Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 6759754, 5 pages
https://doi.org/10.1155/2020/6759754

Research Article

Stein Type Lemmas for Location-Scale Mixture of Generalized Skew-Elliptical Random Vectors

Baishuai Zuo and Chuancun Yin

School of Statistics, Qufu Normal University, Qufu, Shandong 273165, China

Correspondence should be addressed to Chuancun Yin; ccyin@qfnu.edu.cn

Received 17 February 2020; Accepted 20 April 2020; Published 30 May 2020

Academic Editor: Giovanni Falsone

Inspired by the work of Adcock, Landsman, and Shushi (2019) which established the Stein's lemma for generalized skew-elliptical random vectors, we derive Stein type lemmas for location-scale mixture of generalized skew-elliptical random vectors. Some special cases such as the location-scale mixture of elliptical random vectors, the location-scale mixture of generalized skew-normal random vectors, and the location-scale mixture of normal random vectors are also considered. As an application in risk theory, we give a result for optimal portfolio selection.

1. Introduction and Motivation

Since Stein [1] provides an expression $E[h(X)(X-\mu)]$ for normal random variable X, where $h(x)$ is an almost differentiable function, and a number of scholars have generalized the formula. For example, Landsman [2] gives Stein's lemma for 2-dimensional elliptical distributions; Landsman and Nešlehová [3] and Landsman et al. [4] derive Stein's lemma for multivariate elliptical distributions; Landsman et al. [5] establish Stein-type inequality for symmetric generalized hyperbolic distributions; Adcock et al. [6] derive Stein's lemma for generalized skew-elliptical distributions. The result has been applied in statistics, insurance, and finance. For example, Landsman et al. [5] and Landsman et al. [7] apply this lemma in risk theory.

In the study by Kim and Kim [8], the class of normal mean-variance mixture distributions is introduced. The random vector $\mathbf{X}$ is said to be an n-dimensional normal mean-variance mixture variable if $\mathbf{X}=\boldsymbol{\mu}+\Theta\boldsymbol{\gamma}+\Theta^{(1/2)}\mathbf{A}\mathbf{Z}$, where $\mathbf{Z}\sim N_k(\mathbf{0},\mathbf{I}_k)$, the k-dimensional normal random vectors with the identity covariance matrix; $\mathbf{A}$ is an $n\times k$ matrix; Θ is a scalar random variable that follows a nonnegative distribution with the density $\pi(\theta)$, independent of $\mathbf{Z}$; and the following are sontant vectors in R^n:

$$\begin{aligned}\boldsymbol{\mu}&=(\mu_1,\mu_2,\ldots,\mu_n)^T,\\ \boldsymbol{\gamma}&=(\gamma_1,\gamma_2,\ldots,\gamma_n)^T.\end{aligned}\tag{1}$$

These specification implies that conditionally, $\mathbf{X}|\Theta=\theta\sim N_n(\boldsymbol{\mu}+\theta\boldsymbol{\gamma},\theta\boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}=\mathbf{A}\mathbf{A}^T$. Inspired by this, we consider a class of location-scale mixture of generalized skew-elliptical distributions, which is generalization of the class of normal mean-variance mixture distributions. In this paper, we generalize Stein's lemma by Adcock et al. [6] to the case of location-scale mixture of generalized skew-elliptical random vectors.

The rest of the paper is organized as follows. Section 2 introduces the definitions and properties of the location-scale mixture of generalized skew-elliptical distributions. In Section 3, we derive three Stein-type lemmas. In Section 4, we give several special cases. An optimal portfolio selection (a three-fund theorem) for location-scale mixture of generalized skew-elliptical random vectors is given in Section 5.

2. Mixture of Generalized Skew-Elliptical Distributions

In this section, we introduce the class of location-scale mixture of generalized skew-elliptical (LSMGSE) distributions and some of its properties.

Let $\mathbf{Y}$ be an n-dimensional generalized skew-elliptical random vector and denoted by $\mathbf{Y} \sim GSE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n, \pi(\cdot))$. If its probability density function exists, the form will be (see [6])

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{2}{\sqrt{|\boldsymbol{\Sigma}|}} g_n\left\{\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right\} \pi\left(\Sigma^{-(1/2)}(\mathbf{y}-\boldsymbol{\mu})\right), \quad \mathbf{y} \in \mathbb{R}^n, \tag{2}$$

where

$$f_{\mathbf{X}}(\mathbf{x}) := \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} g_n\left\{\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{3}$$

is the density of n-dimensional elliptical random vector $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. Here, $\boldsymbol{\mu}$ is an $n \times 1$ location vector, $\boldsymbol{\Sigma}$ is an $n \times n$ scale matrix, and $g_n(u)$, $u \geq 0$, is the density generator of $\mathbf{X}$. $\pi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, is called the skewing function satisfying $\pi(-\mathbf{x}) = 1 - \pi(\mathbf{x})$ and $0 \leq \pi(\mathbf{x}) \leq 1$. The characteristic function of $\mathbf{X}$ takes the form $\varphi_{\mathbf{X}}(\mathbf{t}) = \exp\{i\mathbf{t}^T\boldsymbol{\mu}\}\psi((1/2)\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$, with function $\psi(t)\colon [0,\infty) \longrightarrow \mathbb{R}$, called the characteristic generator (see [9]). Suppose $\mathbf{A}$ be an $n \times n$ matrix and $\mathbf{b}$ be an $n \times 1$ vector. Then,

$$\mathbf{AY} + \mathbf{b} \sim GSE_n\left(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{A}, g_n, \pi(\cdot)\right). \tag{4}$$

To establish Stein's lemma for n-dimensional generalized skew-elliptical distributions, we use the cumulative generator $\overline{G}_n(u)$. It takes the following form (see [7] or [10]):

$$\overline{G}_n(u) = \int_u^\infty g_n(v)\mathrm{d}v. \tag{5}$$

Let $\mathbf{X}^* \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \overline{G}_n)$ be an elliptical random vector with generator $\overline{G}_n(u)$, whose density function (if it exists) is

$$f_{\mathbf{X}^*}(\mathbf{x}) = \frac{-1}{\psi'(0)\sqrt{|\boldsymbol{\Sigma}|}} \overline{G}_n\left\{\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{6}$$

Let $\mathbf{Y}^* \sim \mathrm{GSE}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \overline{G}_n, \pi(\cdot))$ be a generalized skew-elliptical random vector.

We call $\mathbf{Z} \sim \mathrm{LSMGSE}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\beta}, \Theta, g_n, \pi(\cdot))$ as an n-dimensional LSMGSE distribution with location parameter $\boldsymbol{\mu}$, positive definite scale matrix $\boldsymbol{\Sigma}$, and skew function $\pi(\cdot)$, if

$$\mathbf{Z} = \boldsymbol{\mu} + \Theta\boldsymbol{\beta} + \Theta^{(1/2)}\boldsymbol{\Sigma}^{(1/2)}\mathbf{Y}, \tag{7}$$

where $\boldsymbol{\beta} \in \mathbb{R}^n$ and $\mathbf{Y} \sim \mathrm{GSE}_n(\mathbf{0}, \mathbf{I}_n, g_n, \pi(\cdot))$. Assume that $\mathbf{Y}$ is independent of nonnegative scalar random variable Θ. We have

$$\mathbf{Z}|\Theta = \theta \sim \mathrm{GSE}_n(\boldsymbol{\mu} + \theta\boldsymbol{\beta}, \theta\boldsymbol{\Sigma}, g_n, \pi(\cdot)). \tag{8}$$

3. Main Result

In this section, we consider a random vector

$$\mathbf{Z} \sim \mathrm{LSMGSE}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\beta}, \Theta, g_n, \pi(\cdot)), \tag{9}$$

with location parameter $\boldsymbol{\mu}$, positive definite scale matrix $\boldsymbol{\Sigma}$, and skew function $\pi(\cdot)$ as (7).

Let $\varpi\colon \mathbb{R}^m \longrightarrow \mathbb{R}$, $1 \leq m \leq n$, be an almost everywhere differentiable function, and we write

$$\nabla\varpi\left(\mathbf{z}_{(1)}\right) = \left(\frac{\partial\varpi\left(\mathbf{z}_{(1)}\right)}{\partial z_1}, \frac{\partial\varpi\left(\mathbf{z}_{(1)}\right)}{\partial z_2}, \ldots, \frac{\partial\varpi\left(\mathbf{z}_{(1)}\right)}{\partial z_n}\right)^T. \tag{10}$$

We derive a Stein-type lemma for location-scale mixture of generalized skew-elliptical random vectors below. Partition $\mathbf{Z} = (\mathbf{Z}_{(1)}^T, \mathbf{Z}_{(2)}^T)^T$, where $\mathbf{Z}_{(1)} = (Z_1, Z_2, \ldots, Z_m)^T$ and $\mathbf{Z}_{(2)} = (Z_{m+1}, Z_{m+2}, \ldots, Z_n)^T$. $\boldsymbol{\mu} = (\boldsymbol{\mu}_{(1)}^T, \boldsymbol{\mu}_{(2)}^T)^T$ and $\boldsymbol{\mu} = (\boldsymbol{\mu}_{(1)}^T, \boldsymbol{\mu}_{(2)}^T)$ are also of similar partition.

Theorem 1. *Let $\mathbf{Z} \sim LSMGSE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\beta}, \Theta, g_n, \pi(\cdot))$ be an n-dimensional location-scale mixture of generalized skew-elliptical random vector defined as (7). Assume that function ϖ satisfies $E[\nabla_1\varpi(\mathbf{S}_{(1)}^*)] < \infty$. Then,*

$$\begin{aligned} &E\left[\varpi\left(\mathbf{Z}_{(1)}\right)(\mathbf{Z}-\boldsymbol{\mu})\right] \\ &= E_\theta\Big\{Cov(\mathbf{M})\Big[E\left[\nabla\varpi\left(\mathbf{S}_{(1)}^*\right)\right] + 2E\Big[\varpi\left(\mathbf{M}_{(1)}^*\right)\nabla\pi \\ &\quad\left((\theta\boldsymbol{\Sigma})^{-(1/2)}(\mathbf{M}^* - \boldsymbol{\mu} - \theta\boldsymbol{\beta})\right)\Big]\Big]\Big\} \\ &\quad + E_\theta\left\{E\left[\varpi\left(\mathbf{S}_{(1)}\right)\right]\theta\boldsymbol{\beta}\right\}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \mathbf{S} &= \left(\mathbf{S}_{(1)}^T, \mathbf{S}_{(2)}^T\right)^T = \mathbf{Z}|\Theta = \theta \sim \mathrm{GSE}_n(\boldsymbol{\mu} + \theta\boldsymbol{\beta}, \theta\boldsymbol{\Sigma}, g_n, \pi(\cdot)), \\ \mathbf{M} &= \left(\mathbf{M}_{(1)}^T, \mathbf{M}_{(2)}^T\right)^T \sim E_n(\boldsymbol{\mu} + \theta\boldsymbol{\beta}, \theta\boldsymbol{\Sigma}, g_n), \\ \mathbf{S}^* &= \left(\mathbf{S}_{(1)}^* T, \mathbf{S}_{(2)}^* T\right)^T \sim \mathrm{GSE}_n(\boldsymbol{\mu} + \theta\boldsymbol{\beta}, \theta\boldsymbol{\Sigma}, \overline{G}_n, \pi(\cdot)), \\ \mathbf{M}^* &= \left(\mathbf{M}_{(1)}^* T, \mathbf{M}_{(2)}^* T\right)^T \sim E_n(\boldsymbol{\mu} + \theta\boldsymbol{\beta}, \theta\boldsymbol{\Sigma}, \overline{G}_n). \end{aligned} \tag{12}$$

Proof. Using tower property of expectations, we obtain

$$E\left[\varpi\left(\mathbf{Z}_{(\mathbf{1})}\right)(\mathbf{Z}-\boldsymbol{\mu})\right]=E_{\Theta}\left[E\left[\varpi\left(\mathbf{Z}_{(\mathbf{1})}\right)(\mathbf{Z}-\boldsymbol{\mu})\mid\Theta\right]\right]. \tag{13}$$

while

$$\begin{aligned}
E\left[\varpi\left(\mathbf{Z}_{(\mathbf{1})}\right)(\mathbf{Z}-\boldsymbol{\mu})\mid\Theta=\theta\right] &= E\left[\varpi\left(\mathbf{Z}_{(\mathbf{1})}\mid\Theta=\theta\right)(\mathbf{Z}-\boldsymbol{\mu}\mid\Theta=\theta)\right] \\
&= E\left[\varpi\left(\mathbf{Z}_{(\mathbf{1})}\mid\Theta=\theta\right)((\mathbf{Z}\mid\Theta=\theta)-(\boldsymbol{\mu}+\theta\boldsymbol{\beta})+\theta\boldsymbol{\beta})\right] \\
&= E\left[\varpi\left(\mathbf{Z}_{(\mathbf{1})}\mid\Theta=\theta\right)((\mathbf{Z}\mid\Theta=\theta)-(\boldsymbol{\mu}+\theta\boldsymbol{\beta}))\right]+E\left[\varpi\left(\mathbf{Z}_{(\mathbf{1})}\mid\Theta=\theta\right)\theta\boldsymbol{\beta}\right] \\
&= E\left[\varpi\left(\mathbf{S}_{(\mathbf{1})}\right)(\mathbf{S}-(\boldsymbol{\mu}+\theta\boldsymbol{\beta}))\right]+E\left[\varpi\left(\mathbf{S}_{(\mathbf{1})}\right)\theta\boldsymbol{\beta}\right] \\
&= Cov(\mathbf{M})\left[E\left[\nabla\varpi\left(\mathbf{S}^{*}_{(1)}\right)\right]+2E\left[\varpi\left(\mathbf{M}^{*}_{(1)}\right)\nabla\pi\left((\theta\boldsymbol{\Sigma})^{-(1/2)}(\mathbf{M}^{*}-\boldsymbol{\mu}-\theta\boldsymbol{\beta})\right)\right]\right] \\
&\quad + E\left[\varpi\left(\mathbf{S}_{(1)}\right)\right]\theta\boldsymbol{\beta},
\end{aligned} \tag{14}$$

where the last equality have used (4), (8), and Theorem 3 by Adcock et al. [6]. Therefore, we obtain (11), which completes the proof of Theorem 1. □

Remark 1. From formula (8), we find that $E[\varpi(\mathbf{Z}_{(\mathbf{1})})(\mathbf{Z}-\boldsymbol{\mu})\mid\Theta]$ is a special case of Theorem 3 by Adcock et al. [6].

The following theorems give two special forms of Stein-type lemmas for location-scale mixture of generalized skew-elliptical random vectors.

Theorem 2. *Let $\mathbf{Z}\sim LSMGSE_n(\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\beta},\alpha,g_n,\pi(\cdot))$ be an n-dimensional location-scale mixture of generalized skew-elliptical random vector with*

$$\mathbf{Z}=\boldsymbol{\mu}+V^{-1}\boldsymbol{\beta}+V^{-(1/2)}\boldsymbol{\Sigma}^{(1/2)}\mathbf{Y}, \tag{15}$$

where $V\sim beta(\alpha,1)$. Assume the function ϖ satisfies $E[\nabla_1\varpi(\mathbf{S}^{*}_{(\mathbf{1})})]<\infty$. Then,

$$\begin{aligned}
E\left[\varpi\left(\mathbf{Z}_{(\mathbf{1})}\right)(\mathbf{Z}-\boldsymbol{\mu})\right] &= E_{v}\Big\{Cov(\mathbf{M})\Big[E\left[\nabla\varpi\left(\mathbf{S}^{*}_{(\mathbf{1})}\right)\right] \\
&\quad +2E\left[\varpi\left(\mathbf{M}^{*}_{(\mathbf{1})}\right)\nabla\pi\left(\left(v^{-1}\boldsymbol{\Sigma}\right)^{-(1/2)}\left(\mathbf{M}^{*}-\boldsymbol{\mu}-v^{-1}\boldsymbol{\beta}\right)\right)\right]\Big]\Big\} \\
&\quad + E_{v}\left\{E\left[\varpi\left(\mathbf{S}_{(1)}\right)\right]v^{-1}\boldsymbol{\beta}\right\},
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\mathbf{S} &= \left(\mathbf{S}^{T}_{(\mathbf{1})},\mathbf{S}^{T}_{(\mathbf{2})}\right)^{T}=\mathbf{Z}|V=v\sim \mathrm{GSE}_n\left(\boldsymbol{\mu}+v^{-1}\boldsymbol{\beta},v^{-1}\boldsymbol{\Sigma},g_n,\pi(\cdot)\right), \\
\mathbf{M} &= \left(\mathbf{M}^{T}_{(\mathbf{1})},\mathbf{M}^{T}_{(\mathbf{2})}\right)^{T}\sim E_n\left(\boldsymbol{\mu}+v^{-1}\boldsymbol{\beta},v^{-1}\boldsymbol{\Sigma},g_n\right), \\
\mathbf{S}^{*} &= \left(\mathbf{S}^{*}_{(1)}T,\mathbf{S}^{*}_{(2)}T\right)^{T}\sim \mathrm{GSE}_n\left(\boldsymbol{\mu}+v^{-1}\boldsymbol{\beta},v^{-1}\boldsymbol{\Sigma},\overline{G}_n,\pi(\cdot)\right), \\
\mathbf{M}^{*} &= \left(\mathbf{M}^{*}_{(1)}T,\mathbf{M}^{*}_{(2)}T\right)^{T}\sim E_n\left(\boldsymbol{\mu}+v^{-1}\boldsymbol{\beta},v^{-1}\boldsymbol{\Sigma},\overline{G}_n\right).
\end{aligned} \tag{17}$$

Proof. Letting $\Theta=V^{-1}$ in Theorem 1, we directly obtain (16). This completes the proof of Theorem 2. □

Remark 2. Letting $\pi(\cdot)=(1/2)$ in Theorem 2, we obtain a Stein-type lemma for location-scale mixture of elliptical random vectors:

$$\begin{aligned}
E\left[\varpi\left(\mathbf{Z}_{(\mathbf{1})}\right)(\mathbf{Z}-\boldsymbol{\mu})\right] &= E_{v}\left\{Cov(\mathbf{M})E\left[\nabla\varpi\left(\mathbf{M}^{*}_{(\mathbf{1})}\right)\right]\right\} \\
&\quad + E_{v}\left\{E\left[\varpi\left(\mathbf{M}_{(1)}\right)\right]v^{-1}\boldsymbol{\beta}\right\}.
\end{aligned} \tag{18}$$

Theorem 3. *Let $\mathbf{Z}\sim LSMGSE_n(\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\beta},\Theta,g_n,\pi(\cdot))$ be an n-dimensional location-scale mixture of generalized skew-elliptical random vector defined as (7). Assume the function ϖ satisfies $E[\nabla_1\varpi(\mathbf{S}^{*})]<\infty$. Then,*

$$\begin{aligned}
&E[\varpi(\mathbf{Z})(\mathbf{Z}-\boldsymbol{\mu})] \\
&\quad = E_{\theta}\Big\{Cov(\mathbf{M})\left[E[\nabla\varpi(\mathbf{S}^{*})]+2E\left[\varpi(\mathbf{M}^{*})\nabla\pi\left((\theta\boldsymbol{\Sigma})^{-(1/2)}\right.\right.\right. \\
&\quad \left.\left.\left.(\mathbf{M}^{*}-\boldsymbol{\mu}-\theta\boldsymbol{\beta})\right)\right]\right]\Big\} \\
&\quad + E_{\theta}\{E[\varpi(\mathbf{S})]\theta\boldsymbol{\beta}\}.
\end{aligned} \tag{19}$$

Proof. Letting $\varpi(\mathbf{z}_{(1)}) = \varpi(\mathbf{z})$ in Theorem 1, we obtain (19). This completes the proof of Theorem 3. □

Remark 3. Letting $\pi(\cdot) = (1/2)$ in Theorem 3, we obtain a Stein-type lemma for location-scale mixture of elliptical random vectors:

$$E[\varpi(\mathbf{Z})(\mathbf{Z}-\boldsymbol{\mu})] = E_\theta\{\mathrm{Cov}(\mathbf{M})E[\nabla\varpi(\mathbf{M}^*)]\} + E_\theta\{E[\varpi(\mathbf{M})]\theta\boldsymbol{\beta}\}. \tag{20}$$

4. Special Cases

In this section, we consider several special cases including the location-scale mixture of elliptical distribution, the location-scale mixture of generalized skew-normal distribution, the location-scale mixture of skew-normal distribution, and the location-scale mixture of normal distribution.

Example 1. Letting $\pi(\cdot) = (1/2)$ in Theorem 1, Stein-type lemma for location-scale mixture of elliptical random vector is given by

$$\begin{aligned} E\left[\varpi\left(\mathbf{Z}_{(1)}\right)(\mathbf{Z}-\boldsymbol{\mu})\right] &= E_\theta\left\{Cov(\mathbf{M})E\left[\nabla\varpi\left(\mathbf{M}^*_{(1)}\right)\right]\right\} \\ &\quad + E_\theta\left\{E\left[\varpi\left(\mathbf{M}_{(1)}\right)\right]\theta\boldsymbol{\beta}\right\}. \end{aligned} \tag{21}$$

Remark 4. We find that (21) can be regarded as a special analogue case of Vanduffel and Yao [11].

Example 2. Suppose $\mathbf{Y} \sim \mathrm{GSN}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \pi(\cdot))$ is an n-dimensional generalized skew-normal random vector with probability density function (pdf) as follows:

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{2}{\sqrt{|\boldsymbol{\Sigma}|}(2\pi)^{n/2}}\exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right\}\pi\left(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-(1/2)}(\mathbf{y}-\boldsymbol{\mu})\right), \tag{22}$$

$\mathbf{y} \in \mathbb{R}^n$, where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T$ and function $\pi(\cdot)\colon \mathbb{R} \longrightarrow \mathbb{R}$. Letting $\overline{G}_n(u) = g_n(u) = (2\pi)^{-(n/2)}\exp\{-u\}$ and

$$\pi\left(\boldsymbol{\Sigma}^{-(1/2)}(\mathbf{y}-\boldsymbol{\mu})\right) = \pi\left(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-(1/2)}(\mathbf{y}-\boldsymbol{\mu})\right), \tag{23}$$

in Theorem 1. Assuming that function ϖ satisfies $E[\nabla_1\varpi(\mathbf{S}_{(1)})] < \infty$, Stein-type lemma for location-scale mixture of generalized skew-normal random vector $\mathbf{Z} \sim \mathrm{LSMGSN}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\beta}, \Theta, \pi(\cdot))$ is given by

$$\begin{aligned} &E\left[\varpi\left(\mathbf{Z}_{(1)}\right)(\mathbf{Z}-\boldsymbol{\mu})\right] \\ &= E_\theta\left\{\theta\boldsymbol{\Sigma}E\left[\nabla\varpi\left(\mathbf{S}_{(1)}\right)\right] + 2(\theta\boldsymbol{\Sigma})^{1/2}\boldsymbol{\gamma}E\left[\varpi\left(\mathbf{M}_{(1)}\right)\pi'\right.\right. \\ &\quad \left.\left.\left(\boldsymbol{\gamma}^T(\theta\boldsymbol{\Sigma})^{-1/2}(\mathbf{M}-\boldsymbol{\mu}-\theta\boldsymbol{\beta})\right)\right]\right\} + E_\theta\left\{E\left[\varpi\left(\mathbf{S}_{(1)}\right)\right]\theta\boldsymbol{\beta}\right\}, \end{aligned} \tag{24}$$

where $\pi'(\cdot)$ is the derivative of $\pi(\cdot)$, and

$$\begin{aligned} \mathbf{S} &= \left(\mathbf{S}^T_{(1)}, \mathbf{S}^T_{(2)}\right)^T = \mathbf{Z}|\Theta = \theta \sim \mathrm{GSN}_n(\boldsymbol{\mu}+\theta\boldsymbol{\beta}, \theta\boldsymbol{\Sigma}, \pi(\cdot)), \\ \mathbf{M} &= \left(\mathbf{M}^T_{(1)}, \mathbf{M}^T_{(2)}\right)^T \sim N_n(\boldsymbol{\mu}+\theta\boldsymbol{\beta}, \theta\boldsymbol{\Sigma}). \end{aligned} \tag{25}$$

Example 3. Letting $\pi(\cdot) = \Phi(\cdot)$(the cdf of a standard normal distribution) in Example 2, Stein-type lemma for location-scale mixture of skew-normal random vector is given by

$$\begin{aligned} &E\left[\varpi\left(\mathbf{Z}_{(1)}\right)(\mathbf{Z}-\boldsymbol{\mu})\right] \\ &= E_\theta\left\{\theta\boldsymbol{\Sigma}E\left[\nabla\varpi\left(\mathbf{S}_{(1)}\right)\right] + \sqrt{\frac{2}{\pi}}(\theta\boldsymbol{\Sigma})^{1/2}\boldsymbol{\gamma}E\left[\varpi\left(\mathbf{M}_{(1)}\right)\exp\left\{-\frac{1}{2}\right.\right.\right. \\ &\quad \left.\left.\left.\left(\boldsymbol{\gamma}^T(\theta\boldsymbol{\Sigma})^{-(1/2)}(\mathbf{M}-\boldsymbol{\mu}-\theta\boldsymbol{\beta})\right)^2\right\}\right]\right\} + E_\theta\left\{E\left[\varpi\left(\mathbf{S}_{(1)}\right)\right]\theta\boldsymbol{\beta}\right\}. \end{aligned} \tag{26}$$

Example 4. Letting $\pi(\cdot) = 1$ in Example 2, Stein-type lemma for location-scale mixture of normal random vector is given by

$$E\left[\varpi\left(\mathbf{Z}_{(1)}\right)(\mathbf{Z}-\boldsymbol{\mu})\right] = E_\theta\left\{\theta\boldsymbol{\Sigma}E\left[\nabla\varpi\left(\mathbf{M}_{(1)}\right)\right]\right\} + E_\theta\left\{E\left[\varpi\left(\mathbf{M}_{(1)}\right)\right]\theta\boldsymbol{\beta}\right\}. \tag{27}$$

5. Application in Risk Theory

Considering n risky assets with stochastic returns that are modelled by the n-dimensional random vector,

$$\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)^T \sim \mathrm{LSMGSE}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\beta}, \Theta, g_n, \pi(\cdot)). \tag{28}$$

A risk-free asset bearing a fixed rate of return $r > 0$ is also available. Denote by $\mathbf{t} = (t_1, t_2, \ldots, t_n)^T$ the vector of proportions that are allocated to the different risky assets. The total portfolio return is

$$\mathbf{Z_t} = \sum_{i=1}^n t_i Z_i + \left(1 - \sum_{i=1}^n t_i Z_i\right) r. \tag{29}$$

We define

$$\begin{aligned} \mathbf{S_t} &= \sum_{i=1}^n t_i S_i + \left(1 - \sum_{i=1}^n t_i S_i\right) r, \\ \mathbf{S^*_t} &= \sum_{i=1}^n t_i S^*_i + \left(1 - \sum_{i=1}^n t_i S^*_i\right) r, \\ \mathbf{M^*_t} &= \sum_{i=1}^n t_i M^*_i + \left(1 - \sum_{i=1}^n t_i M^*_i\right) r. \end{aligned} \tag{30}$$

To find an optimal allocation by maximizing the mean return for a given variance risk tolerance, we assume that the investor optimizes

$$L(\mathbf{t}) := E[U(\mathbf{Z_t})], \quad (31)$$

where U: $\mathbb{R} \longrightarrow \mathbb{R}$ is a concave utility function (see [11]).

Theorem 4. *(Three-Fund Separation). Suppose $L(\mathbf{t})$ is a concave continuously differentiable function with $\mathbf{t} \in \mathbb{R}^n$. The solution to problem (31) is given as*

$$\mathbf{t}_{\mathrm{LSMGSE}} = \frac{\mathbf{\Sigma}^{-1}\boldsymbol{\delta}_1}{\psi'(0)E_\theta\left\{\theta E\left[U''\left(\mathbf{S}^*_{\mathbf{t}_{\mathrm{LSMGSE}}}\right)\right]\right\}} + \frac{\boldsymbol{\delta}_2}{E_\theta\left\{\theta E\left[U''\left(\mathbf{S}^*_{\mathbf{t}_{\mathrm{LSMGSE}}}\right)\right]\right\}}, \quad (32)$$

where $\mathbf{e} = (1, 1, \ldots, 1)^T$ *is an* $n \times 1$ *vector whose elements are all equal to 1, and*

$$\begin{aligned}\boldsymbol{\delta}_1 &= E_\theta\left\{E\left[U'\left(\mathbf{S}_{\mathbf{t}_{\mathrm{LSMGSE}}}\right)\right]\theta\boldsymbol{\beta}\right\} + E\left[U'\left(\mathbf{Z}_{\mathbf{t}_{\mathrm{LSMGSE}}}\right)(\boldsymbol{\mu} - r\mathbf{e})\right],\\ \boldsymbol{\delta}_2 &= E_\theta\left\{2\theta E\left[U'\left(\mathbf{M}^*_{\mathbf{t}_{\mathrm{LSMGSE}}}\right)\nabla\pi\left((\theta\mathbf{\Sigma})^{-(1/2)}(\mathbf{M}^* - \boldsymbol{\mu} - \theta\boldsymbol{\beta})\right)\right]\right\}.\end{aligned} \quad (33)$$

Proof. Letting $\nabla L(\mathbf{t}) = \mathbf{0}$, we have

$$\begin{aligned}E[U'(\mathbf{Z_t})(\mathbf{Z} - r\mathbf{e})] &= \mathbf{0},\\ E[U'(\mathbf{Z_t})(\mathbf{Z} - \boldsymbol{\mu})] + E[U'(\mathbf{Z_t})(\boldsymbol{\mu} - r\mathbf{e})] &= \mathbf{0}.\end{aligned} \quad (34)$$

Using (19), we get

$$\begin{aligned}&E_\theta\{Cov(\mathbf{M})[E[\nabla U'(\mathbf{S}^*_\mathbf{t})] + 2E[U'(\mathbf{M}^*_\mathbf{t})\nabla\pi\\ &\left((\theta\mathbf{\Sigma})^{-(1/2)}(\mathbf{M}^* - \boldsymbol{\mu} - \theta\boldsymbol{\beta})\right)]]\}\\ &+ E_\theta\{E[U'(\mathbf{S_t})]\theta\boldsymbol{\beta}\} + E[U(\mathbf{Z_t})(\boldsymbol{\mu} - r\mathbf{e})] = \mathbf{0}.\end{aligned} \quad (35)$$

Note that $E[\partial U'(\mathbf{Z}^*_\mathbf{t})/\partial Z^*_i] = E[t_i U''(\mathbf{Z}^*_\mathbf{t})]$ and $\mathrm{Cov}(\mathbf{M}) = -\psi'(0)\theta\mathbf{\Sigma}$; we have

$$\begin{aligned}&E_\theta\{-\psi'(0)\theta\mathbf{\Sigma}[E[\mathbf{t}U''(\mathbf{S}^*_\mathbf{t})] + 2E[U'(\mathbf{M}^*_\mathbf{t})\nabla\pi\\ &\left((\theta\mathbf{\Sigma})^{-(1/2)}(\mathbf{M}^* - \boldsymbol{\mu} - \theta\boldsymbol{\beta})\right)]]\}\\ &+ E_\theta\{E[U'(\mathbf{S_t})]\theta\boldsymbol{\beta}\} + E[U'(\mathbf{Z_t})(\boldsymbol{\mu} - r\mathbf{e})] = \mathbf{0}.\end{aligned} \quad (36)$$

Therefore, we obtain (32), which completes the proof of Theorem 4. □

Remark 5. When $L(\cdot)$ is only defined on a convex subset A of $\mathbb{R}^n$, a solution $\mathbf{t}$ to (32) is only optimal when $\mathbf{t}$ belongs to the interior of this set A. Otherwise, the optimum has to be found on the boundary of A (see [12] or [11]).

Corollary 1. *Letting $\pi(\cdot) = 1$ in Theorem 4, we obtain*

$$\mathbf{t}_{\mathrm{LSME}} = \frac{\mathbf{\Sigma}^{-1}\boldsymbol{\delta}_1}{\psi'(0)E_\theta\left\{\theta E\left[U''\left(\mathbf{M}^*_{\mathbf{t}_{\mathrm{LSME}}}\right)\right]\right\}}, \quad (37)$$

where $\delta_1 = E_\theta\left\{E[U'(\mathbf{M}_{\mathbf{t}_{LSME}})]\theta\boldsymbol{\beta}\right\} + E[U'(\mathbf{Z}_{\mathbf{t}_{LSME}})(\boldsymbol{\mu} - r\mathbf{e})]$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Nos. 11571198 and 11701319).

References

[1] C. M. Stein, "Estimation of the mean of a multivariate normal distribution," *The Annals of Statistics*, vol. 9, no. 6, pp. 1135–1151, 1981.

[2] Z. Landsman, "On the generalization of Stein's Lemma for elliptical class of distributions," *Statistics & Probability Letters*, vol. 76, no. 10, pp. 1012–1016, 2006.

[3] Z. Landsman and J. Nešlehová, "Stein's Lemma for elliptical random vectors," *Journal of Multivariate Analysis*, vol. 99, no. 5, pp. 912–927, 2008.

[4] Z. Landsman, S. Vanduffel, and J. Yao, "A note on Stein's lemma for multivariate elliptical distributions," *Journal of Statistical Planning and Inference*, vol. 143, no. 11, pp. 2016–2022, 2013.

[5] Z. Landsman, S. Vanduffel, and J. Yao, "Some Stein-type inequalities for multivariate elliptical distributions and applications," *Statistics & Probability Letters*, vol. 97, pp. 54–62, 2015.

[6] C. Adcock, Z. Landsman, and T. Shushi, "Stein's Lemma for generalized skew-elliptical random vectors," *Communications in Statistics—Theory and Methods*, pp. 1–16, 2019.

[7] Z. Landsman, U. Makov, and T. Shushi, "A multivariate tail covariance measure for elliptical distributions," *Insurance: Mathematics and Economics*, vol. 81, pp. 27–35, 2018.

[8] J. H. T. Kim and S.-Y. Kim, "Tail risk measures and risk allocation for the class of multivariate normal mean-variance mixture distributions," *Insurance: Mathematics and Economics*, vol. 86, pp. 145–157, 2019.

[9] K. T. Fang, S. Kotz, and K. W. Ng, *Symmetric Multivariate and Related Distributions*, Chapman and Hall, New York, USA, 1990.

[10] Z. M. Landsman and E. A. Valdez, "Tail conditional expectations for elliptical distributions," *North American Actuarial Journal*, vol. 7, no. 4, pp. 55–71, 2003.

[11] S. Vanduffel and J. Yao, "A Stein type lemma for the multivariate generalized hyperbolic distribution," *European Journal of Operational Research*, vol. 261, no. 2, pp. 606–612, 2017.

[12] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, UK, 2004.