ B(\mathbb{R}) = \mathbb{R}, \quad f\left(u_{n-1}, v_{n-1}\right) = u_{n-1}^{\alpha} v_{n-1}^{\gamma} \in L^2(\Omega), \quad g\left(u_{n-1}, v_{n-1}\right) = u_{n-1}^{\delta} v_{n-1}^{\beta} \in L^2(\Omega). \quad (44)$$

According to the result in [1], we can deduce that system (41) admits a unique solution

$$\left(u_n, v_n\right) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right). \quad (45)$$

By using (41) and the fact that (u_0, v_0) is a supersolution of (1), we have

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_0|^2 dx\right)\triangle u_0 &\geq \lambda u_0^{\alpha} v_0^{\gamma} = -A\left(\int_\Omega |\nabla u_1|^2 dx\right)\triangle u_1, \\ -B\left(\int_\Omega |\nabla v_0|^2 dx\right)\triangle v_0 &\geq \lambda u_0^{\delta} v_0^{\beta} = -B\left(\int_\Omega |\nabla v_1|^2 dx\right)\triangle v_1. \end{aligned} \quad (46)$$

Also, by using Lemma 1, $u_0 \geq u_1$ and $v_0 \geq v_1$, and since $u_0 \geq \underline{u}$, $v_0 \geq \underline{v}$, and the monotonicity of $f(u, v) = u^{\alpha} v^{\gamma}$ and $g(u, v) = u^{\delta} v^{\beta}$, one has

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_1|^2 dx\right)\Delta u_1 &= \lambda u_0^{\alpha} v_0^{\gamma} \geq \lambda \underline{u}\underline{v}^{\alpha\gamma} \geq -A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\Delta \underline{u}, \\ -B\left(\int_\Omega |\nabla v_1|^2 dx\right)\Delta v_1 &= \lambda u_0^{\delta} v_0^{\beta} \geq \lambda \underline{u}\underline{v}^{\delta\beta} \geq -B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\Delta \underline{v}, \end{aligned} \quad (47)$$

from which, according to Lemma 1, $u_1 \geq \underline{u}$ and $v_1 \geq \underline{v}$. For u_2, v_2, we write

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_1|^2 dx\right)\Delta u_1 &= \lambda u_0^{\alpha} v_0^{\gamma} \geq \lambda u_1^{\alpha} v_1^{\gamma} = -A\left(\int_\Omega |\nabla u_2|^2 dx\right)\Delta u_2, \\ -B\left(\int_\Omega |\nabla v_1| dx\right)\Delta v_1 &= \lambda u_0^{\delta} v_0^{\beta} \geq \lambda u_1^{\delta} v_1^{\beta} = -B\left(\int_\Omega |\nabla v_2|^2 dx\right)\Delta v_2, \end{aligned} \quad (48)$$

and then $u_1 \geq u_2$ and $v_1 \geq v_2$. Similarly, $u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ because

$$-A\left(\int_\Omega |\nabla u_2|^2 dx\right)\Delta u_2 = \lambda u_1^{\alpha} v_1^{\gamma} \geq \lambda u_1^{\alpha} \underline{v}_1^{\gamma} \geq -A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\Delta \underline{u},$$

$$-B\left(\int_\Omega |\nabla v_2|^2 dx\right)\Delta v_2 = \lambda u_1^{\delta} v_1^{\beta} \geq \lambda \underline{u}_1^{\delta} \underline{v}_1^{\beta} \geq -B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\Delta \underline{v}. \quad (49)$$

Repeating this argument, we get a bounded monotone sequence $\{(u_n, v_n)\} \subset (H_0^1(\Omega) \times H_0^1(\Omega))$ satisfying

$$\overline{u} = u_0 \geq u_1 \geq u_2 \geq \ldots \geq u_n \geq \ldots \geq \underline{u} > 0, \quad (50)$$

$$\overline{v} = v_0 \geq v_1 \geq v_2 \geq \ldots \geq v_n \geq \ldots \geq \underline{v} > 0. \quad (51)$$

Using the continuity of the functions f and g and the definition of the sequence $\{u_n\}, \{v_n\}$, there exist constants $C_i > 0$, $i = 1, \ldots 4$, independent of n such that

$$\left|f\left(u_{n-1}, v_{n-1}\right)\right| \leq C_1, \quad \left|g\left(u_{n-1}, v_{n-1}\right)\right| \leq C_2, \quad \text{for all } n. \quad (52)$$

From (52), we multiply the first equation of (41) by u_n; in addition, by using the Holder inequality combined with Sobolev embedding, we have

$$\begin{aligned} a_1 \int_\Omega |\nabla u_n|^2 dx &\leq A\left(\int_\Omega |\nabla u_n|^2 dx\right) \int_\Omega |\nabla u_n|^2 dx \\ &= \lambda \int_\Omega f\left(u_{n-1}, v_{n-1}\right) u_n dx \\ &\leq \lambda \int_\Omega \left|f\left(u_{n-1}, v_{n-1}\right)\right| |u_n| dx \\ &\leq C_1 \lambda \left(\int_\Omega |u_n|^2\right)^{(1/2)} dx \\ &\leq C_3 \|u_n\|_{H_0^1(\Omega)} \end{aligned} \quad (53)$$

or $\|u_n\|_{H_0^1(\Omega)} \leq C_3, \forall n,$

where $C_3 > 0$ is a constant independent of n. Similarly, there exists $C_2 > 0$ independent of n such that

$$\|v_n\|_{H_0^1(\Omega)} \leq C_4, \forall n. \quad (54)$$

From (53) and (54), we deduce that the couple $\{(u_n, v_n)\}$ converges weakly in $H_0^1(\Omega, \mathbb{R}^2)$ to the couple $(u, v)$with$u \geq \underline{u} > 0$ and $v \geq v > 0$.

By using a standard regularity argument, $\{(u_n, v_n)\}$ converges to (u, v). Thus, when $n \longrightarrow +\infty$ in (41), we can see that (u, v) is a positive solution of system (1).

The proof is completed. □

5. Conclusion

As a conclusion of this contribution, we have proved the existence of positive solutions of quasi-linear Kirchhoff elliptic systems in bounded smooth domains by using the sub- and super-solution method [20], which is an extension of our recent works of Boulaaras et al. in [18]. In the next work, some other methods such as variational and Galerkin methods (see, for example, [15]) will be used for this problem, and some numerical examples will also be given [9, 22].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

Authors' Contributions

The authors contributed equally to this article. They have all read and approved the final manuscript.

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