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Research Article

On a System of *k*-Difference Equations of Order Three

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In this paper, we deal with the global behavior of the positive solutions of the system of k-difference equations $u_{n+1}^{(1)} = (\alpha_1 u_{n-1}^{(1)}/\beta_1 + \alpha_1 (u_{n-2}^{(2)})^{r_1})$, $u_{n+1}^{(2)} = \alpha_2 u_{n-1}^{(2)}/\beta_2 + \alpha_2 (u_{n-2}^{(3)})^{r_2}, \ldots, u_{n+1}^{(k)} = \alpha_k u_{n-1}^{(k)}/\beta_k + \alpha_k (u_{n-2}^{(1)})^{r_k}, n \in \mathbb{N}_0$, where the initial conditions $u_{-l}^{(i)}\,(l = 0, 1, 2)$ are nonnegative real numbers and the parameters $\alpha_i, \beta_i, \gamma_i$, and r_i are positive real numbers for $i = 1, 2, \ldots, k$, by extending some results in the literature. By the end of the paper, we give three numerical examples to support our theoretical results related to the system with some restrictions on the parameters.

1. Introduction

Recently, many works have been published on rational difference equations, which have an important position in applied sciences. In this process, many rational difference equations have been studied by mathematicians. And so, some equations have frequently been the subject of many articles using generalizations. Many typical examples of these can be found in the literature. For example, in [1], El-Owaidy et al. dealt with global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{p}}, \quad n \in \mathbb{N}_0, \tag{1}$$

with nonnegative parameters and initial conditions. Gumus and Soykan [2] dealt with the dynamical behavior of the positive solutions for a system of rational difference equations of the following form:

$$\begin{aligned} u_{n+1} &= \frac{\alpha u_{n-1}}{\beta + \gamma v_{n-2}^{p}}, \\ -4pt \\ v_{n+1} &= \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_{n-2}^{p}}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{2}$$

where the parameters and initial conditions are positive real numbers. Tollu and Yalcinkaya [3] dealt with the dynamical behavior of the positive solutions for the following three-dimensional system of rational difference equations:

$$\begin{aligned} u_{n+1} &= \frac{\alpha_1 u_{n-1}}{\beta_1 + \gamma_1 v_{n-2}^{p}}, \\ v_{n+1} &= \frac{\alpha_2 v_{n-1}}{\beta_2 + \gamma_2 w_{n-2}^{q}}, \\ w_{n+1} &= \frac{\alpha_3 w_{n-1}}{\beta_3 + \gamma_3 u_{n-2}^{r}}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{3}$$

where the parameters and initial conditions are positive real numbers. For more papers on this topic, see, for example, [4–29].

In the present paper, we investigate the global behavior of the positive solutions of the k-dimensional system of difference equations:

$$u_{n+1}^{(1)} = \frac{\alpha_1 u_{n-1}^{(1)}}{\beta_1 + \gamma_1 \left(u_{n-2}^{(2)}\right)^{r_1}}, u_{n+1}^{(2)} = \frac{\alpha_2 u_{n-1}^{(2)}}{\beta_2 + \gamma_2 \left(u_{n-2}^{(3)}\right)^{r_2}}, \dots, u_{n+1}^{(k)} = \frac{\alpha_k u_{n-1}^{(k)}}{\beta_k + \gamma_k \left(u_{n-2}^{(1)}\right)^{r_k}}, \quad n \in \mathbb{N}_0, \tag{4}$$

where the initial conditions $u_{-l}^{(i)}$ $(l = 0, 1, 2)$ are nonnegative real numbers and the parameters $\alpha_i, \beta_i, \gamma_i$, and r_i are positive real numbers for $i = 1, 2, \dots, k$, by extending some recent results in the literature.

Remark 1. This paper extends the results of studies in the references [1–3]. That is to say, if we take $k = 1$, then system (4) reduces equation (1). If we take $k = 2$, then system (4) reduces system (2). Finally, if we take $k = 3$, then system (4) reduces system (3). So, system (4) is a natural generalization of equation (1), system (2), and system (3).

Note that system (4) can be written as

$$x_{n+1}^{(1)} = \frac{a_1 x_{n-1}^{(1)}}{1 + \left(x_{n-2}^{(2)}\right)^{r_1}}, x_{n+1}^{(2)} = \frac{a_2 x_{n-1}^{(2)}}{1 + \left(x_{n-2}^{(3)}\right)^{r_2}}, \dots, x_{n+1}^{(k)} = \frac{a_k x_{n-1}^{(k)}}{1 + \left(x_{n-2}^{(1)}\right)^{r_k}}, \quad n \in \mathbb{N}_0, \tag{5}$$

by the change of variables $u_n^{(1)} = (\beta_k/\gamma_k)^{1/r_k} x_n^{(1)}$, $u_n^{(2)} = (\beta_1/\gamma_1)^{(1/r_1)} x_n^{(2)}, \dots, u_n^{(k)} = (\beta_{k-1}/\gamma_{k-1})^{(1/r_{k-1})} x_n^{(k)}$ with $a_i = (\alpha_i/\beta_i)$ for $i = 1, 2, \dots k$. So, we will consider system (5) instead of system (4) from now.

2. Preliminaries

Let $I_1, I_2, \dots, I_k$ be some intervals of real numbers and $f_1\colon I_1^3 \times I_2^3 \times \cdots \times I_k^3 \longrightarrow I_1$, $f_2\colon I_1^3 \times I_2^3 \times \cdots \times I_k^3 \longrightarrow I_2$, $\dots, f_k\colon I_1^3 \times I_2^3 \times \cdots \times I_k^3 \longrightarrow I_k$ be continuously differentiable functions. Then, for initial conditions $(u_0^{(1)}, u_{-1}^{(1)}, u_{-2}^{(1)}, u_0^{(2)}, u_{-1}^{(2)}, u_{-2}^{(2)}, \dots, u_0^{(k)}, u_{-1}^{(k)}, u_{-2}^{(k)}) \in I_1^3 \times I_2^3 \times \cdots \times I_k^3$, the system of difference equations,

$$\begin{cases} u_{n+1}^{(1)} = f_1\left(u_n^{(1)}, u_{n-1}^{(1)}, u_{n-2}^{(1)}, u_n^{(2)}, u_{n-1}^{(2)}, u_{n-2}^{(2)}, \dots, u_n^{(k)}, u_{n-1}^{(k)}, u_{n-2}^{(k)}\right), \\ u_{n+1}^{(2)} = f_2\left(u_n^{(1)}, u_{n-1}^{(1)}, u_{n-2}^{(1)}, u_n^{(2)}, u_{n-1}^{(2)}, u_{n-2}^{(2)}, \dots, u_n^{(k)}, u_{n-1}^{(k)}, u_{n-2}^{(k)}\right), \\ \vdots \\ u_{n+1}^{(k)} = f_k\left(u_n^{(1)}, u_{n-1}^{(1)}, u_{n-2}^{(1)}, u_n^{(2)}, u_{n-1}^{(2)}, u_{n-2}^{(2)}, \dots, u_n^{(k)}, u_{n-1}^{(k)}, u_{n-2}^{(k)}\right), \end{cases} \quad n \in \mathbb{N}_0, \tag{6}$$

has the unique solution $\{(u_n^{(1)}, u_n^{(2)}, \dots, u_n^{(k)})\}_{n=-2}^{\infty}$. Also, an equilibrium point of system (6) is a point $(\overline{u}^{(1)}, \overline{u}^{(2)}, \dots, \overline{u}^{(k)})$ that satisfies the following system:

$$\begin{aligned} \overline{u}^{(1)} &= f_1\left(\overline{u}^{(1)}, \overline{u}^{(1)}, \overline{u}^{(1)}, \overline{u}^{(2)}, \overline{u}^{(2)}, \overline{u}^{(2)}, \dots, \overline{u}^{(k)}, \overline{u}^{(k)}, \overline{u}^{(k)}\right), \\ \overline{u}^{(2)} &= f_2\left(\overline{u}^{(1)}, \overline{u}^{(1)}, \overline{u}^{(1)}, \overline{u}^{(2)}, \overline{u}^{(2)}, \overline{u}^{(2)}, \dots, \overline{u}^{(k)}, \overline{u}^{(k)}, \overline{u}^{(k)}\right), \\ &\vdots \\ \overline{u}^{(k)} &= f_k\left(\overline{u}^{(1)}, \overline{u}^{(1)}, \overline{u}^{(1)}, \overline{u}^{(2)}, \overline{u}^{(2)}, \overline{u}^{(2)}, \dots, \overline{u}^{(k)}, \overline{u}^{(k)}, \overline{u}^{(k)}\right). \end{aligned} \tag{7}$$

We rewrite system (6) in the vector form

$$U_{n+1} = F(U_n), \quad n \in \mathbb{N}_0, \tag{8}$$

where $U_n = (u_n^{(1)}, u_{n-1}^{(1)}, u_{n-2}^{(1)}, u_n^{(2)}, u_{n-1}^{(2)}, u_{n-2}^{(2)}, \dots, u_n^{(k)}, u_{n-1}^{(k)}, u_{n-2}^{(k)})^T$, F is a vector map such that $F\colon I_1^3 \times I_2^3 \times \cdots \times I_k^3 \longrightarrow I_1^3 \times I_2^3 \times \cdots \times I_k^3$, and

$$F\left(\begin{pmatrix} v_0^{(1)} \\ v_1^{(1)} \\ v_2^{(1)} \\ v_0^{(2)} \\ v_1^{(2)} \\ v_2^{(2)} \\ \vdots \\ v_0^{(k)} \\ v_1^{(k)} \\ v_2^{(k)} \end{pmatrix}\right) = \begin{pmatrix} f_1\left(v_0^{(1)}, v_1^{(1)}, v_2^{(1)}, v_0^{(2)}, v_1^{(2)}, v_2^{(2)}, \dots, v_0^{(k)}, v_1^{(k)}, v_2^{(k)}\right) \\ v_0^{(1)} \\ v_1^{(1)} \\ f_2\left(v_0^{(1)}, v_1^{(1)}, v_2^{(1)}, v_0^{(2)}, v_1^{(2)}, v_2^{(2)}, \dots, v_0^{(k)}, v_1^{(k)}, v_2^{(k)}\right) \\ v_0^{(2)} \\ v_1^{(2)} \\ \vdots \\ f_k\left(v_0^{(1)}, v_1^{(1)}, v_2^{(1)}, v_0^{(2)}, v_1^{(2)}, v_2^{(2)}, \dots, v_0^{(k)}, v_1^{(k)}, v_2^{(k)}\right) \\ v_0^{(k)} \\ v_1^{(k)} \end{pmatrix}. \tag{9}$$

It is clear that if an equilibrium point of system (6) is $(\overline{u}^{(1)}, \overline{u}^{(2)}, \ldots, \overline{u}^{(k)})$, then the corresponding equilibrium point of system (8) is the point $\overline{U} = (\overline{u}^{(1)}, \overline{u}^{(1)}, \overline{u}^{(1)}, \overline{u}^{(2)}, \overline{u}^{(2)}, \overline{u}^{(2)}, \ldots, \overline{u}^{(k)}, \overline{u}^{(k)}, \overline{u}^{(k)})^T$.

In this study, we denote by $\|\cdot\|$ any convenient vector norm and the corresponding matrix norm. Also, we denote by $U_0 \in I_1^3 \times I_2^3 \times \cdots \times I_k^3$ a initial condition of system (8).

Definition 1. Let $\overline{U}$ be an equilibrium point of system (8). Then,

(i) The equilibrium point $\overline{U}$ is called stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|U_0 - \overline{U}\| < \delta$ implies $\|U_n - \overline{U}\| < \varepsilon$, for all $n \geq 0$. Otherwise, the equilibrium point $\overline{U}$ is called unstable.

(ii) The equilibrium point $\overline{U}$ is called locally asymptotically stable if it is stable and there exists $\gamma > 0$ such that $\|U_0 - \overline{U}\| < \gamma$ and $U_n \longrightarrow \overline{U}$ as $n \longrightarrow \infty$.

(iii) The equilibrium point $\overline{U}$ is called a global attractor if $U_n \longrightarrow \overline{U}$ as $n \longrightarrow \infty$.

(iv) The equilibrium point $\overline{U}$ is called globally asymptotically stable if it is both locally asymptotically stable and global attractor.

The linearized system of (8) evaluated at the equilibrium $\overline{U}$ is

$$Z_{n+1} = J_F Z_n, \quad n \in \mathbb{N}_0, \tag{10}$$

where J_F is the Jacobian matrix of F at the equilibrium $\overline{U}$. The characteristic equation of system (10) about the equilibrium $\overline{U}$ is

$$P(\lambda) = a_0\lambda^{3k} + a_1\lambda^{3k-2} + \cdots + a_{3k-1}\lambda + a_{3k} = 0, \tag{11}$$

with real coefficients and $a_0 > 0$.

Theorem 1 (see [30]). *Assume that $\overline{U}$ is a equilibrium point of system (8). If all eigenvalues of the Jacobian matrix J_F evaluated at $\overline{U}$ lie in the open unit disk $|\lambda| < 1$, then $\overline{U}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\overline{U}$ is unstable.*

3. Global Stability

In this section, we investigate the stability of the two equilibrium points of system (5). When $a_i \in (0,1)$ for $i = 1, 2, \ldots, k$, the point $\overline{X}_0 = (\overline{x}_1^{(1)}, \overline{x}_1^{(2)}, \ldots, \overline{x}_1^{(k)}) = (0, 0, \ldots, 0)$ is the unique nonnegative equilibrium point of system (5). When $a_i \in (1, \infty)$ for $i = 1, 2, \ldots, k$, the unique positive equilibrium point of system (5) is

$$\overline{X}_{a_i} = \left(\overline{x}_2^{(1)}, \overline{x}_2^{(2)}, \ldots, \overline{x}_2^{(k)}\right) = \left(\left(a_k - 1\right)^{(1/r_k)}, \left(a_1 - 1\right)^{(1/r_1)}, \ldots, \left(a_{k-1} - 1\right)^{(1/r_{k-1})}\right). \tag{12}$$

Theorem 2. *The following statements hold:*

(i) If $a_i \in (0,1)$ for $i = 1, 2, \ldots, k$, then the equilibrium point $(\overline{x}_1^{(1)}, \overline{x}_1^{(2)}, \ldots, \overline{x}_1^{(k)})$ of system (5) is locally asymptotically stable

(ii) If $a_i \in (1, \infty)$ for $i = 1, 2, \ldots, k$, then the equilibrium point $(\overline{x}_1^{(1)}, \overline{x}_1^{(2)}, \ldots, \overline{x}_1^{(k)})$ of system (5) is unstable

(iii) If $a_i \in (1, \infty)$ for $i = 1, 2, \ldots, k$, then the positive equilibrium point $(\overline{x}_2^{(1)}, \overline{x}_2^{(2)}, \ldots, \overline{x}_2^{(k)})$ of system (5) is unstable

Proof

(i) The characteristic equation of $J_F(\overline{X}_0)$ is given by

$$P(\lambda) = \lambda^k\left(\lambda^2 - a_1\right)\left(\lambda^2 - a_2\right), \ldots, \left(\lambda^2 - a_k\right) = 0. \tag{13}$$

It is easy to see that if $a_i \in (0,1)$ for $i = 1, 2, \ldots, k$, then all the roots of the characteristic equation (13) lie in the open unit disk $|\lambda| < 1$. So, the equilibrium point $(\overline{x}_1^{(1)}, \overline{x}_1^{(2)}, \ldots, \overline{x}_1^{(k)})$ of (5) is locally asymptotically stable.

(ii) It is clearly seen that if $a_i \in (1, \infty)$ for $i = 1, 2, \ldots, k$, then some roots of characteristic equation (13) have absolute value greater than one. In this case, the equilibrium point $(\overline{x}_1^{(1)}, \overline{x}_1^{(2)}, \ldots, \overline{x}_1^{(k)})$ of (5) is unstable.

(iii) The characteristic polynomial of $J_F(\overline{X}_{a_i})$ is given by

$$P(\lambda) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \lambda^{3k-2j} + (-1)^{k+1} \prod_{i=1}^{k} \frac{r_i(a_i - 1)}{a_i}, \tag{14}$$

where $\binom{k}{j}$ is the binomial coefficient. It is clear that if k is an odd number, then $P(\lambda)$ has a root in interval $(-\infty, -1)$ since

$$P(-1) = \prod_{i=1}^{k} \frac{r_i(a_i - 1)}{a_i} > 0, \qquad \lim_{\lambda \longrightarrow -\infty} P(\lambda) = -\infty. \tag{15}$$

Also, if k is an even number, then $P(\lambda)$ has a root in interval $(1, \infty)$ since

$$P(1) = -\prod_{i=1}^{k} \frac{r_i(a_i - 1)}{a_i} < 0, \qquad \lim_{\lambda \longrightarrow \infty} P(\lambda) = \infty. \tag{16}$$

So, from Theorem 1, we can say that if $a_i \in (1, \infty)$ for $i = 1, 2, \ldots, k$, then the positive equilibrium point $(\overline{x}_2^{(1)}, \overline{x}_2^{(2)}, \ldots, \overline{x}_2^{(k)})$ of system (5) is unstable. □

Theorem 3. *If $a_i \in (0,1)$ for $i = 1,2,\ldots,k$, then the equilibrium point $(\overline{x}_1^{(1)}, \overline{x}_1^{(2)}, \ldots, \overline{x}_1^{(k)})$ of system (5) is globally asymptotically stable.*

Proof. From Theorem 2, we know that if $a_i \in (0,1)$ for $i = 1,2,\ldots,k$, then the equilibrium point $(\overline{x}_1^{(1)}, \overline{x}_1^{(2)}, \ldots, \overline{x}_1^{(k)})$ of system (5) is locally asymptotically stable. Hence, it suffices to show that

$$\lim_{n\longrightarrow\infty}\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)}\right) = (0,0,\ldots,0). \tag{17}$$

From system (5), we have that

$$\begin{aligned}
0 \le x_{n+1}^{(1)} &= \frac{a_1 x_{n-1}^{(1)}}{1+\left(x_{n-2}^{(2)}\right)^{r_1}} \le a_1 x_{n-1}^{(1)},\\
0 \le x_{n+1}^{(2)} &= \frac{a_2 x_{n-1}^{(2)}}{1+\left(x_{n-2}^{(3)}\right)^{r_2}} \le a_2 x_{n-1}^{(2)},\\
&\vdots\\
0 \le x_{n+1}^{(k)} &= \frac{a_k x_{n-1}^{(k)}}{1+\left(x_{n-2}^{(1)}\right)^{r_k}} \le a_k x_{n-1}^{(k)},
\end{aligned} \tag{18}$$

for $n \in \mathbb{N}_0$. From (18), we have by induction

$$0 \le x_{2n-l}^{(i)} \le a_i^n x_{-l}^{(i)}, \tag{19}$$

where $x_{-l}^{(i)}$ $(l = 0,1)$ for $i = 1,2,\ldots,k$ are the initial conditions. Consequently, by taking limits of inequalities in (19) when $a_i \in (0,1)$ for $i = 1,2,\ldots,k$, we have the limit in (17) which completes the proof. □

4. Oscillation Behavior and Existence of Unbounded Solutions

In the following result, we are concerned with the oscillation of positive solutions of system (5) about the equilibrium point $(\overline{x}_2^{(1)}, \overline{x}_2^{(2)}, \ldots, \overline{x}_2^{(k)})$.

Theorem 4. *Assume that $a_i \in (1,\infty)$, and let $\{(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)})\}_{n=-2}^{\infty}$ be a positive solution of system (5) such that*

$$\begin{aligned}
x_{-2}^{(i)}, x_0^{(i)} &\ge \overline{x}_2^{(i)},\\
x_{-1}^{(i)} &< \overline{x}_2^{(i)},
\end{aligned} \tag{20}$$

or

$$\begin{aligned}
x_{-2}^{(i)}, x_0^{(i)} &< \overline{x}_2^{(i)},\\
x_{-1}^{(i)} &\ge \overline{x}_2^{(i)},
\end{aligned} \tag{21}$$

for $i = 1,2,\ldots,k$. Then, $\{(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)})\}_{n=-2}^{\infty}$ oscillates about the equilibrium point $(\overline{x}_2^{(1)}, \overline{x}_2^{(2)}, \ldots, \overline{x}_2^{(k)})$ with semicycles of length one.

Proof. Assume that (20) holds. (The case where (21) holds is similar and will be omitted.) From (5), we have

$$\begin{aligned}
x_1^{(1)} &= \frac{a_1 x_{-1}^{(1)}}{1+\left(x_{-2}^{(2)}\right)^{r_1}} < \frac{a_1 \overline{x}_2^{(1)}}{1+\left(\overline{x}_2^{(2)}\right)^{r_1}} = \overline{x}_2^{(1)}, x_1^{(2)} = \frac{a_2 x_{-1}^{(2)}}{1+\left(x_{-2}^{(3)}\right)^{r_2}} < \frac{a_2 \overline{x}_2^{(2)}}{1+\left(\overline{x}_2^{(3)}\right)^{r_2}} = \overline{x}_2^{(2)}, \vdots\, x_1^{(k)} = \frac{a_k x_{-1}^{(k)}}{1+\left(x_{-2}^{(1)}\right)^{r_k}} < \frac{a_k \overline{x}_2^{(k)}}{1+\left(\overline{x}_2^{(1)}\right)^{r_k}} = \overline{x}_2^{(k)}, x_2^{(1)}\\
&= \frac{a_1 x_0^{(1)}}{1+\left(x_{-1}^{(2)}\right)^{r_1}} \ge \frac{a_1 \overline{x}_2^{(1)}}{1+\left(\overline{x}_2^{(2)}\right)^{r_1}} = \overline{x}_2^{(1)}, x_2^{(2)} = \frac{a_2 x_0^{(2)}}{1+\left(x_{-1}^{(3)}\right)^{r_2}} \ge \frac{a_2 \overline{x}_2^{(2)}}{1+\left(\overline{x}_2^{(3)}\right)^{r_2}} = \overline{x}_2^{(2)}, \vdots\, x_2^{(k)} = \frac{a_k x_0^{(k)}}{1+\left(x_{-1}^{(1)}\right)^{r_k}} \ge \frac{a_k \overline{x}_2^{(k)}}{1+\left(\overline{x}_2^{(1)}\right)^{r_k}} = \overline{x}_2^{(k)}.
\end{aligned} \tag{22}$$

Then, the proof follows by induction.

In the following theorem, we show the existence of unbounded solutions for system (5). □

Theorem 5. *Assume that $a_i \in (1,\infty)$ for $i = 1,2,\ldots,k$, then system (5) possesses an unbounded solution.*

Proof. From Theorem 4, we can assume that, without loss of generality, the solution $\{(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)})\}_{n=-2}^{\infty}$ of system (5) is such that $x_{2n-1}^{(i)} < \overline{x}_2^{(i)}$ and $x_{2n}^{(i)} > \overline{x}_2^{(i)}$ for $i = 1,2,\ldots,k$ and $n \in \mathbb{N}_0$. Then, we have

$$\begin{aligned}
x_{2n+2}^{(1)} &= \frac{a_1 x_{2n}^{(1)}}{1+\left(x_{2n-1}^{(2)}\right)^{r_1}} > \frac{a_1 x_{2n}^{(1)}}{1+\left(\overline{x}_2^{(2)}\right)^{r_1}} = \frac{a_1 x_{2n}^{(1)}}{1+\left(a_1-1\right)} = x_{2n}^{(1)},\\
x_{2n+2}^{(2)} &= \frac{a_2 x_{2n}^{(2)}}{1+\left(x_{2n-1}^{(3)}\right)^{r_2}} > \frac{a_2 x_{2n}^{(2)}}{1+\left(\overline{x}_2^{(3)}\right)^{r_2}} = \frac{a_2 x_{2n}^{(2)}}{1+\left(a_2-1\right)} = x_{2n}^{(2)},\\
&\vdots\\
x_{2n+2}^{(k)} &= \frac{a_k x_{2n}^{(k)}}{1+\left(x_{2n-1}^{(1)}\right)^{r_k}} > \frac{a_k x_{2n}^{(k)}}{1+\left(\overline{x}_2^{(1)}\right)^{r_k}} = \frac{a_k x_{2n}^{(k)}}{1+\left(a_k-1\right)} = x_{2n}^{(k)},\\
x_{2n+3}^{(1)} &= \frac{a_1 x_{2n+1}^{(1)}}{1+\left(x_{2n}^{(2)}\right)^{r_1}} < \frac{a_1 x_{2n+1}^{(1)}}{1+\left(\overline{x}_2^{(2)}\right)^{r_1}} = \frac{a_1 x_{2n+1}^{(1)}}{1+\left(a_1-1\right)} = x_{2n+1}^{(1)},\\
x_{2n+3}^{(2)} &= \frac{a_2 x_{2n+1}^{(2)}}{1+\left(x_{2n}^{(3)}\right)^{r_2}} < \frac{a_2 x_{2n+1}^{(2)}}{1+\left(\overline{x}_2^{(3)}\right)^{r_2}} = \frac{a_2 x_{2n+1}^{(2)}}{1+\left(a_2-1\right)} = x_{2n+1}^{(2)},\\
&\vdots\\
x_{2n+3}^{(k)} &= \frac{a_k x_{2n+1}^{(k)}}{1+\left(x_{2n}^{(1)}\right)^{r_k}} < \frac{a_k x_{2n+1}^{(k)}}{1+\left(\overline{x}_2^{(1)}\right)^{r_k}} = \frac{a_k x_{2n+1}^{(k)}}{1+\left(a_k-1\right)} = x_{2n+1}^{(k)},
\end{aligned} \tag{23}$$

from which it follows that

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\left(x_{2n}^{(1)}, x_{2n}^{(2)}, \ldots, x_{2n}^{(k)}\right) &= (\infty,\infty,\ldots,\infty),\\
\lim_{n\longrightarrow\infty}\left(x_{2n+1}^{(1)}, x_{2n+1}^{(2)}, \ldots, x_{2n+1}^{(k)}\right) &= (0,0,\ldots,0),
\end{aligned} \tag{24}$$

which completes the proof. □

5. Periodicity

In this section, we investigate the existence of period-two solution of system (5).

Theorem 6. *If $a_i = 1$ for $i = 1,2,\ldots,k$, then system (5) possesses the prime period-two solution*

$$\ldots,(0,0,\ldots,0,\varphi),(0,0,\ldots,0,\psi),(0,0,\ldots,0,\varphi),(0,0,\ldots,0,\psi),\ldots, \tag{25}$$

with $\varphi,\psi > 0$. Furthermore, every solution of system (5) converges to a period-two solution.

Proof. Assume that $a_i = 1$ for $i = 1,2,\ldots,k$, and let $\{(x_n^{(1)}, x_n^{(2)},\ldots,x_n^{(k)})\}_{n=-2}^{\infty}$ be a solution of system (5). Then, from system (5), we have

$$\begin{aligned}
x_{2n+1}^{(1)} &= \frac{x_{2n-1}^{(1)}}{1+\left(x_{2n-2}^{(2)}\right)^{r_1}},\\
x_{2n+2}^{(1)} &= \frac{x_{2n}^{(1)}}{1+\left(x_{2n-1}^{(2)}\right)^{r_1}},\\
x_{2n+1}^{(2)} &= \frac{x_{2n-1}^{(2)}}{1+\left(x_{2n-2}^{(3)}\right)^{r_2}},\\
x_{2n+2}^{(2)} &= \frac{x_{2n}^{(2)}}{1+\left(x_{2n-1}^{(3)}\right)^{r_2}},\\
&\vdots\\
x_{2n+1}^{(k)} &= \frac{x_{2n-1}^{(k)}}{1+\left(x_{2n-2}^{(1)}\right)^{r_k}},\\
x_{2n+2}^{(k)} &= \frac{x_{2n}^{(k)}}{1+\left(x_{2n-1}^{(1)}\right)^{r_k}},
\end{aligned} \tag{26}$$

for $n \in \mathbb{N}_0$. From (26), we obtain

$$\begin{aligned}
x_{2n-1}^{(1)} &= x_{-1}^{(1)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-2}^{(2)}\right)^{r_1}}\right),\\
x_{2n}^{(1)} &= x_{0}^{(1)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-1}^{(2)}\right)^{r_1}}\right),\\
x_{2n-1}^{(2)} &= x_{-1}^{(2)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-2}^{(3)}\right)^{r_2}}\right),\\
x_{2n}^{(2)} &= x_{0}^{(2)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-1}^{(3)}\right)^{r_2}}\right),\\
&\vdots\\
x_{2n-1}^{(k)} &= x_{-1}^{(k)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-2}^{(1)}\right)^{r_k}}\right),\\
x_{2n}^{(k)} &= x_{0}^{(k)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-1}^{(1)}\right)^{r_k}}\right),
\end{aligned} \tag{27}$$

for $n \in \mathbb{N}_0$. If $x_{-l}^{(i)} = 0$ for $l = 0,1$ and $i = 1,2,\ldots,k-1$, then $x_n^{(i)} = 0$ for $i = 1,2,\ldots,k-1$ and $(x_{2n-1}^{(k)}, x_{2n}^{(k)}) = (x_{-1}^{(k)}, x_0^{(k)})$ for $n \in \mathbb{N}_0$. Therefore,

$$\ldots,(0,0,\ldots,0,\varphi),(0,0,\ldots,0,\psi),(0,0,\ldots,0,\varphi),(0,0,\ldots,0,\psi),\ldots \tag{28}$$

is a period-two solution of system (5) with $x_{-2}^{(k)} = x_0^{(k)} = \varphi > 0$ and $x_{-1}^{(k)} = \psi > 0$. Furthermore, from (26), we have

$$\begin{aligned}
x_{2n+1}^{(1)} - x_{2n-1}^{(1)} &= -\frac{x_{2n-1}^{(1)}\left(x_{2n-2}^{(2)}\right)^{r_1}}{1+\left(x_{2n-2}^{(2)}\right)^{r_1}} \le 0,\\
x_{2n+1}^{(2)} - x_{2n-1}^{(2)} &= -\frac{x_{2n-1}^{(2)}\left(x_{2n-2}^{(3)}\right)^{r_2}}{1+\left(x_{2n-2}^{(3)}\right)^{r_2}} \le 0,\\
&\vdots\\
x_{2n+1}^{(k)} - x_{2n-1}^{(k)} &= -\frac{x_{2n-1}^{(k)}\left(x_{2n-2}^{(1)}\right)^{r_k}}{1+\left(x_{2n-2}^{(1)}\right)^{r_k}} \le 0,
\end{aligned} \tag{29}$$

$$\begin{aligned}
x_{2n+2}^{(1)} - x_{2n}^{(1)} &= -\frac{x_{2n}^{(1)}\left(x_{2n-1}^{(2)}\right)^{r_1}}{1+\left(x_{2n-1}^{(2)}\right)^{r_1}} \le 0,\\
x_{2n+2}^{(2)} - x_{2n}^{(2)} &= -\frac{x_{2n}^{(2)}\left(x_{2n-1}^{(3)}\right)^{r_2}}{1+\left(x_{2n-1}^{(3)}\right)^{r_2}} \le 0,\\
&\vdots\\
x_{2n+2}^{(k)} - x_{2n}^{(k)} &= -\frac{x_{2n}^{(k)}\left(x_{2n-1}^{(1)}\right)^{r_k}}{1+\left(x_{2n-1}^{(1)}\right)^{r_k}} \le 0.
\end{aligned} \tag{30}$$

From (29) and (30), we obtain $x_{2n+1}^{(i)} \le x_{2n-1}^{(i)}$ and $x_{2n+2}^{(i)} \le x_{2n}^{(i)}$ for $i = 1, 2, \ldots, k$. That is, the sequences $(x_{2n-1}^{(i)})$ and $(x_{2n}^{(i)})$ for $i = 1, 2, \ldots, k$ are nonincreasing. On the other hand, from (26), we have the inequalities

$$\begin{aligned}
x_{2n-1}^{(1)} &= x_{-1}^{(1)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-2}^{(2)}\right)^{r_1}}\right) \le x_{-1}^{(1)},\\
x_{2n}^{(1)} &= x_{0}^{(1)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-1}^{(2)}\right)^{r_1}}\right) \le x_{0}^{(1)},\\
x_{2n-1}^{(2)} &= x_{-1}^{(2)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-2}^{(3)}\right)^{r_2}}\right) \le x_{-1}^{(2)},\\
x_{2n}^{(2)} &= x_{0}^{(2)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-1}^{(3)}\right)^{r_2}}\right) \le x_{0}^{(2)},\\
&\vdots\\
x_{2n-1}^{(k)} &= x_{-1}^{(k)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-2}^{(1)}\right)^{r_k}}\right) \le x_{-1}^{(k)},\\
x_{2n}^{(k)} &= x_{0}^{(k)} \prod_{i=0}^{n-1}\left(\frac{1}{1+\left(x_{2i-1}^{(1)}\right)^{r_k}}\right) \le x_{0}^{(k)},
\end{aligned} \tag{31}$$

which show the boundedness of the solutions. Hence, the odd-index terms tend to one periodic point and the even-index terms tend to another periodic point. This completes the proof. □

6. Numerical Examples

In this section, we give some numerical examples to support our theoretical results related to system (5) with some restrictions on the parameters a_i and r_i for $i = 1, 2, \ldots k$.

Example 1. If $k = 3$, $x_n^{(1)} = x_n$, $x_n^{(2)} = y_n$, $x_n^{(3)} = z_n$, $r_1 = 2$, $r_2 = 3$, and $r_3 = 4$ in system (5), we obtain the following system:

$$\begin{aligned}
x_{n+1} &= \frac{a_1 x_{n-1}}{1+\left(y_{n-2}\right)^2},\\
y_{n+1} &= \frac{a_2 y_{n-1}}{1+\left(z_{n-2}\right)^3},\\
z_{n+1} &= \frac{a_3 z_{n-1}}{1+\left(x_{n-2}\right)^4}.
\end{aligned} \tag{32}$$

We visualize the solutions of system (32) in Figures 1–3 for the initial conditions $x_{-2} = 1.34$, $x_{-1} = 2.13$, $x_0 = 3.1$, $y_{-2} = 0.17$, $y_{-1} = 4.03$, $y_0 = 2.21$, $z_{-2} = 0.32$, $z_{-1} = 2.76$, and $z_0 = 3.12$.

Example 2. If $k = 4$, $x_n^{(1)} = x_n$, $x_n^{(2)} = y_n$, $x_n^{(3)} = z_n$, $x_n^{(4)} = p_n$, $r_1 = 2$, $r_2 = 3$, $r_3 = 4$, and $r_4 = 5$ in system (5), we obtain the following system:

$$\begin{aligned}
x_{n+1} &= \frac{a_1 x_{n-1}}{1+\left(y_{n-2}\right)^2},\\
y_{n+1} &= \frac{a_2 y_{n-1}}{1+\left(z_{n-2}\right)^3},\\
z_{n+1} &= \frac{a_3 z_{n-1}}{1+\left(p_{n-2}\right)^4},\\
p_{n+1} &= \frac{a_4 p_{n-1}}{1+\left(x_{n-2}\right)^5}.
\end{aligned} \tag{33}$$

We visualize the solutions of system (33) in Figures 4–6 for the initial conditions $x_{-2} = 1.34$, $x_{-1} = 2.13$, $x_0 = 3.1$, $y_{-2} = 0.17$, $y_{-1} = 4.03$, $y_0 = 2.21$, $z_{-2} = 0.32$, $z_{-1} = 2.76$, $z_0 = 3.12$, $p_{-2} = 3.27$, $p_{-1} = 1.33$, and $p_0 = 2.78$.

Example 3. If $k = 5$, $x_n^{(1)} = x_n$, $x_n^{(2)} = y_n$, $x_n^{(3)} = z_n$, $x_n^{(4)} = p_n$, and $x_n^{(5)} = q_n$ in system (5), we obtain the following system:

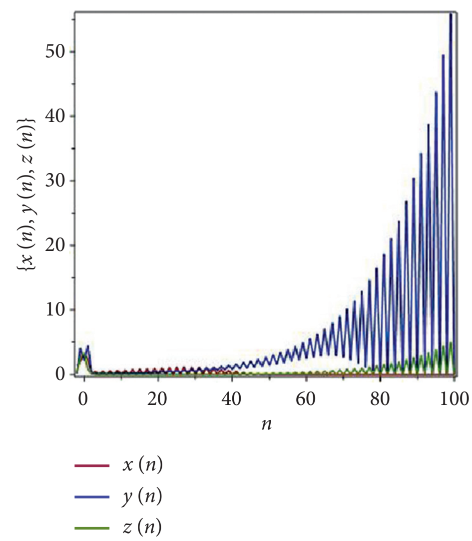

Figure 1: The solutions of system (32) when $a_1 = 1.12$, $a_2 = 1.13$, and $a_3 = 1.14$.

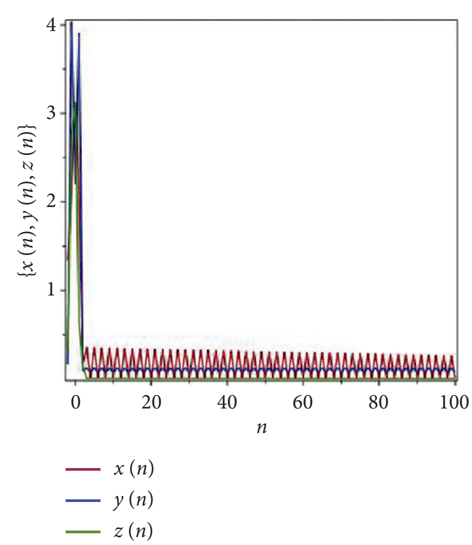

Figure 2: The solutions of system (32) when $a_1 = 1$, $a_2 = 1$, and $a_3 = 1$.

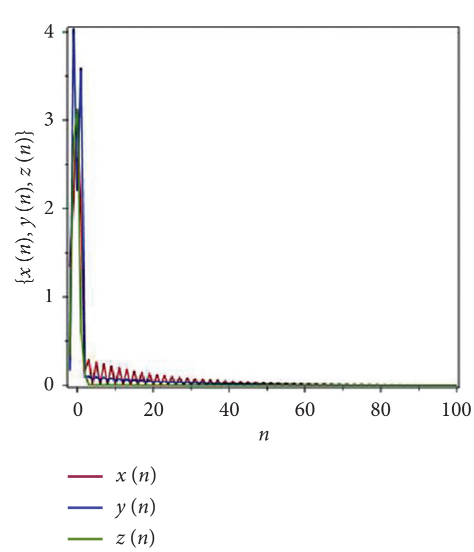

Figure 3: The solutions of system (32) when $a_1 = 0.91$, $a_2 = 0.92$, and $a_3 = 0.93$.

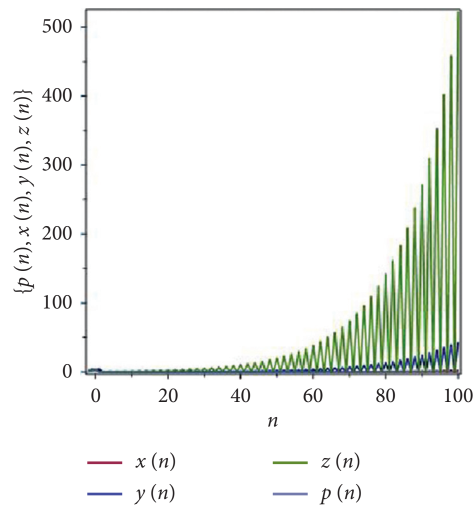

FIGURE 4: The solutions of system (33) when $a_1 = 1.12$, $a_2 = 1.13$, $a_3 = 1.14$, and $a_4 = 1.15$.

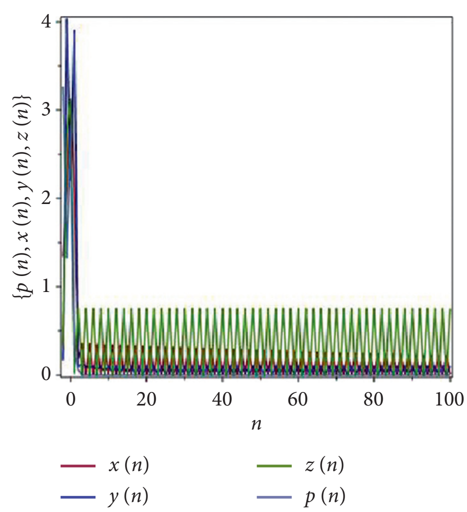

FIGURE 5: The solutions of system (33) when $a_1 = 1$, $a_2 = 1$, $a_3 = 14$, and $a_4 = 1$.

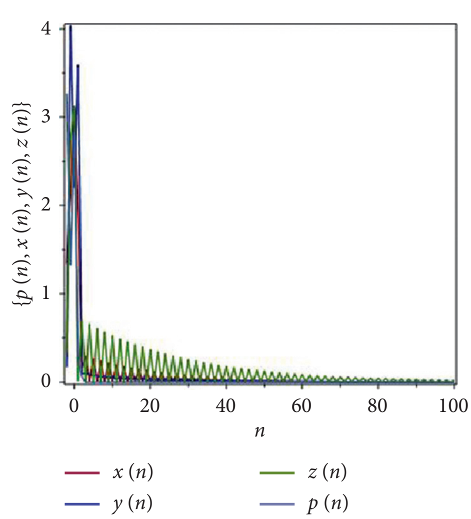

FIGURE 6: The solutions of system (33) when $a_1 = 0.91$, $a_2 = 0.92$, $a_3 = 0.93$, and $a_4 = 0.94$.

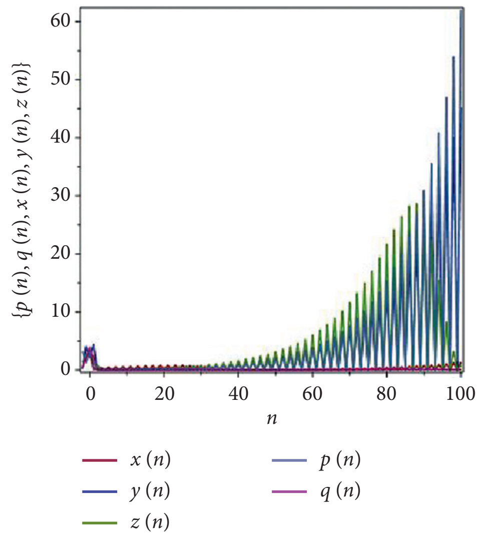

Figure 7: The solutions of system (34) when $a_1 = 1.12$, $a_2 = 1.13$, $a_3 = 1.14$, $a_4 = 1.15$, and $a_5 = 1.16$.

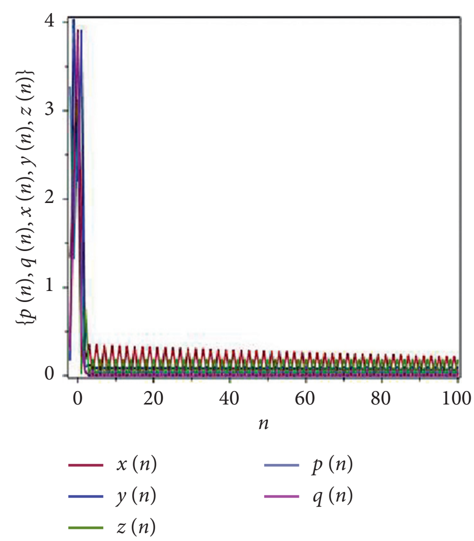

Figure 8: The solutions of system (34) when $a_1 = 1$, $a_2 = 1$, $a_3 = 14$, $a_4 = 1$, and $a_5 = 1$.

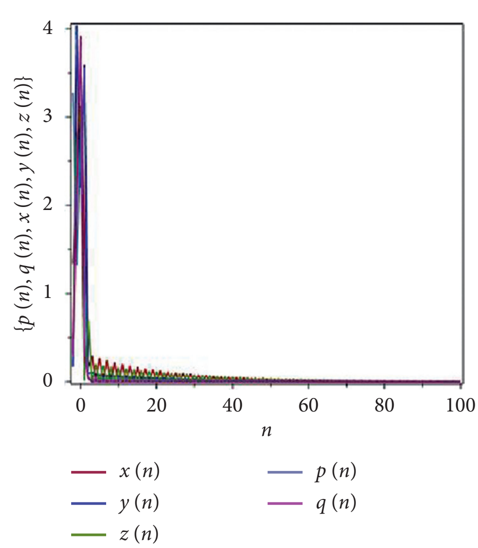

Figure 9: The solutions of system (34) when $a_1 = 0.91$, $a_2 = 0.92$, $a_3 = 0.93$, $a_4 = 0.94$, and $a_1 = 0.95$.

$$\begin{cases} x_{n+1} = \dfrac{a_1 x_{n-1}}{1+(y_{n-2})^2}, \\ y_{n+1} = \dfrac{a_2 y_{n-1}}{1+(z_{n-2})^3}, \\ z_{n+1} = \dfrac{a_3 z_{n-1}}{1+(p_{n-2})^4}, \\ p_{n+1} = \dfrac{a_4 p_{n-1}}{1+(q_{n-2})^5}, \\ q_{n+1} = \dfrac{a_5 q_{n-1}}{1+(x_{n-2})^6}, \end{cases} \quad (34)$$

with $r_1 = 2$, $r_2 = 3$, $r_3 = 4$, $r_4 = 5$, and $r_5 = 6$. We visualize the solutions of system (34) in Figures 7–9 for the initial conditions $x_{-2} = 1.34$, $x_{-1} = 2.13$, $x_0 = 3.1$, $y_{-2} = 0.17$, $y_{-1} = 4.03$, $y_0 = 2.21$, $z_{-2} = 0.32$, $z_{-1} = 2.76$, $z_0 = 3.12$, $p_{-2} = 3.27$, $p_{-1} = 1.33$, $p_0 = 2.78$, $q_{-2} = 0.32$, $q_{-1} = 2.16$, and $q_0 = 3.91$.

7. Conclusion

In this study, we have generalized some of the results in the literature. As shown in Section 1, equation (1) was developed systematically. By this study, we ended this development. More concretely, we investigated the local asymptotic stability, global asymptotic stability, periodicity, and oscillation behavior of system (5) which is the k-dimensional generalization of equation (1). According to our findings, our results are consistent with the results of the paper [1] in the case of $k = 1$. Similarly, our results are in line with the results of the papers [2, 3] in the case of $k = 2$ and $k = 3$, respectively.

Data Availability

The data used to support the findings of this study are available from the first author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest associated with this publication.

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