

Research Article

Cosine Similarity Measure of Complex Fuzzy Sets and Robustness of Complex Fuzzy Connectives

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Complex fuzzy set (CFS), as a generalization of fuzzy set (FS), is characterized by complex-valued membership degrees. By considering the complex-valued membership degree as a vector in the complex unit disk, we introduce the cosine similarity measures between CFSs. Then, we investigate some invariance properties of the cosine similarity measure. Finally, the cosine similarity measure is applied to measure the robustness of complex fuzzy connectives and complex fuzzy inference.

1. Introduction

Similarity measure is an important tool in fuzzy mathematics. It has been successfully applied to many fields [1–3]. Cosine similarity measure is a special type of similarity measure which is viewed as the cosine of the angle between two vectors [4, 5]. In order to define the cosine similarity measure for FSs, membership degree in FSs is used by the vector representation. By this method, some cosine similarity measures are proposed for intuitionistic fuzzy sets and applied to pattern recognition and medical diagnosis [6, 7]. And some cosine similarity measures for interval-valued intuitionistic fuzzy sets [8], Pythagorean fuzzy sets [9], picture fuzzy sets [10], vague sets [11], hesitant fuzzy set [12], and neutrosophic sets [13] have been proposed. However, there is no cosine similarity measures between CFSs. As we have known, in many studies, CFSs [14, 15] are viewed as sets of vectors of the complex unit disk. In [16], Dick considered complex fuzzy logic as a logic of vectors and introduced a feature called rotational invariance for complex fuzzy operations. Later, this feature is examined for some measures and operations of CFSs [17–19] and interval-valued CFSs [20–25]. Ramot et al. [15] considered complex fuzzy aggregation as a vector aggregation on CFSs. Hu et al. [26–28] introduced the orthogonality and parallelity relations

for CFSs based on the geometrical relations of the vectors of complex-valued membership degrees. So, it is natural to define cosine similarity measures between CFSs when they are considered as sets of vectors in the complex unit disk.

The robustness of fuzzy inference has become a particular topic in the research area of fuzzy inference [29–39]. In the research of robustness of complex fuzzy inference, one fundamental problem is how to measure the perturbation of CFSs. Distance and similarity measures between CFSs play an important role in measuring the robustness of complex fuzzy inference methods. Zhang et al. [40] proposed a notion of δ -equality of CFSs. Alkouri et al. [41] and Hu et al. [42] introduced several distance measures for CFSs. As we know, CFSs is defined by a phase term and an amplitude term. These distances are the average or maximum of the difference between the amplitude terms and the difference between the phase terms. As mentioned in [14], the phase term of CFS is the key element that distinguishes CFSs from other extensions of fuzzy sets. We can study the robustness of complex fuzzy operators and complex fuzzy inference methods based on above distances. Zhang et al. [40] studied the robustness of complex fuzzy operations based on the δ -equality of CFSs. However, it cannot provide an effective method for measuring the robustness of the phase term of complex fuzzy operators and complex fuzzy inference. Having

this in mind, we propose the cosine similarity measure for CFSs which just relies on the phase term of CFS. Then, we study the robustness of the phase term of complex fuzzy connectives and complex fuzzy inference based on this similarity of CFSs.

In this paper, we focus on the cosine similarity measure between CFSs. First we review some necessary concepts of CFSs in Section 2. In Section 3, we propose the cosine similarity measure between CFSs and discuss its invariance properties. In Section 4, we compare our measure with other existing measures. In Section 5, the cosine similarity measures are applied to measure the robustness of complex fuzzy connectives and complex fuzzy inference. Finally, conclusions are given in Section 6.

2. Preliminaries

In this paper, our discussion is based on CFS theory. Some basic concepts of CFSs are recalled below, whereas others are given in [14–16].

Let O be the set of complex numbers on complex unit disk, i.e., $O = \{\alpha \in \mathbb{C} \mid |\alpha| \leq 1\}$. Let X be a fixed universe; a mapping $S: X \rightarrow D$ is called a CFS in X , whose its membership degree $\mu_S(x)$ is

$$r_S(x) \cdot e^{j\nu_S(x)}, \quad (1)$$

where $j = \sqrt{-1}$, $r_S(x) \in [0, 1]$ is the amplitude part, and $\nu_S(x) \in \mathbb{R}$ is the phase part.

Three complex fuzzy complement operations are defined by Ramot et al. [14] as follows:

$$(\Gamma_1 S)(x) = (1 - r_S(x)) \cdot e^{j(-\nu_S(x))}, \quad (2)$$

$$(\Gamma_2 S)(x) = (1 - r_S(x)) \cdot e^{j(\nu_S(x))}, \quad (3)$$

$$(\Gamma_3 S)(x) = (1 - r_S(x)) \cdot e^{j(\nu_S(x) + \pi)}. \quad (4)$$

Let A and B be two CFSs; intersection and union of A and B are defined by Ramot et al. [14] as follows:

$$(A \cup B)(x) = (r_A(x) * r_B(x)) \cdot e^{j\nu_{A \otimes B}(x)}, \quad (5)$$

$$(A \cap B)(x) = (r_A(x) * r_B(x)) \cdot e^{j\nu_{A \otimes B}(x)}, \quad (6)$$

where $*$ and $*$ represent a t-conorm and a t-norm, respectively.

Some commonly used functions of $\nu_{A \otimes B}$ are defined as follows (see [14]):

$$\nu_{A+B} = \nu_A + \nu_B, \quad (7)$$

$$\nu_{A \vee B} = \max(\nu_A, \nu_B), \quad (8)$$

$$\nu_{A \wedge B} = \min(\nu_A, \nu_B), \quad (9)$$

$$\nu_{A-B} = \nu_A - \nu_B. \quad (10)$$

Rotation and reflection operations of CFSs are defined by Ramot et al. [14] as follows:

(i) Rotation of a CFS S :

$$\text{Rot}_\theta(S)(x) = r_S(x) \cdot e^{j\nu_S(x) + \theta}, \quad (11)$$

where $\theta \in \mathbb{R}$.

(ii) Reflection of a CFS S :

$$\text{Ref}(S)(x) = r_S(x) \cdot e^{-j\nu_S(x)}. \quad (12)$$

A dependency relation of CFSs is defined by Hu et al. [26] as follows. Two CFSs A and B are said to be orthogonal, denoted by $A \perp B$, if

$$\langle \mu_A(x), \mu_B(x) \rangle = 0, \quad (13)$$

for any $x \in X$, where $\langle \mu_A(x), \mu_B(x) \rangle$ is the inner product of complex numbers $\mu_A(x), \mu_B(x) \in O$.

3. Cosine Similarity Measures for CFSs

In this section, we propose a cosine similarity measure between CFSs.

Let A and B be two CFSs in $X = \{x_1, x_2, \dots, x_n\}$; a cosine similarity measure between two CFSs A and B is defined as follows:

$$C_{\text{CFS}}(A, B) = \bigvee_{i=1}^n \frac{\langle \mu_A(x_i), \mu_B(x_i) \rangle}{|\mu_A(x_i)| \cdot |\mu_B(x_i)|}. \quad (14)$$

Obviously, assume that θ_i ($i = 1, 2, \dots, n$) is the angle of two vector $\mu_A(x_i)$ and $\mu_B(x_i)$; then,

$$C_{\text{CFS}}(A, B) = \bigvee_{i=1}^n \cos(\theta_i). \quad (15)$$

Theorem 1. Suppose that A, B are two CFSs in X ; the cosine similarity measure $C_{\text{CFS}}(A, B)$ of A and B satisfies the following properties:

- (1) $-1 \leq C_{\text{CFS}}(A, B) \leq 1$
- (2) $C_{\text{CFS}}(A, B) = C_{\text{CFS}}(B, A)$
- (3) $C_{\text{CFS}}(A, A) = 1$
- (4) $C_{\text{CFS}}(A, B) = -1$ if $A = -B$, i.e., $\mu_A(x) = -\mu_B(x)$ for all $x \in X$
- (5) $C_{\text{CFS}}(A, B) = 0$ if $A \perp B$

Proof

- (1) Since $-1 \leq \cos(\theta_i) \leq 1$, then $-1 \leq \bigvee_{i=1}^n \cos(\theta_i) \leq 1$.
- (2) It is obvious from $\langle \mu_A(x), \mu_B(x) \rangle = \langle \mu_B(x), \mu_A(x) \rangle$ and $|\mu_A(x)| \cdot |\mu_B(x)| = |\mu_B(x)| \cdot |\mu_A(x)|$ for all $x \in X$.
- (3) It is obvious from $|\mu_A(x)| \cdot |\mu_A(x)| = \langle \mu_A(x), \mu_A(x) \rangle$ for all $x \in X$.
- (4) For any $x \in X$, when $\mu_A(x) = -\mu_B(x)$, $\langle \mu_A(x), \mu_B(x) \rangle = -|\mu_A(x)| \cdot |\mu_B(x)|$. So, we have $C_{\text{CFS}}(A, B) = -1$.
- (5) When $A \perp B$, then the angle of two vector $\mu_A(x)$ and $\mu_B(x)$ is $\pi/2$ for all $x \in X$. So, we have $C_{\text{CFS}}(A, B) = 0$ from $\cos(\pi/2) = 0$.

If we define the distance measure of CFSs by $D_{\text{CFS}}(A, B) = \arccos(C_{\text{CFS}}(A, B))$, then it satisfies the following properties. \square

Theorem 2. Suppose that A, B are two CFSs in X ; the distance measure $D_{CFS}(A, B)$ of A and B satisfies the following properties:

- (1) $0 \leq D_{CFS}(A, B) \leq \pi$.
- (2) $D_{CFS}(A, B) = D_{CFS}(B, A)$
- (3) $D_{CFS}(A, A) = 0$

Proof. Equations (1)–(3) are obvious from $-1 \leq C_{CFS}(A, B) \leq 1$, $C_{CFS}(A, B) = C_{CFS}(B, A)$, and $C_{CFS}(A, A) = 1$, respectively.

The cosine similarity measure for CFSs is reflectionally invariant and rotationally invariant. \square

Theorem 3. Suppose that A, B are two CFSs in X ; the cosine similarity measure $C_{CFS}(A, B)$ of A and B is reflectionally invariant and rotationally invariant, i.e.,

$$C_{CFS}(\text{Ref}(A), \text{Ref}(B)) = C_{CFS}(A, B), \quad (16)$$

$$C_{CFS}(\text{Rot}_\theta(A), \text{Rot}_\theta(B)) = C_{CFS}(A, B), \quad (17)$$

for any θ .

Proof. For any $x_i \in X$, assume that the angle of two vector $\mu_A(x_i)$ and $\mu_B(x_i)$ is θ_i ; then, the angle of two vector $\mu_{\text{Ref}(A)}(x_i)$ and $\mu_{\text{Ref}(B)}(x_i)$ is also θ_i . Thus, $C_{CFS}(\text{Ref}(A), \text{Ref}(B)) = C_{CFS}(A, B)$. Similarly, we have $C_{CFS}(\text{Rot}_\theta(A), \text{Rot}_\theta(B)) = C_{CFS}(A, B)$ for any θ . \square

Theorem 4. Suppose that A, B are two CFSs in X ; the distance measure $D_{CFS}(A, B)$ of A and B is reflectionally invariant and rotationally invariant, i.e.,

$$D_{CFS}(\text{Ref}(A), \text{Ref}(B)) = D_{CFS}(A, B), \quad (18)$$

$$D_{CFS}(\text{Rot}_\theta(A), \text{Rot}_\theta(B)) = D_{CFS}(A, B), \quad (19)$$

for any θ .

Proof. Trivial from Theorem 3.

Moreover, the cosine similarity measure for CFSs is ratio scale invariant. \square

Theorem 5. Suppose that A, B are two CFSs in X ; the cosine similarity measure $C_{CFS}(A, B)$ of A and B is ratio scale invariant, i.e.,

$$C_{CFS}(tA, tB) = C_{CFS}(A, B), \quad (20)$$

for any $t > 0$ such that $tA(x), tB(x) \in O$, where $\mu_{tA}(x) = (t \cdot r_A(x)) \cdot e^{jv_A(x)}$ for all $x \in X$.

Proof. It can be easily obtained from the fact that the angle of two vector $\mu_A(x)$ and $\mu_B(x)$ is same to the angle of two vector $\mu_{tA}(x)$ and $\mu_{tB}(x)$ for any $x \in X$. \square

Theorem 6. Suppose that A, B are two CFSs in X ; the distance measure $D_{CFS}(A, B)$ of A and B is ratio scale invariant, i.e.,

$$D_{CFS}(tA, tB) = D_{CFS}(A, B), \quad (21)$$

for any $t > 0$ such that $tA(x), tB(x) \in O$.

Proof. Trivial from Theorem 5. \square

4. Comparisons of Distance Measures for CFSs

Since there are no other similarity measures of CFSs, in this section, we make a comparison between the proposed distance measure $D_{CFS}(A, B) = \arccos(C_{CFS}(A, B))$ and other existing distance measures in CFSs [40–42]. Let A and B be two CFSs in $X = \{x_1, x_2, \dots, x_n\}$; some existing distance measures between A and B are listed as follows:

$$D_Z(A, B) = \max\left(\sup_{x \in X} |r_A(x) - r_B(x)|, \sup_{x \in X} \frac{|v_A(x) - v_B(x)|}{2\pi}\right), \quad (22)$$

$$D_H(A, B) = \frac{1}{2} \sum_{i=1}^n \left(|r_A(x_i) - r_B(x_i)| + \frac{|v_A(x_i) - v_B(x_i)|}{2\pi} \right), \quad (23)$$

$$D_{nH}(A, B) = \frac{1}{2n} \sum_{i=1}^n \left(|r_A(x_i) - r_B(x_i)| + \frac{|v_A(x_i) - v_B(x_i)|}{2\pi} \right), \quad (24)$$

$$D_E(A, B) = \left[\frac{1}{2} \left(\sum_{i=1}^n \left((r_A(x_i) - r_B(x_i))^2 + \frac{(v_A(x_i) - v_B(x_i))^2}{4\pi^2} \right) \right) \right]^{1/2}, \quad (25)$$

$$D_{nE}(A, B) = \left[\frac{1}{2n} \left(\sum_{i=1}^n \left((r_A(x_i) - r_B(x_i))^2 + \frac{(v_A(x_i) - v_B(x_i))^2}{4\pi^2} \right) \right) \right]^{1/2}, \quad (26)$$

$$D_p(A, B) = \left[\frac{1}{2} \sum_{i=1}^n \left(|r_A(x_i) - r_B(x_i)|^p + \frac{(\min(|v_A(x_i) - v_B(x_i)|, 2\pi - |v_A(x_i) - v_B(x_i)|))^p}{\pi^p} \right) \right]^{1/p}, \quad (27)$$

$$D_{np}(A, B) = \left[\frac{1}{2n} \sum_{i=1}^n \left(|r_A(x_i) - r_B(x_i)|^p + \frac{(\min(|v_A(x_i) - v_B(x_i)|, 2\pi - |v_A(x_i) - v_B(x_i)|))^p}{\pi^p} \right) \right]^{1/p}, \quad (28)$$

where $v_A(x), v_B(x) \in [0, 2\pi]$ for all $x \in X$.

Now, we give a brief summary of some properties of these distance measures for CFSs, as summarized in Table 1, in which \surd and \times , respectively, represent the corresponding invariance property holds or not.

Obviously, the primary difference between $D_{\text{CFS}}(A, B)$ and other distance measures is that it is ratio scale invariant, but others are not.

Example 1. Let $X = \{x_1, x_2, x_3\}$; two CFSs A and B in X are given as follows:

$$\begin{aligned} A &= \frac{0.6e^{j\pi/2}}{x_1} + \frac{1e^{j2\pi/3}}{x_2} + \frac{0.8e^{j\pi/2}}{x_3}, \\ B &= \frac{0.8e^{j3\pi/2}}{x_1} + \frac{0.2e^{j\pi}}{x_2} + \frac{0.4e^{j\pi}}{x_3}. \end{aligned} \quad (29)$$

Let $t = 1/2$; then,

$$\begin{aligned} tA &= \frac{0.3e^{j\pi/2}}{x_1} + \frac{0.5e^{j2\pi/3}}{x_2} + \frac{0.4e^{j\pi/2}}{x_3}, \\ tB &= \frac{0.4e^{j3\pi/2}}{x_1} + \frac{0.1e^{j\pi}}{x_2} + \frac{0.2e^{j\pi}}{x_3}. \end{aligned} \quad (30)$$

Consider the case of $p = 1$; we have

$$\begin{aligned} D_{\text{CFS}}(A, B) &= \frac{\pi}{3}, \\ D_{\text{CFS}}(tA, tB) &= \frac{\pi}{3}, \\ D_Z(A, B) &= 0.8, \\ D_Z(tA, tB) &= 0.5, \\ D_H(A, B) &= \frac{139}{120}, \\ D_H(tA, tB) &= \frac{97}{120}, \\ D_{nH}(A, B) &= \frac{139}{360}, \\ D_{nH}(tA, tB) &= \frac{97}{360}, \\ D_E(A, B) &\approx 0.7682, \\ D_E(tA, tB) &\approx 0.5245, \\ D_{nE}(A, B) &\approx 0.4435, \\ D_{nE}(tA, tB) &\approx 0.3028, \\ D_p(A, B) &= \frac{67}{30}, \\ D_p(tA, tB) &= \frac{23}{15}, \\ D_{nP}(A, B) &= \frac{67}{90}, \\ D_{nP}(tA, tB) &= \frac{23}{45}. \end{aligned} \quad (31)$$

Thus, $D_{\text{CFS}}(A, B) = D_{\text{CFS}}(tA, tB)$ and $D_{\circ}(A, B) \neq D_{\circ}(tA, tB)$ where $\circ \in \{Z, H, nH, E, nE, p, np\}$.

5. Robustness of Complex Fuzzy Connectives and Complex Fuzzy Inference

Now we consider the problem, if there is a small variance of the phase term of inputs, how much might the phase term of output vary?

5.1. Robustness of Complex Fuzzy Connectives

Lemma 1 (see [40]). *Suppose f, g are two bounded, real valued functions in X , then*

$$\begin{aligned} \left| \bigvee_{x \in X} f(x) - \bigvee_{x \in X} g(x) \right| &\leq \bigvee_{x \in X} |f(x) - g(x)|, \\ \left| \bigwedge_{x \in X} f(x) - \bigwedge_{x \in X} g(x) \right| &\leq \bigvee_{x \in X} |f(x) - g(x)|. \end{aligned} \quad (32)$$

Theorem 7. *Suppose A, B are two CFSs in X , then*

$$\begin{aligned} C_{\text{CFS}}(\Gamma_1 A, \Gamma_1 B) &= C_{\text{CFS}}(A, B), \\ C_{\text{CFS}}(\Gamma_2 A, \Gamma_2 B) &= C_{\text{CFS}}(A, B), \\ C_{\text{CFS}}(\Gamma_3 A, \Gamma_3 B) &= C_{\text{CFS}}(A, B). \end{aligned} \quad (33)$$

Proof. Trivial. \square

Example 2. Let $X = \{x_1, x_2, x_3\}$; two CFSs A and B in X are given as follows:

$$\begin{aligned} A &= \frac{0.4e^{j0.2\pi}}{x_1} + \frac{0.2e^{j0.2\pi}}{x_2} + \frac{0.7e^{j0.6\pi}}{x_3}, \\ B &= \frac{0.5e^{j0.4\pi}}{x_1} + \frac{0.3e^{j0.2\pi}}{x_2} + \frac{0.6e^{j0.6\pi}}{x_3}. \end{aligned} \quad (34)$$

Clearly, we have

$$\begin{aligned} \Gamma_1 A &= \frac{0.6e^{-j0.2\pi}}{x_1} + \frac{0.8e^{-j0.2\pi}}{x_2} + \frac{0.3e^{-j0.6\pi}}{x_3}, \\ \Gamma_1 B &= \frac{0.5e^{-j0.4\pi}}{x_1} + \frac{0.7e^{-j0.2\pi}}{x_2} + \frac{0.4e^{-j0.6\pi}}{x_3}, \\ \Gamma_2 A &= \frac{0.6e^{j0.2\pi}}{x_1} + \frac{0.8e^{j0.2\pi}}{x_2} + \frac{0.3e^{j0.6\pi}}{x_3}, \\ \Gamma_2 B &= \frac{0.5e^{j0.4\pi}}{x_1} + \frac{0.7e^{j0.2\pi}}{x_2} + \frac{0.4e^{j0.6\pi}}{x_3}, \\ \Gamma_3 A &= \frac{0.6e^{j1.2\pi}}{x_1} + \frac{0.8e^{j1.2\pi}}{x_2} + \frac{0.3e^{j1.6\pi}}{x_3}, \\ \Gamma_3 B &= \frac{0.5e^{j1.4\pi}}{x_1} + \frac{0.7e^{j1.2\pi}}{x_2} + \frac{0.4e^{j1.6\pi}}{x_3}. \end{aligned} \quad (35)$$

It is easy to verify that $C_{\text{CFS}}(\Gamma_1 A, \Gamma_1 B) = C_{\text{CFS}}(A, B)$, $C_{\text{CFS}}(\Gamma_2 A, \Gamma_2 B) = C_{\text{CFS}}(A, B)$, and $C_{\text{CFS}}(\Gamma_3 A, \Gamma_3 B) = C_{\text{CFS}}(A, B)$.

TABLE 1: Properties of distance measures for CFSs.

	Range	Reflectional invariance	Rotational invariance	Ratio scale invariance
$D_{\text{CFS}}(A, B)$	$[0, \pi]$	\checkmark	\checkmark	\checkmark
$D_Z(A, B)$	$[0, 1]$	\checkmark	\times	\times
$D_H(A, B)$	$[0, n]$	\checkmark	\times	\times
$D_{nH}(A, B)$	$[0, 1]$	\checkmark	\times	\times
$D_E(A, B)$	$[0, \sqrt{n}]$	\checkmark	\times	\times
$D_{nE}(A, B)$	$[0, 1]$	\checkmark	\times	\times
$D_p(A, B)$	$[0, n^{1/p}]$	\checkmark	\checkmark	\times
$D_{np}(A, B)$	$[0, 1]$	\checkmark	\checkmark	\times

Theorem 8. Let $\epsilon_1, \epsilon_2 \in [0, \pi/4]$ and A, B, C be CFSs in X ; if we have $\bigvee_{x \in X} |\nu_A(x) - \nu_B(x)| \leq \epsilon_1$ and $\bigvee_{x \in X} |\nu_B(x) - \nu_C(x)| \leq \epsilon_2$, then

$$C_{\text{CFS}}(A, C) \geq \cos(\epsilon_1 + \epsilon_2). \quad (36)$$

Proof. Since

$$\bigvee_{x \in X} |\nu_A(x) - \nu_C(x)| \leq \bigvee_{x \in X} |\nu_A(x) - \nu_B(x) + \nu_B(x) - \nu_C(x)| \leq \bigvee_{x \in X} (|\nu_A(x) - \nu_B(x)| + |\nu_B(x) - \nu_C(x)|) \leq \epsilon_1 + \epsilon_2, \quad (37)$$

then we have

$$\begin{aligned} C_{\text{CFS}}(A, C) &\geq \bigvee_{i=1}^n \cos(\nu_A(x) - \nu_C(x)) \geq \bigvee_{i=1}^n \cos(\epsilon_1 + \epsilon_2) \\ &\geq \cos(\epsilon_1 + \epsilon_2). \end{aligned} \quad (38)$$

□

Example 3. Let $X = \{x_1, x_2, x_3\}$; two CFSs A and B in X are given as follows:

$$\begin{aligned} A &= \frac{0.7e^{j0.52\pi}}{x_1} + \frac{0.5e^{j0.22\pi}}{x_2} + \frac{0.9e^{j0.61\pi}}{x_3}, \\ B &= \frac{0.7e^{j0.54\pi}}{x_1} + \frac{0.5e^{j0.23\pi}}{x_2} + \frac{0.7e^{j0.62\pi}}{x_3}, \\ C &= \frac{0.9e^{j0.63\pi}}{x_1} + \frac{0.2e^{j0.25\pi}}{x_2} + \frac{0.6e^{j0.55\pi}}{x_3}. \end{aligned} \quad (39)$$

Clearly, we have $\bigvee_{x \in X} |\nu_A(x) - \nu_B(x)| \leq 0.02\pi$ and $\bigvee_{x \in X} |\nu_B(x) - \nu_C(x)| \leq \epsilon_1 \leq 0.09\pi$. Thus, we can verify that

$$\begin{aligned} C_{\text{CFS}}(A, C) &= \max(\cos(0.11\pi), \cos(0.03\pi), \cos(0.06\pi)) \\ &\geq \max(\cos(0.11\pi), \cos(0.11\pi), \cos(0.11\pi)) \\ &= \cos(0.11\pi) = \cos(0.02\pi + 0.09\pi). \end{aligned} \quad (40)$$

Theorem 9. Let $\epsilon_1, \epsilon_2 \in [0, \pi/4]$ and A, B, C, D be CFSs in X ; if we have $\bigvee_{x \in X} |\nu_A(x) - \nu_B(x)| \leq \epsilon_1$ and $\bigvee_{x \in X} |\nu_B(x) - \nu_C(x)| \leq \epsilon_2$, then

$$\begin{aligned} C_{\text{CFS}}(A \cup C, B \cup D) &\geq \min(\cos(\epsilon_1), \cos(\epsilon_2)), \\ C_{\text{CFS}}(A \cap C, B \cap D) &\geq \min(\cos(\epsilon_1), \cos(\epsilon_2)), \end{aligned} \quad (41)$$

where $\otimes \in \{\wedge, \vee\}$.

Proof. Let $\otimes = \vee$; from Lemma 1, we have

$$\nu_A(x) \vee \nu_C(x) - \nu_B(x) \vee \nu_D(x) \leq \max(|\nu_A(x) - \nu_B(x)|, |\nu_C(x) - \nu_D(x)|) \leq \max(\epsilon_1, \epsilon_2). \quad (42)$$

Then, we have

$$\begin{aligned} C_{\text{CFS}}(A \cup C, B \cup D) &\geq \bigvee_{i=1}^n \cos(\nu_{A \cup C}(x) - \nu_{B \cup D}(x)) \\ &\geq \bigvee_{i=1}^n \min(\cos(\epsilon_1), \cos(\epsilon_2)) \\ &\geq \min(\cos(\epsilon_1), \cos(\epsilon_2)). \end{aligned} \quad (43)$$

The other cases can be similarly proved. □

Corollary 1. Suppose that $\epsilon_i \in [0, \pi/4]$ and $\forall_{x \in X} |\nu_{A_i}(x) - \nu_{B_i}(x)| \leq \epsilon_i$ ($i = 1, 2, \dots, n$). For $\otimes \in \{\vee, \wedge\}$, we have

- (i) $C_{CFS}(\bigcup_{i=1}^n A_i, \bigcup_{i=1}^n B_i) \geq \min_i \cos(\epsilon_i)$
- (ii) $C_{CFS}(\bigcap_{i=1}^n A_i, \bigcap_{i=1}^n B_i) \geq \min_i \cos(\epsilon_i)$

Corollary 2. Suppose that $\epsilon_{ij} \in [0, \pi/4]$ and $\forall_{x \in X} |\nu_{A_{ij}}(x) - \nu_{B_{ij}}(x)| \leq \epsilon_{ij}$, ($i = 1, 2, \dots, n_1, j = 1, 2, \dots, n_2$). For $\otimes \in \{\vee, \wedge\}$, we have

- (i) $C_{CFS}(\bigcap_{i=1}^{n_1} \bigcup_{j=1}^{n_2} A_{ij}, \bigcap_{i=1}^{n_1} \bigcup_{j=1}^{n_2} B_{ij}) \geq \min_{i,j} \cos(\epsilon_{i,j})$
- (ii) $C_{CFS}(\bigcup_{i=1}^{n_1} \bigcap_{j=1}^{n_2} A_{ij}, \bigcup_{i=1}^{n_1} \bigcap_{j=1}^{n_2} B_{ij}) \geq \min_{i,j} \cos(\epsilon_{i,j})$

$$\nu_A(x) + \nu_C(x) - \nu_B(x) + \nu_D(x) \leq |\nu_A(x) - \nu_B(x)| + |\nu_C(x) - \nu_D(x)| \leq \epsilon_1 + \epsilon_2. \quad (45)$$

Then, we have

$$\begin{aligned} C_{CFS}(A \cup C, B \cup D) &\geq \bigvee_{i=1}^n \cos(\nu_{AUC}(x) - \nu_{BUD}(x)) \\ &\geq \bigvee_{i=1}^n (\cos(\epsilon_1) + \cos(\epsilon_2)) \geq \cos(\epsilon_1 + \epsilon_2). \end{aligned} \quad (46)$$

The other cases can be similarly proved. \square

Corollary 3. Suppose that $\epsilon_i \in [0, \pi/4]$ and $\forall_{x \in X} |\nu_{A_i}(x) - \nu_{B_i}(x)| \leq \epsilon_i$ ($i = 1, 2, \dots, n$). For $\otimes \in \{+, -\}$, we have

- (i) $C_{CFS}(\bigcup_{i=1}^n A_i, \bigcup_{i=1}^n B_i) \geq \cos(\sum_i \epsilon_i)$
- (ii) $C_{CFS}(\bigcap_{i=1}^n A_i, \bigcap_{i=1}^n B_i) \geq \cos(\sum_i \epsilon_i)$

Corollary 4. Suppose that $\epsilon_{ij} \in [0, \pi/4n_1n_2]$ and $\forall_{x \in X} |\nu_{A_{ij}}(x) - \nu_{B_{ij}}(x)| \leq \epsilon_{ij}$, ($i = 1, 2, \dots, n_1, j = 1, 2, \dots, n_2$). For $\otimes \in \{+, -\}$, we have

- (i) $C_{CFS}(\bigcap_{i=1}^{n_1} \bigcup_{j=1}^{n_2} A_{ij}, \bigcap_{i=1}^{n_1} \bigcup_{j=1}^{n_2} B_{ij}) \geq \cos(\sum_{i,j} \epsilon_{i,j})$
- (ii) $C_{CFS}(\bigcup_{i=1}^{n_1} \bigcap_{j=1}^{n_2} A_{ij}, \bigcup_{i=1}^{n_1} \bigcap_{j=1}^{n_2} B_{ij}) \geq \cos(\sum_{i,j} \epsilon_{i,j})$

Example 4. Let $X = \{x_1, x_2, x_3\}$; two CFSs A and B in X are given as follows:

$$\begin{aligned} A &= \frac{0.6e^{j0.52\pi}}{x_1} + \frac{0.1e^{j0.22\pi}}{x_2} + \frac{0.8e^{j0.61\pi}}{x_3}, \\ B &= \frac{0.5e^{j0.51\pi}}{x_1} + \frac{0.3e^{j0.23\pi}}{x_2} + \frac{0.8e^{j0.62\pi}}{x_3}, \\ C &= \frac{0.8e^{j1.13\pi}}{x_1} + \frac{0.2e^{j1.25\pi}}{x_2} + \frac{0.4e^{j1.45\pi}}{x_3}, \\ D &= \frac{0.8e^{j1.23\pi}}{x_1} + \frac{0.3e^{j1.25\pi}}{x_2} + \frac{0.4e^{j1.55\pi}}{x_3}. \end{aligned} \quad (47)$$

Theorem 10. Let $\epsilon_1, \epsilon_2 \in [0, \pi/4]$ and A, B, C, D be CFSs in X ; if we have $\forall_{x \in X} |\nu_A(x) - \nu_B(x)| \leq \epsilon_1$ and $\forall_{x \in X} |\nu_C(x) - \nu_D(x)| \leq \epsilon_2$, then

$$\begin{aligned} C_{CFS}(A \cup C, B \cup D) &\geq \min(\cos(\epsilon_1), \cos(\epsilon_2)), \\ C_{CFS}(A \cap C, B \cap D) &\geq \min(\cos(\epsilon_1), \cos(\epsilon_2)), \end{aligned} \quad (44)$$

where $\otimes \in \{+, -\}$.

Proof. Let $\otimes = +$; from Lemma 1, we have

Clearly, we have $\forall_{x \in X} |\nu_A(x) - \nu_B(x)| \leq 0.01\pi$ and $\forall_{x \in X} |\nu_C(x) - \nu_D(x)| \leq 0.1\pi$. Let \star be the min t-norm, $\otimes = +$; then,

$$\begin{aligned} A \cap C &= \frac{0.6e^{j1.65\pi}}{x_1} + \frac{0.1e^{j1.47\pi}}{x_2} + \frac{0.4e^{j2.06\pi}}{x_3}, \\ B \cap D &= \frac{0.5e^{j1.74\pi}}{x_1} + \frac{0.3e^{j1.48\pi}}{x_2} + \frac{0.4e^{j2.17\pi}}{x_3}. \end{aligned} \quad (48)$$

Thus, we can verify that

$$\begin{aligned} C_{CFS}(A, B) &= \max(\cos(0.09\pi), \cos(0.01\pi), \cos(0.11\pi)) \\ &= \cos(0.11\pi) = \cos(0.01\pi + 0.1\pi). \end{aligned} \quad (49)$$

Let \star be the min t-norm, $\otimes = \wedge$; then,

$$\begin{aligned} A \cap C &= \frac{0.6e^{j0.52\pi}}{x_1} + \frac{0.1e^{j0.22\pi}}{x_2} + \frac{0.4e^{j0.61\pi}}{x_3}, \\ B \cap D &= \frac{0.5e^{j0.51\pi}}{x_1} + \frac{0.3e^{j0.23\pi}}{x_2} + \frac{0.4e^{j0.62\pi}}{x_3}. \end{aligned} \quad (50)$$

Thus, we can verify that

$$\begin{aligned} C_{CFS}(A, B) &= \max(\cos(0.01\pi), \cos(0.01\pi), \cos(0.01\pi)) \\ &= \cos(0.01\pi) = \min(\cos(0.01\pi), \cos(0.01\pi)). \end{aligned} \quad (51)$$

5.2. Robustness of Ramot et al.'s Complex Fuzzy Inference Method. Now, we study the robustness of Ramot et al.'s [15] complex fuzzy inference method. We only consider fuzzy modus ponens (FMP) of complex fuzzy inference, i.e., given an input CFS A^* and a complex fuzzy rule $A \rightarrow B$; then, infer an output CFS B^* , where A, A^* are CFSs on X and B, B^* are CFSs on Y .

In [15], $A \rightarrow B$ is represented by a complex fuzzy relation $R(X, Y)$, and the output B^* is denoted by $FMP(R, A^*)$.

Then, B^* is obtained as follows:

$$r_{B^*}(y) = \sup_{x \in X} [r_{A^*}(x) * r_R(x, y)], \quad (52)$$

$$v_{B^*}(y) = g(v_{A^*}(x) \otimes v_R(x, y)), \quad (53)$$

where $\otimes \in \{+, -, \vee, \wedge\}$ and g is one of the following functions:

$$v_{B^*}(y) = \sup_{x \in X} (v_{A^*}(x) \otimes v_R(x, y)), \quad (54)$$

$$v_{B^*}(y) = \inf_{x \in X} (v_{A^*}(x) \otimes v_R(x, y)), \quad (55)$$

$$v_{B^*}(y) = \sum_{x \in X} (v_{A^*}(x) \otimes v_R(x, y)). \quad (56)$$

Theorem 11. Let $\epsilon \in [0, \pi/4]$, g be the function of (54) or (55), and $\otimes \in \{\vee, \wedge\}$; if $\forall_{x \in X} |v_{A^*}(x) - v_{A^{*'}}(x)| \leq \epsilon$, then

$$C_{\text{CFS}}(\text{FMP}(R, A^*), \text{FMP}(R, A^{*'})) \geq \cos(\epsilon). \quad (57)$$

Proof. Trivial from Corollaries 1 and 2. \square

Theorem 12. Let $|X| = n$, $\epsilon \in [0, \pi/4n]$, g be the function of (56), and $\otimes \in \{+, -\}$; if $\forall_{x \in X} |v_{A^*}(x) - v_{A^{*'}}(x)| \leq \epsilon$, then

$$C_{\text{CFS}}(\text{FMP}(R, A^*), \text{FMP}(R, A^{*'})) \geq \cos(n\epsilon). \quad (58)$$

$$C_{\text{CFS}}(\text{FMP}(R, A^*), \text{FMP}(R, A^{*'})) = \max(\cos(0.1\pi), \cos(0), \cos(0.1\pi)) = \cos(0.1\pi). \quad (62)$$

Let g be the function of (56) and $\otimes = +$; then,

$$\begin{aligned} \text{FMP}(R, A^*) &= \frac{e^{j4.2\pi}}{y_1} + \frac{e^{j3.7\pi}}{y_2} + \frac{e^{j3.4\pi}}{y_3}, \\ \text{FMP}(R, A^{*'}) &= \frac{e^{j4.5\pi}}{y_1} + \frac{e^{j3.9\pi}}{y_2} + \frac{e^{j3.7\pi}}{y_3}. \end{aligned} \quad (63)$$

$$\begin{aligned} C_{\text{CFS}}(\text{FMP}(R, A^*), \text{FMP}(R, A^{*'})) &= \max(\cos(0.3\pi), \cos(0.2\pi), \cos(0.3\pi)) \\ &= \cos(0.3\pi) = \cos(3 \times 0.1\pi). \end{aligned} \quad (64)$$

6. Conclusion

In this paper, a cosine similarity measure was proposed for CFSs by considering CFSs as sets of vectors in complex unit disk. In particular, the proposed cosine similarity measure is rotationally invariant, reflectionally invariant, and ratio scale invariant. All the existing measures of CFS in [40–42] are not ratio scale invariant. Finally, we applied the proposed similarity to measure the robustness of complex fuzzy connectives and Ramot et al.'s complex fuzzy inference.

Proof. Trivial from Corollaries 3 and 4. \square

Example 6. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$; a complex fuzzy relation $R(X, Y)$ on $X \times Y$ is given as follows:

$$R(X, Y) = \begin{bmatrix} e^{j0.5\pi} & e^{j0.2\pi} & e^{j0.3\pi} \\ e^{j0.6\pi} & e^{j0.7\pi} & e^{j0.3\pi} \\ e^{j0.7\pi} & e^{j0.4\pi} & e^{j0.4\pi} \end{bmatrix}. \quad (59)$$

Two CFSs are, respectively, given as follows:

$$\begin{aligned} A^* &= \frac{e^{j1.1\pi}}{x_1} + \frac{e^{j0.3\pi}}{x_2} + \frac{e^{j1.0\pi}}{x_3} \\ A^{*'} &= \frac{e^{j1.2\pi}}{x_1} + \frac{e^{j0.4\pi}}{x_2} + \frac{e^{j1.1\pi}}{x_3}. \end{aligned} \quad (60)$$

Clearly, we have $\forall_{x \in X} |v_{A^*}(x) - v_{A^{*'}}(x)| \leq 0.1\pi$. Let g be the function of (55) and $\otimes = \wedge$; then,

$$\begin{aligned} \text{FMP}(R, A^*) &= \frac{e^{j0.3\pi}}{y_1} + \frac{e^{j0.2\pi}}{y_2} + \frac{e^{j0.3\pi}}{y_3} \\ \text{FMP}(R, A^{*'}) &= \frac{e^{j0.4\pi}}{y_1} + \frac{e^{j0.2\pi}}{y_2} + \frac{e^{j0.2\pi}}{y_3}. \end{aligned} \quad (61)$$

Thus, we can verify that

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We should note that the similarity measure presented in this paper mostly depends on the phase term of CFSs. Our robustness results for complex fuzzy connectives are estimated based on the perturbation of the phase term of CFSs. It may be a little extreme. So, how to apply to these results in the real world is another problem of interest. In [43], Ma et al. proposed a CFS-based prediction method which can handle the uncertainty and periodicity simultaneously, in which the modulus part is used to describe the semantic uncertainty feature and the phase part is for the temporal periodicity

feature. In [21, 22], Bi et al. studied target selection application of CFSs, in which the modulus part is used to describe the distance and the phase part is for the direction of the target. Therefore, it will be meaningful to further investigate the real world application of these approaches with periodic (or direction) perturbations.

In the future, the proposed similarity measure can be extended to different complex fuzzy environments such as complex intuitionistic fuzzy set [44], complex Pythagorean fuzzy set [45], complex neutrosophic set [46], and complex q-rung orthopair fuzzy set [47] environments.

Data Availability

The data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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