Research Article

A Derivative-Free Liu–Storey Method for Solving Large-Scale Nonlinear Systems of Equations

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Received 22 June 2020; Revised 6 October 2020; Accepted 16 October 2020; Published 29 October 2020

Academic Editor: Chaudry M. Khalique

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In this paper, a descent Liu–Storey conjugate gradient method is extended to solve large-scale nonlinear systems of equations. Based on certain assumptions, the global convergence property is obtained with a nonmonotone line search. The proposed method is suitable to solve large-scale problems for the low-storage requirement. Numerical experiment results show that the new method is practically effective.

1. Introduction

Consider the following nonlinear system of equations:

$$F(x) = 0, \quad (1)$$

where $(F: R^n \rightarrow R^n)$ has continuous partial derivatives. In the past few decades, Newton's methods have been widely used to solve problem (1) for their fast convergence speeds, see [1–10]. Newton's methods determine the search direction by solving a linear system of equations, in which the coefficient matrix is the Jacobian matrix of F or an approximation of it. It is expensive to compute the coefficient matrix and solve the linear system of equations at each iteration. Therefore, Newton's methods are not suitable for solving a large-scale problem in which the Jacobian matrix of F is unavailable or needs massive storage space. To overcome this shortcoming of Newton's methods, in this paper, we attempt to develop a numerical algorithm based on a descent Liu–Storey conjugate gradient method and a nonmonotone line research.

Recently, La Cruz and Raydan [11] designed the *Spectral Algorithm for Nonlinear Equations* (SANE) to solve (1) and analyzed the convergence property based on a modified Grippo–Lampariello–Lucidi (GLL) [12] nonmonotone line

search. However, at each iteration, SANE needs to compute a directional derivative or an approximation of it. To improve the efficiency of the SANE algorithm, La Cruz et al. [13] proposed the derivative-free SANE (DF-SANE) algorithm by using a derivative-free nonmonotone line search method. Numerical experiment results show that the DF-SANE is efficient for solving large-scale problems. Similar to the DF-SANE method, Cheng and Li [14] developed the N-DF-SANE method for solving large-scale nonlinear systems of equations by combining the spectral residual method and the Zhang–Hager [15] nonmonotone line search. The global convergence of the method was established based on some mild conditions.

If the continuously differentiable mapping $F: R^n \rightarrow R^n$ is the gradient of $f: R^n \rightarrow R$, then (1) is the first-order necessary optimality condition of the optimization problem

$$\min f(x), \quad x \in R^n, \quad (2)$$

namely, we can get the solution of equation (1) by solving optimization problem (2). Therefore, it is reasonable to extend the numerical algorithms for solving (2) to solve large-scale nonlinear systems of equations. In this paper, we develop a derivative-free method for solving (1) based on a descent nonlinear conjugate gradient method.

Nonlinear conjugate gradient methods are efficient for solving large-scale unconstrained optimization problems for their low-memory requirements and excellent convergence properties. Among the nonlinear conjugate gradient methods, the Hestenes–Stiefel (HS), Polak–Ribière–Polyak (PRP), and Liu–Storey (LS) method [16–19] are the most famous methods for the excellent performance in practice. These methods have been studied and used extensively [20–28].

Recently, Li and Feng proposed a modified Liu–Storey conjugate gradient (MLS) method [24]. A good property of the MLS method is that, with any line search method, it can always generate descent search directions for objective functions. When the strong Wolfe line search is used, Li and Feng proved that their method is convergent globally.

The numerical algorithm for solving nonlinear systems of equations has been widely studied in the last decades. There are some numerical algorithms for nonlinear systems based on conjugate gradient methods [29–35]. Yuan and Zhang [29] proposed a numerical method for large-scale nonlinear systems of equations based on a three-term Polak–Ribière–Polyak conjugate gradient method and the hyperplane projection strategy given by Solodov and Svaiter [8]. Abubakar and Kumam [32] improved this three-term method by combining the idea of Yan et al. [36]. Yuan et al. [30] also extended the Hestenes–Stiefel conjugate gradient method to solve large-scale nonlinear equations and establish its global convergence. Bala and Kumam [31] proposed a numerical method for nonlinear systems by combining a descent Dai–Liao conjugate gradient and hyperplane projection method. Yu [34] extended the PRP method to solve large-scale nonlinear systems of equations based on the Grippo–Lampariello–Lucidi (GLL) [12] and Li–Fukushima (LF) [5] nonmonotone line search.

Inspired by the CG-type methods for the nonlinear system of equations, in this paper, we attempt to extend the MLS conjugate gradient method to solve problem (1) for the sufficient descent property, global convergence, and excellent numerical performance. To guarantee the global convergence of the new method, we determine the step length by a nonmonotone line search method which is a modification of the line search in [14, 15].

Our paper is organized as follows: in the second section, we introduce the algorithm. In Section 3, we analyze the global convergence of the proposed method. Some numerical experiment results are reported and analyzed in the last section.

Throughout this paper, we denote the Euclidean norm of vectors by $\|\cdot\|$, and $J(x)$ means the Jacobian matrix of F at x .

2. Algorithm

In the remainder of this paper, we let

$$f(x) = \frac{1}{2} \|F(x)\|^2. \quad (3)$$

Based on the analysis in Section 1, we now extend the MLS method [24] to solve (1), and the steps of the algorithm are stated as follows.

$$\sum_{k=0}^{\infty} \eta_k = \eta < \infty. \quad (4)$$

Step 1: compute d_k by

$$d_k = -F_k + \beta_k^{\text{MLS}} d_{k-1}, \quad d_0 = -F_0, \quad (5)$$

$$\beta_k^{\text{MLS}} = -\frac{F_k^T y_{k-1}}{F_{k-1}^T d_{k-1}} - t \frac{\|y_{k-1}\|^2 F_k^T d_{k-1}}{(F_{k-1}^T d_{k-1})^2}, \quad (6)$$

where $F_k = F(x_k)$, $(y_{k-1} = F_k - F_{k-1})$ and $t > (1/4)$ is a constant.

Step 2: select a suitable initial step length $\alpha_{k_0} \in [\alpha_{\min}, \alpha_{\max}]$, and set $\alpha_k = \alpha_{k_0}$.

Step 3: nonmonotone line search step:

If

$$f(x_k + \alpha_k d_k) \leq C_k + \eta_k - \lambda_1 \alpha_k^2 \|d_k\|^4 - \lambda_2 \alpha_k^2 f(x_k), \quad (7)$$

then set $x_{k+1} = x_k + \alpha_k d_k$, and go to Step 4.

Else, if

$$f(x_k - \alpha_k d_k) \leq C_k + \eta_k - \lambda_1 \alpha_k^2 \|d_k\|^4 - \lambda_2 \alpha_k^2 f(x_k), \quad (8)$$

then set $x_{k+1} = x_k - \alpha_k d_k$, and go to Step 4.

Else, set $(\alpha_k = \rho \alpha_k)$, and go to Step 3.

Step 4: stop if $\|F(x_{k+1})\| = 0$.

Step 5: select $\tau_k \in [\tau_{\min}, \tau_{\max}]$, and let

$$Q_{k+1} = \tau_k Q_k + 1, \quad C_{k+1} = \frac{\tau_k Q_k (C_k + \eta_k) + f(x_{k+1})}{Q_{k+1}}. \quad (9)$$

Set $k = k + 1$, and go to Step 1.

The line search is the modification of the nonmonotone line search of Cheng and Li [14]. Define

$$A_k = \frac{\sum_{i=0}^k (f(x_i) + i\eta_{i-1})}{k+1}, \quad \eta_{-1} = 0. \quad (10)$$

The following lemma shows that if $\tau_k \in [0, 1]$, the parameter C_k lies between f_k and A_k .

Lemma 1. Suppose that $\{f(x_k)\}$, $\{C_k\}$ and $\{A_k\}$ are infinite sequences generated by Algorithm 1. Then, $(f(x_k) \leq C_k \leq A_k)$ for all $k \geq 0$, and

$$C_{k+1} \leq C_k + \eta_k. \quad (11)$$

We omit the proof of Lemma 1 since it is similar to Lemma 2.2 in [14]. It is clear that Step 3 will be finished in a finite number of iterations for $k = 0$ since $C_0 = f(x_0)$ and f is continuously differentiable. Suppose that the line search step finishes in a finite number of iterations for index k . Then, we get from the line search conditions that

Step 0: given $x_0 \in R^n$, which is an initial guess of the solution. Let $0 < \rho < 1$, $\lambda_1, \lambda_2 > 0$, $0 < \tau_{\min} < \tau_{\max} < 1$, and $0 < \alpha_{\min} < \alpha_{\max}$ be given positive constants. Set $k = 0$, $C_0 = f(x_0)$, and $Q_0 = 1$. Select a positive sequence $\{\eta_k\}$ such that

ALGORITHM 1: Derivative-free MLS method:

$$f(x_{k+1}) \leq C_k + \eta_k. \quad (12)$$

From the definition of Q_k and C_k , we have

$$\begin{aligned} C_{k+1} &= \frac{\tau_k Q_k (C_k + \eta_k) + f(x_{k+1})}{Q_{k+1}}, \\ &= \frac{\tau_k Q_k (C_k + \eta_k) + f(x_{k+1})}{\tau_k Q_k + 1}, \\ &> f(x_{k+1}). \end{aligned} \quad (13)$$

This means condition (7) or (8) will be satisfied when α_k is small enough. Therefore, the algorithm is well defined.

The formula for direction d_k is based on the MLS method proposed by Li and Feng [24]. From the definition of d_k , we have that if $F_k \neq 0$ and $F_{k-1}^T d_{k-1} \neq 0$ for some $k \geq 1$, then

$$\begin{aligned} F_k^T d_k &= -\|F_k\|^2 + \beta_k^{\text{MLS}} F_k^T d_{k-1}, \\ &= -\|F_k\|^2 - \left(\frac{F_k^T y_{k-1}}{F_{k-1}^T d_{k-1}} + t \frac{\|y_{k-1}\|^2 F_k^T d_{k-1}}{(F_{k-1}^T d_{k-1})^2} \right) F_k^T d_{k-1}, \\ &= \frac{-\|F_k\|^2 (F_{k-1}^T d_{k-1})^2 - (F_k^T y_{k-1})(F_{k-1}^T d_{k-1}) - t \|y_{k-1}\|^2 (F_k^T d_{k-1})^2}{(F_{k-1}^T d_{k-1})^2}, \\ &\leq \frac{-\|F_k\|^2 (F_{k-1}^T d_{k-1})^2 + (1/2)(1/2t)\|F_k\|^2 (F_{k-1}^T d_{k-1})^2 + (1/2)2t\|y_{k-1}\|^2 (F_k^T d_{k-1})^2}{(F_{k-1}^T d_{k-1})^2}, \\ &\quad - \frac{t\|y_{k-1}\|^2 (F_k^T d_{k-1})^2}{(F_{k-1}^T d_{k-1})^2}, \\ &= \left(\frac{1}{4t} - 1 \right) \|F_k\|^2, \end{aligned} \quad (14)$$

where the inequality follows from an upper bound for the second term of the third equation, which is obtained from $u^T v \leq (1/2)(\|u\|^2 + \|v\|^2)$ with $u = (-1/\sqrt{2t})(F_{k-1}^T d_{k-1})F_k$ and $v = \sqrt{2t}(F_k^T d_{k-1})y_{k-1}$. Obviously, inequality (14) also holds for $k = 0$. Therefore, the search direction d_k will always satisfy

$$F_k^T d_k \leq \left(\frac{1}{4t} - 1 \right) \|F_k\|^2 < 0. \quad (15)$$

This means that the scalar β_k^{MLS} in (6) is well defined as long as $F_{k-1} \neq 0$ for $k \geq 1$ since $F_{k-1}^T d_{k-1} \neq 0$ by (15). Moreover, Algorithm 1 does not need to calculate the Jacobian matrix of F and occupy low-memory space; therefore, it is suitable for large-scale problems. For convenience, we call this algorithm as the DF-MLS method.

3. Convergence Property

In this section, we devote to analyze the global convergence of our method under the following assumptions.

Assumption 1.

- (i) The level set $\Omega = \{x \in R^n | f(x) \leq f(x_0) + \eta\}$ is bounded, where $x_0 \in R^n$ is an arbitrary initial point and η is a constant such that

$$\sum_{k=0}^{\infty} \eta_k = \eta < \infty. \quad (16)$$

- (ii) The function $F(x)$ is continuously differentiable on some neighborhood Γ of Ω . Hence, $F(x)$ is Lipschitz continuous on Γ . That is, there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in \Gamma. \quad (17)$$

The assumption implies that there is a constant $\gamma_1 > 0$ such that

$$\|F(x)\| \leq \gamma_1, \quad \forall x \in \Omega. \quad (18)$$

Lemma 2. Suppose that $\{x_k\}$ is an infinite sequence generated by the DF-MLS method and Assumption 1 holds. Then, $\{x_k\}$ is contained in Ω .

Proof. Based on the line search step of the algorithm, we have

$$f(x_k) \leq C_k + \eta_k, \quad \forall k \geq 0. \quad (19)$$

Combining with (11), we have

$$\begin{aligned} f(x_k) &\leq C_k + \eta_k \leq C_{k-1} + \eta_{k-1} + \eta_k \\ &\leq \dots \leq f(x_0) + \sum_{i=0}^k \eta_i \leq f(x_0) + \eta. \end{aligned} \quad (20)$$

The above inequality draws the conclusion.

Lemma 3. Suppose that $\{x_k\}$ is an infinite sequence generated by the DF-MLS method and Assumption 1 holds. Then,

$$\begin{aligned} \lim_{k \rightarrow \infty} \alpha_k \|d_k\|^2 &= 0, \\ \lim_{k \rightarrow \infty} \alpha_k^2 f(x_k) &= 0. \end{aligned} \quad (21)$$

Proof. We get from the third step of the algorithm that

$$f(x_{k+1}) \leq C_k + \eta_k - \lambda_1 \alpha_k^2 \|d_k\|^4 - \lambda_2 \alpha_k^2 f(x_k). \quad (22)$$

Combining this with (9), we have

$$\begin{aligned} C_{k+1} &= \frac{\tau_k Q_k (C_k + \eta_k) + f(x_{k+1})}{Q_{k+1}}, \\ &\leq \frac{\tau_k Q_k (C_k + \eta_k) + C_k + \eta_k - \lambda_1 \alpha_k^2 \|d_k\|^4 - \lambda_2 \alpha_k^2 f(x_k)}{Q_{k+1}}, \\ &= C_k + \eta_k - \frac{\lambda_1 \alpha_k^2 \|d_k\|^4 + \lambda_2 \alpha_k^2 f(x_k)}{Q_{k+1}}. \end{aligned} \quad (23)$$

By (4), we have

$$\sum_{k=0}^{\infty} \frac{\lambda_1 \alpha_k^2 \|d_k\|^4 + \lambda_2 \alpha_k^2 f(x_k)}{Q_{k+1}} < \infty. \quad (24)$$

Since $0 < \tau_{\min} \leq \tau_k \leq \tau_{\max} < 1$ and $Q_0 = 1$, we get from (9) that

$$Q_{k+1} = \tau_k Q_k + 1 = 1 + \sum_{j=0}^k \prod_{i=0}^j \tau_{k-i} \leq 1 + \sum_{j=0}^k \tau_{\max}^{j+1} \leq \frac{1}{1 - \tau_{\max}}. \quad (25)$$

This together with (24) implies

$$\sum_{k=0}^{\infty} \lambda_1 \alpha_k^2 \|d_k\|^4 + \lambda_2 \alpha_k^2 f(x_k) < \infty, \quad (26)$$

which gives (21). This completes the proof.

Lemma 4. Suppose that $\{d_k\}$ is an infinite sequence generated by the DF-MLS method and Assumption 1 holds. If there exists a constant $\gamma > 0$ such that

$$\|F_k\| \geq \gamma, \quad \forall k \geq 0, \quad (27)$$

then there exists a constant $B > 0$ such that

$$\|d_k\| \leq B, \quad \forall k \geq 0, \quad (28)$$

$$\lim_{k \rightarrow \infty} \beta_k^{\text{MLS}} \|d_{k-1}\| = 0. \quad (29)$$

Proof. We get from (15), (17), (18), and (5) that

$$\begin{aligned} \|d_k\| &\leq \|F_k\| + |\beta_k^{\text{MLS}}| \|d_{k-1}\|, \\ &\leq \|F_k\| + \frac{\|F_k\| \|y_{k-1}\| \|d_{k-1}\|}{|F_{k-1}^T d_{k-1}|} + t \frac{\|y_{k-1}\|^2 \|F_k\| \|d_{k-1}\|}{|F_{k-1}^T d_{k-1}|^2} \|d_{k-1}\|, \\ &\leq \gamma_1 + \frac{4t\gamma_1 L \alpha_{k-1} \|d_{k-1}\|^2}{(4t-1)\gamma^2} + \frac{t(4t)^2 2\gamma_1^2 L \alpha_{k-1} \|d_{k-1}\|^2}{(4t-1)^2 \gamma^4} \|d_{k-1}\|, \\ &= \gamma_1 + \frac{4t\gamma_1 L}{(4t-1)\gamma^2} \alpha_{k-1} \|d_{k-1}\|^2 \\ &\quad + \frac{32t^3 \gamma_1^2 L}{(4t-1)^2 \gamma^4} \alpha_{k-1} \|d_{k-1}\|^2 \|d_{k-1}\|. \end{aligned} \quad (30)$$

For any constant $b \in (0, 1)$, we get from (21) that there exists an index k_0 such that

$$\frac{32t^3 \gamma_1^2 L}{(4t-1)^2 \gamma^4} \alpha_{k-1} \|d_{k-1}\|^2 < b, \quad \forall k > k_0. \quad (31)$$

Then,

$$\|d_k\| \leq \gamma_1 + \frac{(4t-1)\gamma^2 b}{8t^2 \gamma_1} + b \|d_{k-1}\| = c + b \|d_{k-1}\|, \quad (32)$$

where $c = \gamma_1 + ((4t-1)\gamma^2 b / (8t^2 \gamma_1))$ is a constant. For any $k > k_0$, we have

$$\|d_k\| \leq c(1 + b + b^2 + \dots + b^{k-k_0+1}) + b^{k-k_0}\|d_{k_0}\| \leq \frac{c}{1-b} + \|d_{k_0}\|. \quad (33)$$

So, we can let $B = \max\{\|d_1\|, \|d_2\|, \dots, \|d_{k_0}\|, c/1-b + \|d_{k_0}\|\}$ to get (28). By (17), (27), (21), and (28), we have

$$\begin{aligned} |\beta_k^{\text{MLS}}| \|d_{k-1}\| &= |\beta_k^{\text{MLS}}| \|d_{k-1}\|, \\ &\leq \frac{\|F_k\| \|y_{k-1}\| \|d_{k-1}\|}{|F_{k-1}^T d_{k-1}|} + t \frac{\|y_{k-1}\|^2 \|F_k\| \|d_{k-1}\|}{|F_{k-1}^T d_{k-1}|^2} \|d_{k-1}\|, \\ &\leq \left(\frac{4t\gamma_1 L}{(4t-1)\gamma^2} + \frac{32t^3\gamma_1^2 L}{(4t-1)^2\gamma^4} \|d_{k-1}\| \right) \alpha_{k-1} \|d_{k-1}\|^2, \end{aligned} \quad (34)$$

which shows (29).

Analogy to Theorem 1 in [13], we establish the global convergence of the DF-MLS method as follows.

Theorem 1. Suppose that $\{x_k\}$ is an infinite sequence generated by the DF-MLS method and Assumption 1 holds. Then,

$$\liminf_{k \rightarrow \infty} \|F_k\| = 0, \quad (35)$$

or every limit point x^* of $\{x_k\}$ satisfies

$$F(x^*)^T J(x^*) F(x^*) = 0, \quad (36)$$

where $J(x^*)$ is the Jacobian matrix of $F(x)$ at x^* .

Proof. Let x^* be a limit point of $\{x_k\}$ and $\mathbb{K}$ be an infinite sequence of indices such that

$$\lim_{k \in \mathbb{K}} x_k = x^*. \quad (37)$$

(21) implies

$$\lim_{k \in \mathbb{K}} \alpha_k^2 f(x_k) = \lim_{k \in \mathbb{K}} \frac{1}{2} \alpha_k^2 \|F_k\|^2 = 0. \quad (38)$$

If $\lim_{k \in \mathbb{K}} \alpha_k \neq 0$, then there exists an infinite subsequence $\mathbb{K}_1 \subseteq \mathbb{K}$ such that α_k is bounded away from zero for any $k \in \mathbb{K}_1$. Therefore, in this case, we get from (38) that

$$\lim_{k \in \mathbb{K}_1} \|F_k\| = 0. \quad (39)$$

This implies $\|F(x^*)\| = 0$, and (35) holds since F is continuous and $\lim_{k \in \mathbb{K}_1} x_k = x^*$.

Next, we analyze the case when

$$\lim_{k \in \mathbb{K}} \alpha_k = 0. \quad (40)$$

Assume that (35) does not hold; then, there exists an infinite subsequence $\mathbb{K}_2 \subseteq \mathbb{K}$ such that $\|F_k\|$ is bounded away from zero for any $k \in \mathbb{K}_2$. From the third step of the DF-MLS method, the step length $\rho^{-1}\alpha_k$ will satisfy neither (7) nor (8) when $k \in \mathbb{K}_2$ is large enough. This means

$$f(x_k + \rho^{-1}\alpha_k d_k) > C_k + \eta_k - \lambda_1 \rho^{-2} \alpha_k^2 \|d_k\|^4 - \lambda_2 \rho^{-2} \alpha_k^2 f(x_k), \quad (41)$$

$$f(x_k - \rho^{-1}\alpha_k d_k) > C_k + \eta_k - \lambda_1 \rho^{-2} \alpha_k^2 \|d_k\|^4 - \lambda_2 \rho^{-2} \alpha_k^2 f(x_k). \quad (42)$$

We get from the conclusion of Lemma 1 that $C_k \geq f(x_k) \geq 0$. Combining this with (41), we have

$$f(x_k + \rho^{-1}\alpha_k d_k) > f(x_k) - \lambda_1 \rho^{-2} \alpha_k^2 \|d_k\|^4 - \lambda_2 \rho^{-2} \alpha_k^2 f(x_k). \quad (43)$$

It follows from Lemma 2 that

$$f(x_k) \leq f(x_0) + \eta. \quad (44)$$

Combining this with (18) and (28), we obtain

$$f(x_k + \rho^{-1}\alpha_k d_k) - f(x_k) > -M\alpha_k^2, \quad (45)$$

where $M = \lambda_1 \rho^{-2} B^4 + \lambda_2 \rho^{-2} (f(x_0) + \eta)$. Therefore,

$$\frac{f(x_k + \rho^{-1}\alpha_k d_k) - f(x_k)}{\alpha_k} > -M\alpha_k. \quad (46)$$

By the mean value theorem and (3), there exists a parameter $\xi_k \in (0, 1)$ such that

$$\rho^{-1} F(x_k + \xi_k \rho^{-1} \alpha_k d_k)^T J(x_k + \xi_k \rho^{-1} \alpha_k d_k) d_k > -M\alpha_k, \quad (47)$$

namely,

$$\begin{aligned} \rho^{-1} F(x_k + \xi_k \rho^{-1} \alpha_k d_k)^T J(x_k + \xi_k \rho^{-1} \alpha_k d_k) (-F_k + \beta_k^{\text{MLS}} d_{k-1}) \\ > -M\alpha_k. \end{aligned} \quad (48)$$

Combining this with (21), (29), and (40) and taking limits in (48), we obtain

$$F(x^*)^T J(x^*) F(x^*) \leq 0. \quad (49)$$

Using (42) and proceeding in a similar way, we can get

$$F(x^*)^T J(x^*) F(x^*) \geq 0. \quad (50)$$

From the last two inequalities, we get (36) which completes the proof.

Corollary 1. Suppose that $\{x_k\}$ is an infinite sequence generated by the DF-MLS method and Assumption 1 holds. If x^* is a limit point of $\{x_k\}$ and

$$y^T J(x^*) y \neq 0, \quad \forall y \in \mathbb{R}^n, y \neq 0, \quad (51)$$

then

$$F(x^*) = 0. \quad (52)$$

TABLE 1: Numerical results.

Prob	Dim	DF-SANE Iter/Nfun/Time	N-DF-SANE Iter/Nfun/Time	DF-SCGNE Iter/Nfun/Time	DF-MLS Iter/Nfun/Time
ARGLINA	50	3/3/0.001	3/5/0.001	3/6/0.001	2/5/0.001
ARGLINA	200	3/3/0.005	3/5/0.007	3/6/0.009	2/5/0.008
ARGLINB	100	47/1151/0.298	3/29/0.008	3/30/0.008	31/817/0.209
ARGLINB	200	867/24347/25.89	3/33/0.039	4/35/0.041	31/923/0.988
ARGLINC	100	231/5707/1.441	3/29/0.008	3/30/0.009	33/865/0.216
ARGLINC	200	39/1071/1.133	3/33/0.037	4/35/0.04	34/1023/1.071
ARWHEAD	500	7/15/0.002	5/17/0.003	5/17/0.003	5/17/0.002
ARWHEAD	5000	6/16/0.021	5/19/0.027	5/19/0.026	5/19/0.025
BDQRTIC	500	67/131/0.02	86/297/0.046	21/52/0.009	16/41/0.006
BDQRTIC	5000	50/110/0.192	194/1019/1.758	16/45/0.083	2/35/0.062
BOX	100	66/198/0.006	309/873/0.028	78/372/0.012	37/153/0.005
BOX	1000	—	706/4367/1.344	124/743/0.23	157/1255/0.384
BROWNAL	100	3/13/0.002	3/15/0.002	3/15/0.002	3/15/0.002
BROYDN7D	1000	—	68/137/0.138	24/49/0.049	25/51/0.051
BRYBND	1000	18/29/0.012	39/85/0.036	102/388/0.16	13/31/0.013
BRYBND	5000	23/33/0.08	36/75/0.186	17/37/0.093	16/37/0.089
CHAINWOO	4000	170/355/0.431	94/332/0.414	105/476/0.586	64/201/0.247
CHAINWOO	10000	41/67/0.235	—	49/178/0.605	22/51/0.175
CHNROSNB	50	—	—	—	—
CRAGGLVY	500	28/44/0.015	35/81/0.025	19/45/0.016	19/45/0.015
CRAGGLVY	1000	32/50/0.032	43/125/0.072	22/51/0.034	18/43/0.029
CURLY10	1000	—	185/752/0.161	100/216/0.05	118/357/0.074
CURLY20	1000	—	182/942/0.324	121/475/0.168	182/919/0.313
CURLY30	100	—	809/4720/0.198	621/4448/0.187	757/5431/0.225
DECONVU	61	19/23/0.001	24/51/0.002	27/57/0.002	22/47/0.002
DIXMAANA	1500	6/8/0.003	5/11/0.004	6/13/0.005	5/11/0.004
DIXMAANA	9000	6/8/0.021	5/11/0.029	6/13/0.036	7/17/0.042
DIXMAANB	3000	7/9/0.007	6/13/0.01	10/21/0.017	7/17/0.014
DIXMAANB	9000	7/9/0.023	6/13/0.035	10/21/0.054	7/17/0.043
DIXMAANC	3000	9/13/0.011	7/17/0.013	9/21/0.017	7/17/0.013
DIXMAANC	9000	9/13/0.032	7/17/0.046	9/21/0.058	7/17/0.042
DIXMAAND	1500	7/11/0.004	6/15/0.005	7/17/0.009	5/13/0.005
DIXMAAND	3000	7/11/0.008	6/15/0.013	7/17/0.014	5/13/0.01
DIXMAANE	3000	22/26/0.019	47/95/0.075	20/41/0.032	26/55/0.041
DIXMAANE	9000	22/26/0.064	47/95/0.244	20/41/0.105	26/55/0.138
DIXMAANF	3000	17/19/0.015	25/51/0.041	17/35/0.028	13/29/0.022
DIXMAANF	9000	17/19/0.047	25/51/0.13	17/35/0.09	13/29/0.073
DIXMAANG	3000	14/18/0.014	13/29/0.023	12/27/0.021	11/25/0.019
DIXMAANG	9000	14/18/0.044	13/29/0.076	12/27/0.07	11/25/0.062
DIXMAANH	3000	9/13/0.01	9/21/0.017	9/21/0.017	8/19/0.014
DIXMAANH	9000	9/13/0.034	9/21/0.054	9/21/0.054	11/27/0.067
DIXMAANI	3000	29/37/0.028	84/169/0.133	29/59/0.048	37/77/0.058
DIXMAANI	9000	29/37/0.093	84/169/0.4319	29/59/0.15	37/77/0.19
DIXMAANJ	3000	18/20/0.016	33/67/0.054	23/47/0.037	15/33/0.025
DIXMAANJ	9000	18/20/0.051	33/67/0.172	23/47/0.121	15/33/0.083
DIXMAANK	1500	14/18/0.006	15/33/0.013	14/31/0.011	12/27/0.01
DIXMAANK	3000	14/18/0.013	15/33/0.027	14/31/0.025	12/27/0.021
DIXMAANL	1500	12/16/0.006	9/21/0.009	9/21/0.007	8/19/0.007
DIXMAANL	3000	12/16/0.012	9/21/0.018	9/21/0.019	8/19/0.014
DIXON3DQ	10000	—	—	609/1235/1.459	—
DQDRTIC	500	12/18/0.001	56/175/0.017	7/19/0.001	7/19/0.002
DQDRTIC	1000	12/18/0.003	40/91/0.02	7/19/0.004	7/19/0.004
EDENSCH	2000	19/25/0.014	12/29/0.018	17/39/0.023	13/33/0.019
EG2	1000	9/15/0.004	6/13/0.004	16/37/0.011	6/13/0.006
ENGVAL1	100	12/16/0.001	14/31/0.001	9/21/0.001	9/21/0.001
ENGVAL1	5000	8/12/0.016	7/17/0.025	9/21/0.03	6/15/0.02
ERRINROS	50	20/32/0.001	12/33/0.001	16/41/0.001	12/33/0.001
EXTROSNB	100	40/55/0.001	36/81/0.001	19/44/0.001	17/43/0
EXTROSNB	1000	22/28/0.004	17/39/0.007	6/19/0.003	18/51/0.008

TABLE 1: Continued.

Prob	Dim	DF-SANE Iter/Nfun/Time	N-DF-SANE Iter/Nfun/Time	DF-SCGNE Iter/Nfun/Time	DF-MLS Iter/Nfun/Time
FLETGBV2	10000	1/1/0.005	1/1/0.006	1/1/0.005	1/1/0.005
FLETGBV3	100	—	—	—	—
FREUROTH	1000	816/3457/1.068	78/266/0.085	64/235/0.074	37/97/0.031
GENHUMPS	5000	—	—	—	—
GENROSE	500	—	—	—	—
HILBERTA	10	12/12/0	93/193/0.001	7/14/0	6/11/0.001
HILBERTB	50	5/7/0.001	5/11/0.002	6/13/0.002	5/11/0.001
HYDC20LS	99	777/3215/0.543	342/1701/0.264	314/1874/0.291	107/447/0.069
INDEF	100	—	—	—	—
LIARWHD	5000	43/97/0.109	957/7417/8.453	35/94/0.116	16/43/0.052
LIARWHD	10000	—	—	419/3534/8.681	22/59/0.153
MANCINO	100	5/19/0.118	5/23/0.146	5/23/0.145	63/351/2.1817
MODBEALE	2000	587/2400/2.926	224/1261/1.576	55/183/0.235	31/78/0.098
MOREBV	1000	20/22/0.005	49/97/0.021	21/42/0.01	14/27/0.005
MOREBV	5000	5/5/0.006	6/11/0.012	19/38/0.043	3/5/0.006
MSQRTALS	1024	—	—	—	—
NONCVXUN	100	37/45/0.002	42/97/0.004	30/63/0.003	39/101/0.004
NONDIA	5000	3/15/0.016	3/17/0.019	4/19/0.022	3/17/0.018
NONDIA	10000	4/18/0.041	4/21/0.052	4/21/0.05	4/21/0.048
NONDQUAR	1000	24/36/0.004	22/51/0.008	30/69/0.009	23/55/0.007
NONDQUAR	10000	15/27/0.034	17/45/0.061	30/73/0.098	11/33/0.042
NONMSQRT	100	—	177/805/0.059	80/285/0.022	483/2815/0.198
OSCIPATH	500	11/13/0.001	12/25/0.002	10/22/0.002	8/17/0.001
PENALTY1	50	10/20/0	9/27/0.001	9/28/0	10/31/0
PENALTY2	200	38/72/0.026	58/181/0.056	33/79/0.03	29/71/0.026
POWELLSG	5000	48/64/0.035	251/591/0.336	36/77/0.046	24/53/0.03
POWELLSG	10000	48/64/0.073	159/685/0.737	36/77/0.096	24/53/0.062
POWER	10000	15/321/0.26	9/41/0.039	10/43/0.042	22/517/0.413
SCHMVETT	10000	—	—	—	39/96/0.930
SCOSINE	100	—	—	—	—
SENSORS	100	439/598/4.047	13/33/0.209	17/39/0.246	38/103/0.650
SINQUAD	5000	—	305/2441/5.297	—	27/83/0.175
SPARSQUR	1000	13/21/0.008	9/25/0.01	14/35/0.014	9/25/0.01
SPARSQUR	5000	13/23/0.049	10/29/0.066	12/33/0.073	10/29/0.061
SPMSRTLS	10000	—	—	—	—
SROSENBR	1000	12/20/0.003	120/637/0.077	6/17/0.002	9/23/0.003
SROSENBR	5000	12/20/0.012	82/345/0.223	6/17/0.012	9/23/0.015
TESTQUAD	1000	43/63/0.006	154/319/0.039	38/87/0.009	39/89/0.008
TESTQUAD	5000	43/65/0.03	159/329/0.198	43/97/0.053	43/97/0.05
TOINTGOR	50	171/319/0.005	130/433/0.007	62/129/0.002	65/145/0.003
TOINTGSS	1000	2/4/0.002	3/7/0.004	4/9/0.005	3/7/0.004
TOINTPSP	50	642/1771/0.015	198/836/0.007	162/676/0.006	220/880/0.007
TOINTQOR	50	26/30/0.001	87/177/0.002	22/47/0.001	22/47/0
TQUARTIC	1000	—	538/4539/0.713	—	416/4104/0.642
TQUARTIC	10000	—	—	—	—
TRIDIA	5000	655/2455/1.118	390/2181/1.075	206/419/0.245	195/397/0.216
TRIDIA	10000	724/2890/2.760	395/2219/2.251	205/417/0.522	205/419/0.484
VARDIM	100	5/103/0.001	9/43/0.001	10/45/0.001	4/83/0.001
VARDIM	200	16/426/0.009	10/49/0.001	12/54/0.001	7/187/0.004
VAREIGVL	1000	26/42/0.018	27/59/0.028	17/39/0.019	19/43/0.019
VAREIGVL	5000	116/290/0.745	124/371/1.000	42/187/0.487	36/117/0.303
WOODS	4000	31/43/0.033	37/81/0.06	42/117/0.083	14/37/0.025
WOODS	10000	31/43/0.081	37/81/0.163	40/113/0.218	14/37/0.071

Proof. We draw the conclusion straightforwardly by using Theorem 1.

If the Jacobian matrix of $F(x)$ is positive definite for all $x \in \mathbb{R}^n$, we say that the mapping F is strictly monotone, namely, the Jacobian matrix $J(x)$ satisfies

$$x^T J(x) x > 0, \quad \forall x \in \mathbb{R}^n. \quad (53)$$

If mapping $F(x)$ is strictly monotone and $F(x) = 0$ has a solution, its solution must be unique. See Chapter 5 of [37].

Corollary 2. Suppose that $\{x_k\}$ is an infinite sequence generated by the DF-MLS method and Assumption 1 holds. If F or $-F$ is strictly monotone, then every bounded subsequence $\{x_k\}_{k \in \mathbb{N}}$ converges to the solution of (1).

Proof. We draw the conclusion straightforwardly by using Theorem 1.

4. Numerical Results

In this section, we report the numerical experimental results and analyze the efficiency of our method. We compare the numerical performance of our method with some other methods such as the DF-SANE method [13], the N-DF-SANE method [14], and the DF-SCGNE method [34]. We used the aforementioned methods to find the stationary points of the test problems by using their gradients only. We tested 112 unconstrained optimization problems in the CUTEr library [38] with dimensions varying from 50 to 10,000. We ran two versions of some problems for which the dimensions can be chosen randomly.

We downloaded the DF-SANE code from Professor Raydan's home page at http://kuainasi.ciens.ucv.ve/mraydan/mraydan_pub.html. These four methods were coded in Fortran 77 and carried out on a personal computer with 2.8 GHz CPU processor and 2 GB RAM and Linux operating system. We terminated the program when

$$\frac{\|F(x_k)\|}{\sqrt{n}} \leq e_a + e_r \frac{\|F(x_0)\|}{\sqrt{n}}, \quad (54)$$

where $e_a = 10^{-5}$ and $e_r = 10^{-4}$.

Table 1 lists the detailed experimental results, which include the problem names (Prob), dimensions (Dim), the total number of iterations (Iter), function evaluations (Nfun), and the CPU time (Time) in seconds, respectively. The symbol “—” means that the number of iterations exceeds 1000, or the line search iterates more than 100 times. In other words, this means that the method failed to solve that problem.

We compared the numerical performance of the methods by using the profiles designed by Dolan and Moré [39]. Figures 1 and 2 describe the numerical performance of the methods relative to the CPU time (in seconds), the total number of iterations, and function value evaluations, respectively. The curves in the figures have the following meanings.

- (1) “DF-SANE” denotes the DF-SANE method proposed by La Cruz et al. [13] with $n_{\text{exp}} = 2$, $\sigma_{\min} = 10^{-10}$, $\sigma_{\max} = 10^{10}$, $\sigma_0 = 1$, $\tau_{\min} = 0.1$, $\tau_{\max} = 0.5$, $\gamma = 10^{-4}$, $M = 10$, and $\eta_k = (\|F(x_0)\|/2^k)$ for all k .
- (2) “N-DF-SANE” means the N-DF-SANE method developed by Cheng and Li [14] with $\eta_k = 0.6$, $\sigma_{\min} = 10^{-10}$, $\sigma_{\max} = 10^{10}$, $\rho_{\min} = 0.1$, $\rho_{\max} \gamma = 10^{-4}$, and $\varepsilon_k = (\|F(x_0)\|/2^k)$ for all k .
- (3) “DF-MLS” stands for Algorithm 1. We set the parameters as follows: $\rho = 0.5$, $\tau_k = 0.6$, $t = 1$, $\lambda_1 = \lambda_2 = 10^{-4}$, $\alpha_{\min} = 10^{-10}$, $\alpha_{\max} = 10^{10}$, and $\eta_k = (\|F(x_0)\|/$

FIGURE 1: Performance profiles related to the number of iterations.

FIGURE 2: Performance profiles related to the number of function evaluations.

FIGURE 3: Performance profiles relative to the CPU time.

2^k) for all k . In the line search, the initial step length α_{k_0} is determined by

$$\alpha_{k_0} = \begin{cases} \alpha_{\max}, & \text{if } \sigma > \alpha_{\max}, \\ \sigma, & \text{if } \sigma \in [\alpha_{\min}, \alpha_{\max}], \\ \alpha_{\min}, & \text{if } \sigma < \alpha_{\min}, \end{cases}$$

$$\sigma = \frac{-F_k^T d_k}{d_k^T z_k}, \quad (55)$$

$$z_k = \frac{F(x_k + \varepsilon d_k) - F(x_k)}{\varepsilon},$$

$$\varepsilon = 10^{-8}.$$

- (4) “DF-SCGNE” means the DF-SCGNE method proposed by Yu [34] using the same line search as DF-MLS.

For each method, the corresponding curve in Figure 1 plots the fraction P of problems for which the method is within a factor τ of the smallest total number of iterations. We observe from Figure 1 that DF-MLS solves 60.71% (68 out of 112) of the test problems with the smallest total number of iterations. This may be because the DF-MLS method can generate more efficiently than other methods. We observe from Figures 1 and 2 that the performance of DF-MLS is better than that of N-DF-SANE and DF-SCGNE. However, the DF-MLS method requires more function evaluations and CPU time than DF-SANE. This might be because that the DF-MLS method needs an additional function evaluation to determine the initial step length α_{k_0} at each iteration. Figure 3 compares the performance of these four methods relative to the number of function evaluations. How to improve the efficiency of the DF-MLS method will be an interesting topic for us in the future.

5. Conclusion

Many practical applications give rise to nonlinear systems of equations. In the past few decades, Newton’s methods [1–10] have been widely used to solve this problem for their fast convergence speeds. However, Newton’s methods are not suitable for solving a large-scale problem in which the Jacobian matrix of the objective function is unavailable or needs massive storage space. To overcome this shortcoming of Newton’s methods, some derivative-free numerical algorithms have been developed such as the SANE, DF-SANE, and some CG-type methods [11, 13, 29–34].

Inspired by the CG-type methods, we design a derivative-free method for a large-scale nonlinear system of equations based on the MLS [24] conjugate gradient method. The line search in the proposed method is a modification of the Zhang–Hager [15] and Cheng–Li [14] nonmonotone line searches. We established the global convergence based on some mild conditions. The proposed method is suitable to solve large-scale problems because it does not need the Jacobian matrix of the objective function.

In the numerical experiment, we tested 112 unconstrained optimization problems in the CUTER library [38] and compared the numerical performance of our method with that of DF-SANE [13], N-DF-SANE [14], and DF-SCGNE [34]. We used the methods to find the stationary points of the test problems by using their gradients only. The numerical results show that the proposed method is efficient and has good numerical stability.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The authors are very grateful to Prof. Raydan for providing them the codes.

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