Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 6916925, 14 pages
https://doi.org/10.1155/2020/6916925

Research Article

Optimal Reinsurance Strategy for an Insurer and a Reinsurer with Generalized Variance Premium Principle

Danping Li[1] **and Chaohai Shen**[2]

[1]*Key Laboratory of Advanced Theory and Application in Statistics and Data Science-MOE, School of Statistics, Faculty of Economics and Management, East China Normal University, Shanghai 200062, China*
[2]*Faculty of Economics and Management, East China Normal University, Shanghai 200062, China*

Correspondence should be addressed to Chaohai Shen; chshen@dbm.ecnu.edu.cn

Received 27 January 2020; Accepted 28 April 2020; Published 11 May 2020

Academic Editor: Sergio Ortobelli

This paper focuses on the optimal reinsurance problem with consideration of joint interests of an insurer and a reinsurer. In our model, the risk process is assumed to follow a Brownian motion with drift. The insurer can transfer the risk to the reinsurer via proportional reinsurance, and the reinsurance premium is calculated according to the variance and standard deviation premium principles. The objective is to maximize the expected exponential utility of the weighted sum of the insurer's and the reinsurer's terminal wealth, where the weight can be viewed as a regularization parameter to measure the importance of each party. By applying stochastic control theory, we establish the Hamilton–Jacobi–Bellman equation and obtain explicit expressions of optimal reinsurance strategies and optimal value functions. Furthermore, we provide some numerical simulations to illustrate the effects of model parameters on the optimal reinsurance strategies.

1. Introduction

Since reinsurance is an effective way to spread risk in the insurance business, the problem of optimal reinsurance for insurers has drawn great attention in recent years. For example, Cai and Tan [1], Tan and Weng [2], Chi [3], and Román et al. [4] considered the optimal reinsurance problems in the static models under the criteria of minimizing value at risk, conditional tail expectation, or multiobjectives.

In a dynamic model, the technique of stochastic control theory is frequently used to deal with the problem of optimal reinsurance. For instance, Schmidli [5], Promislow and Young [6], and Liang and Young [7] investigated the problem of optimal reinsurance and investment for insurers in the sense of minimizing the ruin probability. Bäuerle [8], Delong and Gerrard [9], and Chen and Yao [10] studied the problem of optimal reinsurance and investment for insurers under the mean-variance criterion. Since traditional dynamic mean-variance optimization problem is a time-inconsistent problem, more and more literature develops time-consistent strategies for mean-variance insurers, e.g., Li et al. [11], Lin and Qian [12], Wang et al. [13], and references therein. For the objective of expected utility maximization, Liu et al. [14] considered the optimal reinsurance and investment problem with dynamic risk constraint and regime switching. Huang et al. [15] introduced the constrained control variables into the optimal reinsurance and investment problem for a jump-diffusion risk model. Zheng et al. [16] considered the robust optimal proportional reinsurance and investment problem for an insurer under the constant elasticity of variance model. Li et al. [17] studied the robust optimal excess-of-loss reinsurance and investment problem in a model with jumps. Zhang and Zheng [18] investigated an optimal investment-reinsurance policy with stochastic interest and inflation rates.

Although research on the problem of optimal reinsurance increases rapidly, only a few papers deal with the problem with consideration of the joint interest of an insurer and a reinsurer. Actually, the reinsurer also aims to increase her profit, so great attention should also be paid to the reinsurer. Fang and Qu [19] obtained the optimal reinsurance strategy to maximize the joint survival probability of an insurer and a reinsurer. Cai et al. [20] focused on the problem of optimal reinsurance to maximize not only the joint survival probability but also the joint profitable probability. In a dynamic model, Zeng and Luo [21] modeled reinsurance as a stochastic cooperation game and discussed the stochastic Pareto-optimal reinsurance problem. Li et al. [11] derived the time-consistent reinsurance-investment strategy for an insurer and a reinsurer under the mean-variance criterion. Huang et al. [22] discussed the robust optimal investment and reinsurance problem for the product of an insurer's and a reinsurer's utilities. Li et al. [23], Li et al. [24], and Zhao et al. [25] considered optimal reinsurance and investment problems of a general company including an insurer and a reinsurer under the expected utility maximization and mean-variance criterion, respectively.

In this paper, we investigate the optimal reinsurance strategies taking into account the joint interests of both an insurer and a reinsurer in a continuous-time model. Different from Li et al. [23], Li et al. [24], and Zhao et al. [25], the reinsurance premium principles used in this paper are the variance and standard deviation premium principles, instead of expected value principle used in most of the literature. Both variance and standard deviation premium principles are popular in the actuarial science, and they have received considerable attention in recent years, such as Chi [3], Zeng and Luo [21], and Liang and Yuen [26]. Particularly, Zeng and Luo [21] presented that proportional reinsurance is the optimal reinsurance way under variance and standard deviation premium principles. In more detail, the risk process in this paper is assumed to follow a Brownian motion with drift, and the insurer can purchase proportional reinsurance from the reinsurer. By applying stochastic control theory, we establish the corresponding Hamilton–Jacobi–Bellman (HJB) equation and obtain the explicit expressions of optimal reinsurance strategies and optimal value functions. Finally, we present some numerical simulations to show the effects of model parameters on the optimal reinsurance strategies.

Our paper has three main contributions to the literature on the optimal reinsurance strategy. (1) The explicit optimal reinsurance strategy for a dynamic model considering the joint interests of both an insurer and a reinsurer is derived. In the special case with the weight being equal to 1 or 0, our results can reduce to the corresponding problem for an insurer or a reinsurer only. Thus, our model is more general than other studies in this aspect. (2) Different from expected value premium principle, we consider variance and standard deviation premium principles in this paper. From the mathematical computation point of view, the solving process under variance premium principle is more difficult. (3) The effects of weight coefficients on the reinsurance strategies are analyzed, which can give some suggestions to investors who hold shares of an insurer and a reinsurer.

This paper is organized as follows. In Section 2, model formulation is introduced and the corresponding verification theorem for a general case is provided. In Section 3, by solving the HJB equation, optimal reinsurance strategies and optimal value functions under variance and standard deviation premium principles are derived, respectively. In Section 4, sensitivity analysis and numerical simulations are presented to illustrate our results. Section 5 concludes this paper.

2. Model Formulation

Consider a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ satisfying the usual condition, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration with $\mathcal{F} = \mathcal{F}_T$, and T is a fixed and finite time horizon. All stochastic processes introduced below are supposed to be adapted processes in this space.

According to the classical Cramér–Lundberg (C-L) model, the surplus process of an insurer can be described by

$$X_1(t) = x_1 + ct - \sum_{i=1}^{N(t)} Z_i, \tag{1}$$

where x_1 is the initial wealth and c is the premium rate. Suppose that the premium rate c is calculated according to the expected value principle, i.e., $c = (1+\eta)\lambda E[Z_i]$, where η is the safety loading of the insurer, $N(t)$ is a homogeneous Poisson process with intensity λ, and the claim sizes $\{Z_i, i \geq 1\}$ are independent and identically distributed positive random variables with finite expectation and variance.

To proceed, suppose that the insurer can purchase reinsurance from a reinsurer to transfer the risk. According to Taksar and Zeng [27], a risk share function can be described by a nondecreasing function $g\colon [0,\infty) \longrightarrow [0,\infty)$ with $g(x) \in [0,x]$. Let $\mathcal{G}$ be the set of all such functions and $\Theta \subset R$ be a parameter space. For each $\nu(t) \in \Theta$, we denote $g(x;\nu(t))\colon = g(x) \in \mathcal{G}$, representing the insurance business retained by the insurer. In other words, when the claim arrives at time t, the insurer pays $g(Z_i;\nu(t))$, while the reinsurer pays the rest part $Z_i - g(Z_i;\nu(t))$. Corresponding to the risk share function, there is a premium share function $\pi(\cdot;\lambda)$ in each reinsurance contract.

Denote a reinsurance strategy by $\nu\colon = \{\nu(t), t \in [0,T]\}$. Once the reinsurance strategy ν is chosen, the dynamics of the wealth processes for the insurer and the reinsurer become

$$\begin{aligned} X_1^{\nu}(t) &= x_1 + \int_0^t [c - \pi(Z_i - g(Z_i;\nu(s));\lambda)]\mathrm{d}s - \sum_{i=1}^{N(t)} g(Z_i;\nu(t)), \\ X_2^{\nu}(t) &= x_2 + \int_0^t \pi(Z_i - g(Z_i;\nu(s));\lambda)\mathrm{d}s - \sum_{i=1}^{N(t)} (Z_i - g(Z_i;\nu(t))). \end{aligned} \tag{2}$$

In addition, we assume that the wealth of both the insurer and the reinsurer increases with the interest rate r_0. Then, the diffusion approximation (cf. Promislow and Young [6]) for the wealth processes of the insurer and the reinsurer is

$$\mathrm{d}X_1^{\nu}(t) = [r_0 X_1^{\nu}(t) + c - \pi(Z_i - g(Z_i;\nu(t));\lambda) - \lambda E[g(Z_i;\nu(t))]]\mathrm{d}t + \sqrt{\lambda E[(g(Z_i;\nu(t)))^2]}\mathrm{d}W(t), \tag{3}$$

$$\mathrm{d}X_2^{\nu}(t) = [r_0 X_2^{\nu}(t) + \pi(Z_i - g(Z_i;\nu(t));\lambda) - \lambda E[Z_i - g(Z_i;\nu(t))]]\mathrm{d}t + \sqrt{\lambda E[(Z_i - g(Z_i;\nu(t)))^2]}\mathrm{d}W(t), \tag{4}$$

respectively, where $W(t)$ is a standard Brownian motion.

Taking into account the interests of both the insurer and the reinsurer, the weighted sum process can be described as follows:

$$X^{\nu}(t) = \alpha X_1^{\nu}(t) + \beta X_2^{\nu}(t), \tag{5}$$

where weight parameters satisfy $\alpha, \beta \in [0,1]$.

Remark 1. The weighted sum process $X^{\nu}(t)$ has abundant practical implications. On the one hand, the weight in $X^{\nu}(t)$ can be viewed as a regularization parameter to measure the importance of each party, and in the extreme case with the weight being equal to 1 or 0, our model can be reduced to the corresponding problem only for an insurer or a reinsurer. On the other hand, our model can also be viewed as the optimal investment problem for an investor who holds shares of an insurer and a reinsurer. $X^{\nu}(t)$ can be interpreted as the total surplus of the investor, i.e., the investor owns $100\,\alpha\%$ shares of the insurer and $100\,\beta\%$ shares of the reinsurer. In reality, there are some such investors, for example, in 2015, Central Huijin Investment Limited held 38.8% shares of New China Life Insurance Company Limited and 85.5% shares of China Reinsurance (Group) Company; Munich Reinsurance Group held 94.7% shares of ERGO Insurance Company and 100% shares of Munich Reinsurance America. In addition, from the risk management point of view, it is a natural risk management way for the investor to implement a risk transfer from the insurer to the reinsurer. Therefore, our model is more general.

Combining equations (3) and (4) yields the weighted sum process as follows:

$$\begin{aligned}\mathrm{d}X^{\nu}(t) = {} & [r_0 X^{\nu}(t) + \alpha c - (\alpha-\beta)\pi(Z_i - g(Z_i;\nu(t));\lambda) \\ & - (\alpha-\beta)\lambda E[g(Z_i;\nu(t))] - \beta\lambda E[Z_i]]\mathrm{d}t \\ & + \Big[\alpha\sqrt{\lambda E[(g(Z_i;\nu(t)))^2]} \\ & + \beta\sqrt{\lambda E[(Z_i - g(Z_i;\nu(t)))^2]}\Big]\mathrm{d}W(t).\end{aligned} \tag{6}$$

In this paper, for an admissible strategy $\nu \in \Theta$, the value function V^{ν} from state x at time t is defined as

$$V^{\nu}(t,x) = E\big[U(X^{\nu}(T)) \,\big|\, X^{\nu}(t) = x\big], \tag{7}$$

where $U(x)$ is strictly concave and continuously differentiable on $(-\infty,\infty)$. Then, the optimal value function is

$$V(t,x) = \sup_{\nu\in\Theta} V^{\nu}(t,x), \tag{8}$$

with the boundary condition $V(T,x) = U(x)$. The objective is to find an optimal reinsurance strategy $\{\nu^*(t)\colon t \in [0,T]\}$ such that $V^{\nu^*}(t,x) = V(t,x)$.

Denote by $C^{1,2}([0,T]\times\mathbb{R})$ as

$$C^{1,2}([0,T]\times\mathbb{R}) = \{\phi(t,x) \mid \phi(t,\cdot) \text{ is once continuously differentiable on } [0,T], \text{ and } \phi(\cdot,x) \text{ is twice continuously differentiable on } \mathbb{R}\}. \tag{9}$$

From equation (6), for any $\varphi(t,x) \in C^{1,2}([0,T]\times\mathbb{R})$, it is clear that the infinitesimal generator of $X^{\nu}(t)$ is given by

$$\begin{aligned}\mathscr{A}^{\nu}\varphi(t,x) := {} & \lim_{\varepsilon\downarrow 0}\frac{\mathbb{E}_{t,x}[\varphi(t+\varepsilon, X^{\nu}(t+\varepsilon))] - \varphi(t,x)}{\varepsilon} \\ = {} & \varphi_t(t,x) + [r_0 x + \alpha c - (\alpha-\beta)\pi(Z_i \\ & - g(Z_i;\nu(t));\lambda) - (\alpha-\beta)\lambda E[g(Z_i;\nu(t))] \\ & - \beta\lambda E[Z_i]]\varphi_x(t,x) + \frac{1}{2}\Big[\alpha\sqrt{\lambda E[(g(Z_i;\nu(t)))^2]} \\ & + \beta\sqrt{\lambda E[(Z_i - g(Z_i;\nu(t)))^2]}\Big]^2\varphi_{xx}(t,x),\end{aligned} \tag{10}$$

where φ_t, φ_x, and φ_{xx} denote the corresponding first-order and second-order partial derivatives with respect to (w.r.t.) the corresponding variables. To solve the above problem, we use the dynamic programming approach described in Fleming and Soner [28]. From the standard arguments, we see that if the value function $V(t,x) \in C^{1,2}([0,T]\times\mathbb{R})$, then $V(t,x)$ satisfies the following HJB equation:

$$\sup_{\nu\in\Theta}\mathscr{A}^{\nu}V(t,x) = 0, \tag{11}$$

with the boundary condition $V(T,x) = U(x)$.

Using the standard methods of Fleming and Soner [28], we have the following verification theorem.

Theorem 1. *Let $W(t,x) \in C^{1,2}([0,T]\times\mathbb{R})$ be a classical solution to equation (11) with the boundary condition $W(T,x) = U(x)$. Then, $V(t,x) = W(t,x)$. Furthermore, set ν^* such that $\mathscr{A}^{\nu^*}V(t,x) = 0$ holds for all $(t,x) \in [0,T]\times\mathbb{R}$; then, $\{\nu^*(t)\colon t \in [0,T]\}$ is the optimal strategy.*

In this paper, we consider the exponential utility function:

$$U(x) = -\frac{1}{\gamma}e^{-\gamma x}, \tag{12}$$

where $\gamma > 0$ is the constant absolute risk aversion parameter. As we know, exponential utility function plays an important role in insurance mathematics and actuarial practice. It is the only utility function under the principle of "zero utility" giving a fair premium that is independent of the level of an insurer's reserves (see [29]).

3. Optimal Reinsurance Strategy with Generalized Variance Premium Principle

In this paper, we consider that the premium share function $\pi(\cdot;\lambda)$ is based on the generalized variance premium principle. So, the premium rate is

$$\begin{aligned}\pi(Z_i - g(Z_i;\nu(t));\lambda) &= \lambda E[Z_i - g(Z_i;\nu(t))] \\ &\quad + \theta\psi\left(\lambda E\left[(Z_i - g(Z_i;\nu(t)))^2\right]\right),\end{aligned} \tag{13}$$

where θ represents the safety loading of the reinsurer and $\psi(x)$ is a positive increasing function. Besides, we assume that the insurer purchases proportional reinsurance from the reinsurer, i.e., $g(x;\nu(t)) = \nu(t)x$, and parameter space $\Theta = [0,1]$. Denote $a = \lambda E[Z_i]$ and $b^2 = \lambda E[Z_i^2]$; then, equation (6) becomes

$$\begin{aligned}\mathrm{d}X^{\nu}(t) &= \left[r_0X^{\nu}(t) + \alpha c - \alpha a - (\alpha-\beta)\theta\psi\left((1-\nu(t))^2b^2\right)\right]\mathrm{d}t \\ &\quad + [\alpha\nu(t)b + \beta(1-\nu(t))b]\mathrm{d}W(t),\end{aligned} \tag{14}$$

and HJB equation (11) can be rewritten as

$$\begin{aligned}\sup_{\nu\in[0,1]}\Big\{&V_t + \left[r_0x + \alpha c - (\alpha-\beta)\theta\psi\left((1-\nu(t))^2b^2\right) - \alpha a\right]V_x \\ &+ \frac{1}{2}[\alpha\nu(t)b + \beta(1-\nu(t))b]^2V_{xx}\Big\} = 0.\end{aligned} \tag{15}$$

From a mathematical point of view, when $\alpha = \beta$, equation (14) reduces to

$$\mathrm{d}X^{\nu}(t) = [r_0X^{\nu}(t) + \alpha c - \alpha a]\mathrm{d}t + \alpha b\mathrm{d}W(t), \tag{16}$$

which is irrelevant to the reinsurance strategy $\{\nu(t)\colon t\in[0,T]\}$. Thus, any measurable function $\nu^*(t)\colon [0,T] \longrightarrow [0,1]$ is an optimal reinsurance treaty. From a practical point of view, for an investor, the role of reinsurance is transferring the wealth between the insurer and the reinsurer, and if the investor has the same shares on the insurer and the reinsurer or the investor pays the same attention to the insurer and the reinsurer, reinsurance has no effect on the wealth of the investor. In the following section, to derive the explicit solutions to the optimization problem, we discuss two cases: variance and standard deviation premium principles, respectively, only with $\alpha \neq \beta$.

3.1. Variance Premium Principle. Under variance premium principle, $\psi(x) = x$, and denote $A = 2\theta/\gamma$ and $B = Ae^{-r_0T}$. By using stochastic control theory, the optimal reinsurance strategy $\{\nu^*(t)\colon t\in[0,T]\}$ for $\alpha \neq \beta$ can be obtained analytically as summarized in Theorem 2. The expression of the optimal reinsurance strategy $\{\nu^*(t)\colon t\in[0,T]\}$ is given for different cases as outlined in Tables 1 and 2.

Theorem 2. *Denote*

$$\begin{aligned}t_1 &= T - \frac{1}{r_0}\ln\left(\frac{A}{\beta}\right), \\ t_2 &= T - \frac{1}{r_0}\ln\left(\frac{A}{\beta-\alpha}\right), \\ t_3 &= T - \frac{1}{r_0}\ln\left(\frac{A}{\alpha+\beta}\right), \\ K_\nu &= \frac{\alpha(c-a) - (\alpha-\beta)\theta b^2}{r_0}.\end{aligned} \tag{17}$$

The optimal reinsurance strategy and the corresponding optimal value function for problem (8) with equation (12) and $\alpha \neq \beta$ under variance premium principle are as follows.

(1) For Cases I, IV, and VI in Table 1, the optimal reinsurance strategy is

$$\nu^*(t) = 1 - \frac{\alpha\gamma e^{r_0(T-t)}}{(\alpha-\beta)\gamma e^{r_0(T-t)} + 2\theta}, \quad 0 \le t \le T, \tag{18}$$

and the optimal value function is

$$V(t,x) = -\frac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-d_1(t))\right]}, \quad 0 \le t \le T, \tag{19}$$

where

$$\begin{aligned}d_1(t) &= -K_\nu\left(1 - e^{-r_0(T-t)}\right) + \frac{\beta^2b^2\gamma}{4r_0}\left(e^{r_0(T-t)} - e^{-r_0(T-t)}\right) \\ &\quad - e^{r_0t}\int_t^T\left[\frac{(\alpha-\beta)e^{-r_0s}}{2}\cdot\frac{\left(\beta\gamma e^{r_0(T-s)} - 2\theta\right)^2b^2}{(\alpha-\beta)\gamma e^{r_0(T-s)} + 2\theta}\right]\mathrm{d}s.\end{aligned} \tag{20}$$

(2) For Cases II and V in Table 1, the optimal reinsurance strategy is

$$\nu^*(t) = \begin{cases} 0, & 0 \le t < t_1, \\ 1 - \dfrac{\alpha\gamma e^{r_0(T-t)}}{(\alpha-\beta)\gamma e^{r_0(T-t)} + 2\theta}, & t_1 \le t \le T,\end{cases} \tag{21}$$

and the optimal value function is

Table 1: Different cases with $\beta<\alpha$ and variance premium principle.

Parameters	Case
$A\le 1$	
$\beta\le B$	I
$B<\beta\le A$	II
$A<\beta$	III
$B\le 1<A$	
$\beta\le B$	IV
$B<\beta$	V
$1<B$	
$\beta<\alpha$	VI

Table 2: Different cases with $\alpha<\beta$ and variance premium principle.

Parameters	Case
$A\le 2$	
$\beta\le B-\alpha$	VII
$B-\alpha<\beta\le \min\{B+\alpha, A-\alpha\}$	VIII
$A-\alpha<\beta\le B+\alpha$	IX
$B+\alpha<\beta\le A-\alpha$	X
$\max\{B+\alpha, A-\alpha\}<\beta\le A+\alpha$	XI
$A+\alpha<\beta$	XII
$\beta\le B-\alpha$	
$B\le 2<A$	XIII
$B-\alpha<\beta\le B+\alpha$	XIV
$B+\alpha<\beta$	XV
$2<B$	
$\alpha<\beta$	XVI

$$V(t,x)=\begin{cases}-\dfrac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(x-d_2(t))]}, & 0\le t<t_1,\\[2ex] -\dfrac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(x-d_1(t))]}, & t_1\le t\le T,\end{cases} \tag{22}$$

where

$$\begin{aligned}d_2(t)={}&-K_\nu\left(1-e^{-r_0(T-t)}\right)+\frac{\beta^2b^2\gamma}{4r_0}\left(e^{r_0(T-t)}-e^{-r_0(T-t)}\right)\\ &-e^{r_0t}\int_{t_1}^{T}\left[\frac{(\alpha-\beta)e^{-r_0s}}{2}\cdot\frac{\left(\beta\gamma e^{r_0(T-s)}-2\theta\right)^2b^2}{(\alpha-\beta)\gamma e^{r_0(T-s)}+2\theta}\right]ds,\end{aligned} \tag{23}$$

and $d_1(t)$ is given by equation (20).

(3) For Case III in Table 1 and Cases VII, XIII, and XVI in Table 2, the optimal reinsurance strategy is

$$\nu^*(t)=0,\quad 0\le t\le T, \tag{24}$$

and the optimal value function is

$$V(t,x)=-\frac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(x-d_3(t))]},\quad 0\le t\le T, \tag{25}$$

where

$$d_3(t)=-K_\nu\left(1-e^{-r_0(T-t)}\right)+\frac{\beta^2b^2\gamma}{4r_0}\left(e^{r_0(T-t)}-e^{-r_0(T-t)}\right). \tag{26}$$

(4) For Cases VIII, X, XIV, and XV in Table 2, the optimal reinsurance strategy is

$$\nu^*(t)=\begin{cases}1, & 0\le t<t_3,\\ 0, & t_3\le t\le T,\end{cases} \tag{27}$$

and the optimal value function is

$$V(t,x)=\begin{cases}-\dfrac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(x-d_4(t))]}, & 0\le t<t_3,\\[2ex] -\dfrac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(x-d_3(t))]}, & t_3\le t\le T,\end{cases} \tag{28}$$

where

$$\begin{aligned}d_4(t)={}&-\frac{\alpha(c-a)}{r_0}\left(1-e^{-r_0(t_3-t)}\right)+\frac{\alpha^2b^2\gamma}{4r_0}\\ &\cdot\left(e^{r_0(T-t)-e^{r_0(T+t-2t_3)}}\right)-K_\nu\left(e^{-r_0(t_3-t)}-e^{-r_0(T-t)}\right)\\ &+\frac{\beta^2b^2\gamma}{4r_0}\left(e^{r_0(T+t-2t_3)}-e^{r_0(T-t)}\right),\end{aligned} \tag{29}$$

and $d_3(t)$ is given in equation (26).

(5) For Cases IX, XI, and XII in Table 2, the optimal reinsurance strategy is

$$\nu^*(t)=1,\quad 0\le t\le T, \tag{30}$$

and the optimal value function is

$$V(t,x)=-\frac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(x-d_5(t))]},\quad 0\le t\le T, \tag{31}$$

where

$$d_5(t)=-\frac{\alpha(c-a)}{r_0}\left(1-e^{-r_0(T-t)}\right)+\frac{\beta^2b^2\gamma}{4r_0}\left(e^{r_0(T-t)}-e^{-r_0(T-t)}\right). \tag{32}$$

Proof. See Appendix A. □

Remark 2. We consider two special cases:

(1) If $\alpha=1,\beta=0$, the optimal reinsurance problem under variance premium principle reduces to the case only for an insurer. The optimal reinsurance strategy is

$$\nu^*(t)=1-\frac{\gamma e^{r_0(T-t)}}{\gamma e^{r_0(T-t)}+2\theta},\quad 0\le t\le T, \tag{33}$$

and the optimal value function is

$$V(t,x) = -\frac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}\left(x-d_6(t)\right)\right]}, \quad 0 \le t \le T, \tag{34}$$

where

$$d_6(t) = \frac{c-a-\theta b^2}{r_0}\left(e^{-r_0(T-t)}-1\right) - e^{r_0 t}\int_t^T \frac{2\theta^2 b^2}{\gamma e^{r_0 T} + 2\theta e^{r_0 s}}\mathrm{d}s. \tag{35}$$

From equation (33), we find that the optimal reinsurance strategy is similar to those in Liang and Yuen [26], Lin and Yang [30], and Wen [31], which considered the optimal reinsurance strategies under variance premium principle only for an insurer.

(2) If $\alpha = 0, \beta = 1$, the optimal reinsurance problem under variance premium principle becomes the case only for a reinsurer. There is almost no literature considering this case. For Cases I, IV, and VI in Table 3, the optimal reinsurance strategy is

$$\nu^*(t) = 0, \quad 0 \le t \le T, \tag{36}$$

and the optimal value function is

$$V(t,x) = -\frac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}\left(x-d_7(t)\right)\right]}, \quad 0 \le t \le T, \tag{37}$$

where

$$d_7(t) = -\frac{\theta b^2}{r_0}\left(1-e^{-r_0(T-t)}\right) + \frac{b^2\gamma}{4r_0}\left(e^{r_0(T-t)} - e^{-r_0(T-t)}\right). \tag{38}$$

For Cases II and V in Table 3, the optimal reinsurance strategy is

$$\nu^*(t) = \begin{cases} 1, & 0 \le t < T - \dfrac{1}{r_0}\ln A, \\ 0, & T - \dfrac{1}{r_0}\ln A \le t \le T, \end{cases} \tag{39}$$

and the optimal value function is

$$V(t,x) = \begin{cases} -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}\left(x-d_8(t)\right)\right]}, & 0 \le t < T - \dfrac{1}{r_0}\ln A, \\ -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}\left(x-d_7(t)\right)\right]}, & T - \dfrac{1}{r_0}\ln A \le t \le T, \end{cases} \tag{40}$$

where

$$\begin{aligned} d_8(t) &= -\frac{\theta b^2}{r_0}\left(e^{-r_0(T-t)+\ln A} - e^{-r_0(T-t)}\right) + \frac{\beta^2 b^2\gamma}{4r_0} \\ &\quad \cdot\left(e^{-r_0(T-t)+2\ln A} - e^{r_0(T-t)}\right), \end{aligned} \tag{41}$$

Table 3: Different cases with $\alpha = 0$ and $\beta = 1$ and variance premium principle.

Parameters	Case
$A \le 2$	
$1 \le B$	I
$B < 1 \le A$	II
$A < 1$	III
$B \le 2 < A$	
$1 \le B$	IV
$B < 1$	V
$2 < B$	VI

and $d_7(t)$ is given in equation (38).

For Case III in Table 3, the optimal reinsurance strategy is

$$\nu^*(t) = 1, \quad 0 \le t \le T, \tag{42}$$

and the optimal value function is

$$V(t,x) = -\frac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}\left(x-d_9(t)\right)\right]}, \quad 0 \le t \le T, \tag{43}$$

where

$$d_9(t) = \frac{b^2\gamma}{4r_0}\left(\mathrm{e}^{r_0(T-t)} - \mathrm{e}^{-r_0(T-t)}\right). \tag{44}$$

3.2. Standard Deviation Premium Principle. Under standard deviation premium principle, $\psi(x) = \sqrt{x}$, and denote $C = \theta/b\gamma$ and $D = Ce^{-r_0 T}$. By stochastic control theory, the optimal reinsurance strategy $\{\nu^*(t): t \in [0,T]\}$ for $\alpha \ne \beta$ can be obtained analytically as summarized in Theorem 3. The expression of the optimal reinsurance strategy $\{\nu^*(t): t \in [0,T]\}$ is given for different cases as outlined in Tables 4 and 5.

Theorem 3. *Denote*

$$\begin{aligned} t_4 &= T - \frac{1}{r_0}\ln\left(\frac{C}{\beta}\right), \\ t_5 &= T - \frac{1}{r_0}\ln\left(\frac{C}{\alpha}\right), \\ K_{s_1} &= \frac{\alpha(c-a) - (\alpha-\beta)\theta b}{r_0}, \\ K_{s_2} &= \frac{\alpha(c-a) - \alpha\theta b}{r_0}. \end{aligned} \tag{45}$$

For problem (8) with equation (12) and $\alpha \ne \beta$ under standard deviation premium principle, the optimal reinsurance strategy and the corresponding optimal value function are as follows.

(1) For Cases I, VII, and X in Table 4 and Case XVI in Table 5, the optimal reinsurance strategy is

$$\nu^*(t) = 1, \quad 0 \le t \le T, \tag{46}$$

Table 4: Different cases with $\beta<\alpha$ and standard deviation premium principle.

Parameters	Case
$C\le 1$	
$\alpha\le D$	I
$\beta\le D<\alpha\le C$	II
$\beta\le D<C<\alpha$	III
$D<\beta<\alpha\le C$	IV
$D<\beta\le C<\alpha$	V
$C<\beta$	VI
$\alpha\le D$	
$D\le 1<C$	VII
$\beta\le D<\alpha$	VIII
$D<\beta$	IX
$1<D$	
$\beta<\alpha$	X

Table 5: Different cases with $\alpha<\beta$ and standard deviation premium principle.

Parameters	Case
$C\le 1$	
$\beta\le D$	XI
$\alpha\le D<\beta\le C$	XII
$\alpha\le D<C<\beta$	XIII
$D<\alpha<\beta\le C$	XIV
$D<\alpha\le C<\beta$	XV
$C<\alpha$	XVI
$\beta\le D$	
$D\le 1<C$	XVII
$\alpha\le D<\beta$	XVIII
$D<\alpha$	XIX
$1<D$	
$\alpha<\beta$	XX

and the optimal value function is

$$V(t,x)=-\frac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_1(t))\right]},\quad 0\le t\le T,\tag{47}$$

where

$$f_1(t)=-\frac{\alpha(c-a)}{r_0}\left(1-e^{-r_0(T-t)}\right)+\frac{\alpha^2b^2\gamma}{4r_0}\cdot\left(e^{r_0(T-t)}-e^{-r_0(T-t)}\right).\tag{48}$$

(2) For Cases II and VIII in Table 4, the optimal reinsurance strategy is

$$\nu^*(t)=\begin{cases}-\dfrac{\beta}{\alpha-\beta}+\dfrac{\theta}{(\alpha-\beta)b\gamma e^{r_0(T-t)}}, & 0\le t<t_5,\\[2ex] 1, & t_5\le t\le T,\end{cases}\tag{49}$$

and the optimal value function is

$$V(t,x)=\begin{cases}-\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_2(t))\right]}, & 0\le t<t_5,\\[2ex] -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_1(t))\right]}, & t_5\le t\le T,\end{cases}\tag{50}$$

where

$$f_2(t)=-K_{s_2}\left(1-e^{-r_0(t_5-t)}\right)-\frac{\theta^2}{2\gamma e^{r_0(T-t)}}(t_5-t)-\frac{\alpha(c-a)}{r_0}\left(e^{-r_0(t_5-t)}-e^{-r_0(T-t)}\right)+\frac{\alpha^2b^2\gamma}{4r_0}\cdot\left(e^{r_0(T+t-2t_5)}-e^{-r_0(T-t)}\right),\tag{51}$$

and $f_1(t)$ is given by equation (48).

(3) For Case III in Table 4 and Case XIII in Table 5, the optimal reinsurance strategy is

$$\nu^*(t)=-\frac{\beta}{\alpha-\beta}+\frac{\theta}{(\alpha-\beta)b\gamma e^{r_0(T-t)}},\quad 0\le t\le T,\tag{52}$$

and the optimal value function is

$$V(t,x)=-\frac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_3(t))\right]},\quad 0\le t\le T,\tag{53}$$

where

$$f_3(t)=-K_{s_2}\left(1-e^{-r_0(T-t)}\right)-\frac{\theta^2}{2\gamma e^{r_0(T-t)}}(T-t).\tag{54}$$

(4) For Cases IV and IX in Table 4, the optimal reinsurance strategy is

$$\nu^*(t)=\begin{cases}0, & 0\le t<t_4,\\[2ex] -\dfrac{\beta}{\alpha-\beta}+\dfrac{\theta}{(\alpha-\beta)b\gamma e^{r_0(T-t)}}, & t_4\le t<t_5,\\[2ex] 1, & t_5\le t\le T,\end{cases}\tag{55}$$

and the optimal value function is

$$V(t,x)=\begin{cases}-\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_4(t))\right]}, & 0\le t<t_4,\\[2ex] -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_2(t))\right]}, & t_4\le t<t_5,\\[2ex] -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_1(t))\right]}, & t_5\le t\le T,\end{cases}\tag{56}$$

where

$$f_4(t) = -K_{s_1}\left(1 - e^{-r_0(t_4-t)}\right) + \frac{\beta^2 b^2 \gamma}{4r_0} \cdot \left(e^{r_0(T-t_4)} - e^{r_0(T+t-2t_4)}\right) - K_{s_2} \cdot \left(e^{-r_0(t_4-t)} - e^{-r_0(t_5-t)}\right) - \frac{\theta^2}{2\gamma e^{r_0(T-t)}}(t_5 - t_4) - \frac{\alpha(c-a)}{r_0}\left(e^{-r_0(t_5-t)} - e^{-r_0(T-t)}\right) + \frac{\alpha^2 b^2 \gamma}{4r_0} \cdot \left(e^{r_0(T+t-2t_5)} - e^{-r_0(T-t)}\right), \quad (57)$$

and $f_1(t)$, $f_2(t)$ are given in equations (48) and (51), respectively.

(5) For Case V in Table 4, the optimal reinsurance strategy is

$$\nu^*(t) = \begin{cases} 0, & 0 \le t < t_4, \\ -\dfrac{\beta}{\alpha-\beta} + \dfrac{\theta}{(\alpha-\beta)b\gamma e^{r_0(T-t)}}, & t_4 \le t \le T, \end{cases} \quad (58)$$

and the optimal value function is

$$V(t,x) = \begin{cases} -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_5(t))\right]}, & 0 \le t < t_4, \\ -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_3(t))\right]}, & t_4 \le t \le T, \end{cases} \quad (59)$$

where

$$f_5(t) = -K_{s_1}\left(1 - e^{-r_0(t_4-t)}\right) + \frac{\beta^2 b^2 \gamma}{4r_0} \cdot \left(e^{r_0(T-t)} - e^{r_0(T+t-2t_4)}\right) - K_{s_2} \cdot \left(e^{-r_0(t_4-t)} - e^{-r_0(T-t)}\right) - \frac{\theta^2}{2\gamma e^{r_0(T-t)}}(T - t_4), \quad (60)$$

and $f_3(t)$ is given by equation (54).

(6) For Case VI in Table 4 and Cases XI, XVII, and XX in Table 5, the optimal reinsurance strategy is

$$\nu^*(t) = 0, \quad 0 \le t \le T, \quad (61)$$

and the optimal value function is

$$V(t,x) = -\frac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_6(t))\right]}, \quad 0 \le t \le T, \quad (62)$$

where

$$f_6(t) = -K_{s_1}\left(1 - e^{-r_0(T-t)}\right) + \frac{\beta^2 b^2 \gamma}{4r_0}\left(e^{r_0(T-t)} - e^{-r_0(T-t)}\right). \quad (63)$$

(7) For Cases XII and XVIII in Table 5, the optimal reinsurance strategy is

$$\nu^*(t) = \begin{cases} -\dfrac{\beta}{\alpha-\beta} + \dfrac{\theta}{(\alpha-\beta)b\gamma e^{r_0(T-t)}}, & 0 \le t < t_4, \\ 0, & t_4 \le t \le T, \end{cases} \quad (64)$$

and the optimal value function is

$$V(t,x) = \begin{cases} -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_7(t))\right]}, & 0 \le t < t_4, \\ -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_6(t))\right]}, & t_4 \le t \le T, \end{cases} \quad (65)$$

where

$$f_7(t) = -K_{s_2}\left(1 - e^{-r_0(t_4-t)}\right) - \frac{\theta^2}{2\gamma e^{r_0(T-t)}}(t_4 - t) - K_{s_1} \cdot \left(e^{-r_0(t_4-t)} - e^{-r_0(T-t)}\right) + \frac{\beta^2 b^2 \gamma}{4r_0} \cdot \left(e^{r_0(T+t-2t_4)} - e^{-r_0(T-t)}\right), \quad (66)$$

and $f_6(t)$ is given in equation (63).

(8) For Cases XIV and XIX in Table 5, the optimal reinsurance strategy is

$$\nu^*(t) = \begin{cases} 1, & 0 \le t < t_5, \\ -\dfrac{\beta}{\alpha-\beta} + \dfrac{\theta}{(\alpha-\beta)b\gamma e^{r_0(T-t)}}, & t_5 \le t < t_4, \\ 0, & t_4 \le t \le T, \end{cases} \quad (67)$$

and the optimal value function is

$$V(t,x) = \begin{cases} -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_8(t))\right]}, & 0 \le t < t_5, \\ -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_7(t))\right]}, & t_5 \le t < t_4, \\ -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_6(t))\right]}, & t_4 \le t \le T, \end{cases} \quad (68)$$

where

$$f_8(t) = -\frac{\alpha(c-a)}{r_0}\left(1-e^{-r_0(t_5-t)}\right) + \frac{\alpha^2 b^2 \gamma}{4r_0} \cdot \left(e^{r_0(T-t)} - e^{r_0(T+t-2t_5)}\right) - K_{s_2} \cdot \left(e^{-r_0(t_5-t)} - e^{-r_0(t_4-t)}\right) - \frac{\theta^2}{2\gamma e^{r_0(T-t)}}(t_4 - t_5) - K_{s_1}\left(e^{-r_0(t_4-t)} - e^{-r_0(T-t)}\right) + \frac{\beta^2 b^2 \gamma}{4r_0} \cdot \left(e^{r_0(T+t-2t_4)} - e^{-r_0(T-t)}\right), \quad (69)$$

and $f_6(t)$ and $f_7(t)$ are given by equations (63) and (66), respectively.

(9) For Case XV in Table 5, the optimal reinsurance strategy is

$$\nu^*(t) = \begin{cases} 1, & 0 \le t < t_5, \\ -\dfrac{\beta}{\alpha-\beta} + \dfrac{\theta}{(\alpha-\beta)b\gamma e^{r_0(T-t)}}, & t_5 \le t \le T, \end{cases} \quad (70)$$

and the optimal value function is

$$V(t,x) = \begin{cases} -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_9(t))\right]}, & 0 \le t < t_5, \\ -\dfrac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f_3(t))\right]}, & t_5 \le t \le T, \end{cases} \quad (71)$$

where

$$f_9(t) = -\frac{\alpha(c-a)}{r_0}\left(1-e^{-r_0(t_5-t)}\right) + \frac{\alpha^2 b^2 \gamma}{4r_0} \cdot \left(e^{r_0(T-t)} - e^{r_0(T+t-2t_5)}\right) - K_{s_2}\left(e^{-r_0(t_5-t)} - e^{-r_0(T-t)}\right) - \frac{\theta^2}{2\gamma e^{r_0(T-t)}}(T - t_5), \quad (72)$$

and $f_3(t)$ is given in equation (54).

Proof. See Appendix B. □

4. Sensitivity Analysis and Numerical Illustration

This section illustrates the effects of parameters on the optimal reinsurance strategy by sensitivity analysis and numerical examples. For the following numerical illustrations, unless otherwise stated, the basic parameters are given by $r_0 = 0.05$, $\theta = 0.2$, $\gamma = 0.6$, and $b = 2$.

Firstly, we provide sensitivity analysis about the effects of parameters on the optimal reinsurance strategy under variance and standard deviation premium principles, respectively. From Theorem 2, we find that for $\alpha < \beta$, when $0 \le t < t_2$, $\nu^0(t) \ge 1$, then $\nu^*(t) = 1$, and when $t_2 \le t \le T$, the left side of HJB equation (11) is a parabola opening upwards w.r.t. ν; then, $\nu^*(t)$ equals to 0 or 1. Therefore, $\nu^*(t) \ne 0, 1$ only appears in the case of $\alpha > \beta$, and we mainly analyze this case. Under variance premium principle, according to equation (A.5), for $\nu^*(t) \ne 0, 1$, we can derive partial derivatives of the optimal reinsurance strategy (see Table 6) w.r.t. different parameters, which imply that the optimal reinsurance strategy $\nu^*(t)$ increases w.r.t. time t and safety loading of the reinsurer θ, but decreases w.r.t. risk-free interest rate r_0 and risk aversion coefficient of the investor γ.

Different from variance premium principle, the optimal reinsurance strategy under standard deviation premium principle is dependent on b. From equation (B.4), for $\nu^*(t) \ne 0, 1$, we can obtain partial derivatives of the optimal reinsurance strategy (see Table 7) w.r.t. different parameters, which imply that when $\alpha > \beta$, the optimal reinsurance strategy $\nu^*(t)$ increases w.r.t time t and safety loading of the reinsurer θ, but decreases w.r.t. risk-free interest rate r_0, risk aversion coefficient of the investor γ, and volatility rate b, and when $\alpha < \beta$, the effects of parameters on the optimal reinsurance strategy are opposite to the cases of $\alpha > \beta$.

From the above sensitivity analysis, we derive that the effects of parameters t, r_0, θ, and γ on $\nu^*(t)$ under variance premium principle are the same as those under standard deviation premium principle for the case of $\alpha > \beta$. The economic explanation for $\alpha > \beta$ has been given in most existing studies which consider the case only with an insurer, i.e., $\alpha = 1, \beta = 0$ (cf. Lin and Yang [30]), and the results for $\alpha < \beta$ are opposite to those for $\alpha > \beta$. Therefore, we omit numerical simulations about the impacts of parameters t, r_0, θ, and γ on $\nu^*(t)$ under variance premium principle and standard deviation premium principle. For the effect of b on $\nu^*(t)$ under standard deviation premium principle, since larger b implies larger claims, when $\alpha > \beta$, the investor holds more shares in the insurer, and he/she will follow the insurer's preference to purchase more reinsurance and undertake less risks. When $\alpha < \beta$, the investor holds more shares in the reinsurer, and the reinsurer's preference is paid more attention and the reinsurer would like to accept less reinsurance; then, the risk retained by the insurer increases w.r.t. b. Since using variance premium principle neutralizes the effect of volatility of claim, the optimal reinsurance strategy under variance premium principle is independent of b.

In the following, we will discuss the influence of weight coefficients α and β on the optimal reinsurance strategy $\nu^*(t)$ in detail with numerical simulations. Under variance premium principle, we have

Table 6: Partial derivatives of the optimal reinsurance strategy under variance premium principle.

Derivatives	$\partial\nu^*(t)/\partial t$	$\partial\nu^*(t)/\partial r_0$	$\partial\nu^*(t)/\partial\theta$	$\partial\nu^*(t)/\partial\gamma$
$\alpha > \beta$	>0	<0	>0	<0

Table 7: Partial derivatives of the optimal reinsurance strategy under standard deviation premium principle.

Derivatives	$\partial\nu^*(t)/\partial t$	$\partial\nu^*(t)/\partial r_0$	$\partial\nu^*(t)/\partial\theta$	$\partial\nu^*(t)/\partial\gamma$	$\partial\nu^*(t)/\partial b$
$\alpha > \beta$	>0	<0	>0	<0	<0
$\alpha < \beta$	<0	>0	<0	>0	>0

$$\frac{\partial\nu^*(t)}{\partial\alpha} = -\frac{\left[2\theta - \beta\gamma e^{r_0(T-t)}\right]\gamma e^{r_0(T-t)}}{\left[(\alpha-\beta)\gamma e^{r_0(T-t)} + 2\theta\right]^2},$$
$$\frac{\partial\nu^*(t)}{\partial\beta} = -\frac{\alpha\gamma^2 e^{2r_0(T-t)}}{\left[(\alpha-\beta)\gamma e^{r_0(T-t)} + 2\theta\right]^2}, \quad (73)$$

and thus when $\beta > Ae^{-r_0(T-t)}$, $\nu^*(t)$ is an increasing function of α, and when $\beta < Ae^{-r_0(T-t)}$, $\nu^*(t)$ is a decreasing function of α. Meanwhile, $\nu^*(t)$ decreases with β all the time. For the case under standard deviation premium principle, we derive

$$\frac{\partial\nu^*(t)}{\partial\alpha} = \frac{\beta b\gamma e^{r_0(T-t)} - \theta}{(\alpha-\beta)^2 b\gamma e^{r_0(T-t)}},$$
$$\frac{\partial\nu^*(t)}{\partial\beta} = -\frac{\alpha b\gamma e^{r_0(T-t)} - \theta}{(\alpha-\beta)^2 b\gamma e^{r_0(T-t)}}. \quad (74)$$

So, when $\beta > Ce^{-r_0(T-t)}$, $\nu^*(t)$ increases w.r.t. α, and when $\beta < Ce^{-r_0(T-t)}$, $\nu^*(t)$ decreases w.r.t. α. Meanwhile, when $\alpha > Ce^{-r_0(T-t)}$, β exerts a negative influence on $\nu^*(t)$, and when $\alpha < Ce^{-r_0(T-t)}$, β exerts a positive impact on $\nu^*(t)$.

The following figures also provide the relationships. Figure 1 shows the impacts of weight coefficients α and β on the optimal reinsurance strategy $\nu^*(t)$ under variance premium principle. As mentioned above, $\nu^*(t) \neq 0, 1$ exists in the case of $\alpha > \beta$. With $\alpha > \beta$, the investor holds more shares of the insurer. If β is small enough ($\beta < Ae^{-r_0(T-t)}$), a larger α yields a smaller $\nu^*(t)$, and later, the optimal reinsurance strategy becomes 0. Meanwhile, $\nu^*(t)$ decreases with β. Figure 2 illustrates the effects of weight coefficients α and β on the optimal reinsurance strategy $\nu^*(t)$ under standard deviation premium principle. In Figure 2(a), since parameters in the numerical simulation satisfy $\beta > Ce^{-r_0(T-t)}$ for $t \in [0, T]$, $\nu^*(t)$ ($\nu^*(t) \neq 0, 1$) increases w.r.t. α. As α increases, $\nu^*(t)$ becomes 1 and 0, respectively. In Figure 2(b), parameters satisfy $\alpha > Ce^{-r_0(T-t)}$ for $t \in [0, T]$, and thus $\nu^*(t)$ ($\nu^*(t) \neq 0, 1$) is a decreasing function of β. With the increase of β, $\nu^*(t)$ also becomes 0 and 1, respectively.

From the analysis under variance premium principle, $\nu^*(t)$ only takes the extremes strategy, 1 or 0 under the case of $\alpha < \beta$, and thus we consider the case of $\alpha > \beta$. $\alpha > \beta$ implies that the investor holds more shares of the insurer, and the investor pays more attention to the insurer. If β is small enough, $\beta < Ae^{-r_0(T-t)}$, to maximize the utility of the investor, the investor is suggested to cede more risk from the

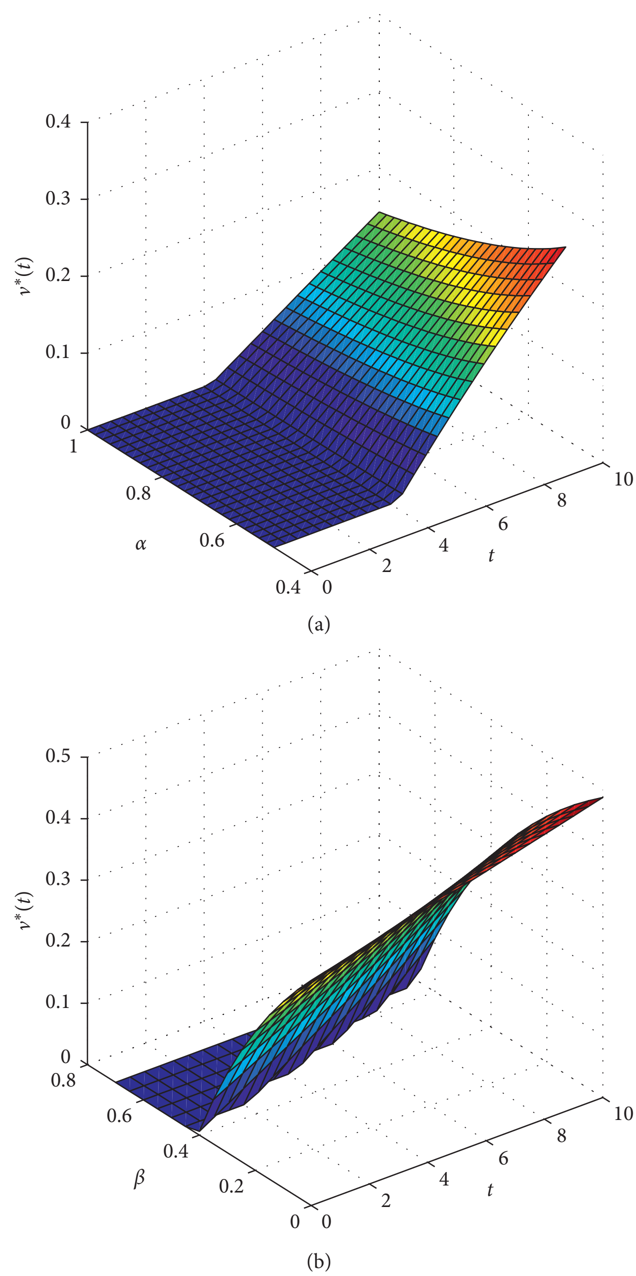

Figure 1: (a) The effect of α on $\nu^*(t)$ under variance premium principle ($\beta = 0.5$). (b) The effect of β on $\nu^*(t)$ under variance premium principle ($\alpha = 0.5$).

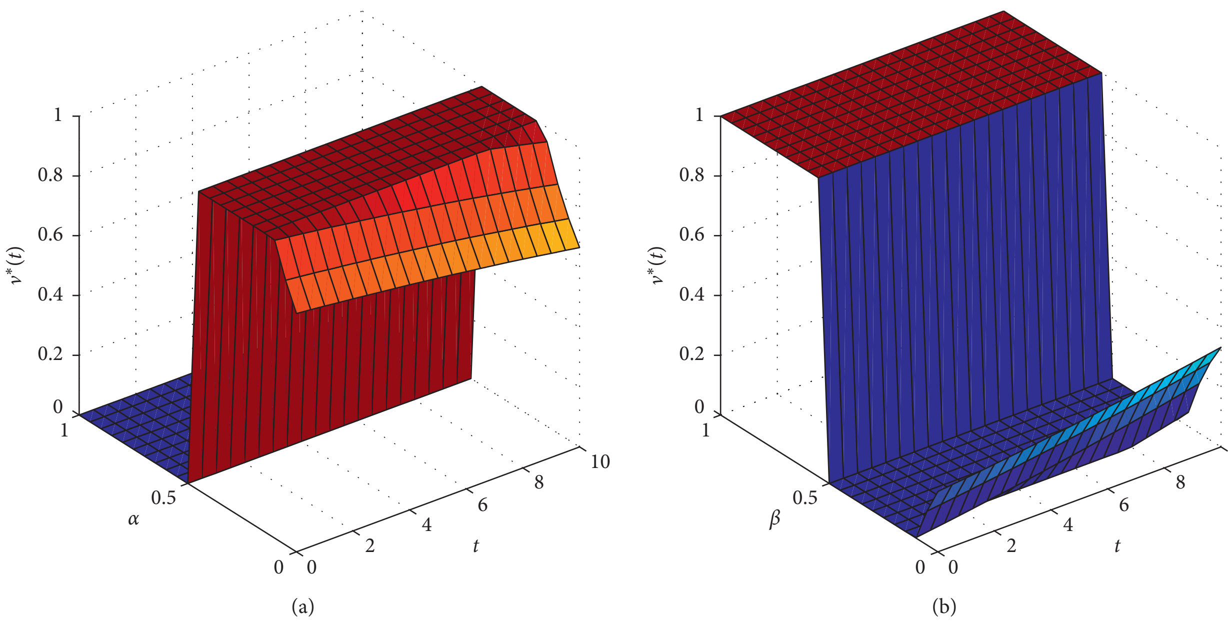

Figure 2: (a) The effect of α on $\nu^*(t)$ under standard deviation premium principle ($\beta = 0.5$). (b) The effect of β on $\nu^*(t)$ under standard deviation premium principle ($\alpha = 0.5$).

insurer to the reinsurer as the shares of the insurer, α, increase, while if $\beta > Ae^{-r_0(T-t)}$, the investor is suggested to let the insurer undertake more claim risks as α increases. Moreover, to gain more benefits, the investor is suggested to transfer more claim risks from insurer to reinsurer, when the shares of reinsurer, β, increase, no matter the value of α.

From the analysis under standard deviation premium principle, $\nu^*(t) \neq 0, 1$ appears in both cases that the investor holds more or less shares of the insurer, i.e., the cases of both $\alpha > \beta$ and $\alpha < \beta$. The effect of α on $\nu^*(t)$ under standard deviation premium principle is similar to that under variance premium principle. Therefore, the investor is suggested to make a decision like the case of variance premium principle. The main reason is that the aim of the investor is to maximize the utility of his/her total wealth (the weighted sum of the insurer's and the reinsurer's wealth). Different from variance premium principle, the effect of β on $\nu^*(t)$ under standard deviation premium principle is also dependent on the value of α. If the investor holds less shares of the insurer, $\alpha < Ce^{-r_0(T-t)}$, the investor is suggested to let the insurer accept more claim risks as the shares of the reinsurer, β, increase to maximize his/her utility, while if the investor holds more shares of the insurer, $\alpha > Ce^{-r_0(T-t)}$, the investor is suggested to cede more risk from the insurer to the reinsurer with β increasing.

5. Conclusion

In this paper, we consider the optimal reinsurance problem with joint interests of both an insurer and a reinsurer. The risk process is assumed to follow a Brownian motion with drift and the insurer transfers part of the risk to the reinsurer via proportional reinsurance. Meanwhile, the reinsurance premium is calculated according to the variance and standard deviation premium principles. The objective is to maximize the expected exponential utility of the weighted sum of the insurer's and the reinsurer's terminal wealth, where the weight can be viewed as shares held by the investor in the insurer and the reinsurer or a regularization parameter to measure the importance of each party. By applying stochastic control theory, we establish the HJB equation and obtain the explicit expressions of optimal reinsurance strategies and optimal value functions. Furthermore, we provide some sensitivity analyses and numerical simulations to illustrate the effects of model parameters on the optimal reinsurance strategies.

In future work, a more general premium principle will be considered in the optimal reinsurance problem for an insurer and a reinsurer. Moreover, we will consider the optimal reinsurance problem among multiple insurers and reinsurers, which is also an interesting extension of this paper.

Appendix

A. Proof of Theorem 2

In the case of variance premium principle, the differential operator $\mathcal{A}^\nu$ becomes

$$\begin{aligned}\mathcal{A}^\nu V(t,x) := {} & V_t(t,x) + \left[r_0 x + \alpha c - \alpha a - (\alpha-\beta)\theta \right.\\ & \left. \cdot (1-\nu(t))^2 b^2\right] V_x(t,x) + \frac{1}{2}[\alpha\nu(t)b \\ & + \beta(1-\nu(t))b]^2 V_{xx}(t,x).\end{aligned} \tag{A.1}$$

Differentiating equation (15) w.r.t. ν, from first-order optimality condition, we have

$$\nu^0 = 1 - \frac{\alpha V_{xx}}{(\alpha-\beta)V_{xx} - 2\theta V_x}. \tag{A.2}$$

To proceed, we conjecture the solution in the following form

$$V(t,x) = -\frac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-d(t))\right]}, \tag{A.3}$$

with the boundary condition $d(T)=0$. Then,

$$\begin{aligned} V_t &= -\gamma\left[-r_0 e^{r_0(T-t)}(x-d(t)) - d_t(t)e^{r_0(T-t)}\right]V,\\ V_x &= -\gamma e^{r_0(T-t)}V,\\ V_{xx} &= \gamma^2 e^{2r_0(T-t)}V. \end{aligned} \tag{A.4}$$

Substituting the above derivatives into equation (A.2) implies

$$\nu^0 = 1 - \frac{\alpha\gamma e^{r_0(T-t)}}{(\alpha-\beta)\gamma e^{r_0(T-t)} + 2\theta}. \tag{A.5}$$

When $\alpha>\beta$, we have $((\partial^2\mathscr{A}^\nu V(t,x))/\partial\nu^2)<0$ and $\nu^0\le 1$. Let us denote $F(\nu)\colon = \mathscr{A}^\nu V(t,x)$. Therefore, $F(\nu)$ is a parabola opening downwards w.r.t. ν. To proceed, define

$$\begin{aligned} \mathscr{A}_1 &= \left\{(t,x)\in[0,T]\times R;\quad 0\le\nu^0\le 1\right\},\\ \mathscr{A}_2 &= \left\{(t,x)\in[0,T]\times R;\quad \nu^0<0\right\}. \end{aligned} \tag{A.6}$$

For $(t,x)\in\mathscr{A}_1$, the supremum of equation (15) over ν is attained at ν^0 given by equation (A.5). Substituting equation (A.2) into equation (15) gives

$$\begin{aligned} &V_t + \left[r_0x + \alpha c - \alpha a - (\alpha-\beta)\theta b^2\right]V_x + \frac{1}{2}\beta^2 b^2 V_{xx}\\ &\quad - \frac{\alpha-\beta}{2}\cdot\frac{(\beta V_{xx} + 2\theta V_x)^2 b^2}{(\alpha-\beta)V_{xx} - 2\theta V_x} = 0. \end{aligned} \tag{A.7}$$

For $(t,x)\in\mathscr{A}_2$, equation (15) reaches its maximum at $\nu^*=0$. Consequently, equation (15) becomes

$$V_t + \left[r_0x + \alpha c - \alpha a - (\alpha-\beta)\theta b^2\right]V_x + \frac{1}{2}\beta^2 b^2 V_{xx} = 0. \tag{A.8}$$

When $\alpha<\beta$, we have $\partial^2\mathscr{A}^\nu V(t,x)/\partial\nu^2<0$ if and only if $t<t_2$. At this time $\nu^0\ge 1$, the maximum in equation (15) is attained at $\nu^*=1$. If $\partial^2\mathscr{A}^\nu V(t,x)/\partial\nu^2=0$, then $F(\nu)$ is an increasing function w.r.t. ν. Thus, $\nu^*=1$. When $t>t_2$, we derive $\partial^2\mathscr{A}^\nu V(t,x)/\partial\nu^2>0$ and $F(\nu)$ is a parabola opening upwards. To proceed, define

$$\begin{aligned} \mathscr{A}_3 &= \left\{(t,x)\in[0,T]\times R; \nu^0\ge\frac{1}{2}\right\},\\ \mathscr{A}_4 &= \left\{(t,x)\in[0,T]\times R; \nu^0<\frac{1}{2}\right\}. \end{aligned} \tag{A.9}$$

For $(t,x)\in\mathscr{A}_3$, the supremum of equation (15) over ν is attained at $\nu^*=0$. Introducing $\nu^*=0$ into equation (15) reduces to equation (A.8). Similarly, the maximum of equation (15) on $\mathscr{A}_4$ is $\nu^*=1$. Substituting $\nu^*=1$ into equation (15) yields

$$V_t + \left[r_0x + \alpha c - \alpha a\right]V_x + \frac{1}{2}\alpha^2 b^2 V_{xx} = 0. \tag{A.10}$$

Equations (A.7)–(A.10) can be solved by the same procedure, and we demonstrate the procedure with equation (A.7) only. Substituting V_t, V_x,and V_{xx} into equation (A.7), we derive

$$\begin{aligned} &d_t(t) - r_0 d(t) - \alpha c + \alpha a + (\alpha-\beta)\theta b^2 + \frac{\beta^2 b^2\gamma e^{r_0(T-t)}}{2}\\ &\quad - \frac{\alpha-\beta}{2}\cdot\frac{\left(\beta\gamma e^{r_0(T-t)} - 2\theta\right)^2 b^2}{(\alpha-\beta)\gamma e^{r_0(T-t)} + 2\theta} = 0. \end{aligned} \tag{A.11}$$

Considering the boundary condition, we obtain

$$\begin{aligned} d(t) &= -K_\nu\left(1 - e^{-r_0(T-t)}\right) + \frac{\beta^2 b^2\gamma}{4r_0}\left(e^{r_0(T-t)} - e^{-r_0(T-t)}\right)\\ &\quad - e^{r_0 t}\int_t^T \frac{(\alpha-\beta)e^{-r_0 s}}{2}\cdot\frac{\left(\beta\gamma e^{r_0(T-s)} - 2\theta\right)^2 b^2}{(\alpha-\beta)\gamma e^{r_0(T-s)} + 2\theta}\,\mathrm{d}s. \end{aligned} \tag{A.12}$$

Based on the above discussion, we can derive the optimal reinsurance strategy for each case outlined in Tables 1 and 2. According to the procedure demonstrated above for solving equations (A.7)–(A.10), as well as the continuity of the function $V(t,x)$, we can obtain the corresponding optimal value function under variance premium principle which is summarized in Theorem 2.

B. Proof of Theorem 3

In the case of standard derivation premium principle, the differential operator $\mathscr{A}^\nu$ becomes

$$\begin{aligned} \mathscr{A}^\nu V(t,x) &:= V_t(t,x) + \left[r_0x + \alpha c - \alpha a - (\alpha-\beta)\theta\right.\\ &\quad \left.\cdot(1-\nu(t))b\right]V_x(t,x) + \frac{1}{2}\left[\alpha\nu(t)b + \beta\right.\\ &\quad \left.\cdot(1-\nu(t))b\right]^2 V_{xx}(t,x). \end{aligned} \tag{B.1}$$

The first-order condition for the optimal reinsurance strategy gives

$$\nu^0 = -\frac{\beta}{\alpha-\beta} - \frac{\theta V_x}{(\alpha-\beta)bV_{xx}}, \tag{B.2}$$

and according to $V_{xx}<0$, we find $\partial^2\mathscr{A}^\nu V(t,x)/\partial\nu^2<0$ holds all the time. Similar to the case of variance premium principle, we try to find the solution with the following structure:

$$V(t,x) = -\frac{1}{\gamma}e^{-\gamma\left[e^{r_0(T-t)}(x-f(t))\right]}, \tag{B.3}$$

and the boundary condition is $f(T)=0$. Then, ν^0 becomes

$$\nu^0 = -\frac{\beta}{\alpha-\beta} + \frac{\theta}{(\alpha-\beta)b\gamma e^{r_0(T-t)}}. \tag{B.4}$$

To proceed, define

$$\begin{aligned}
\mathscr{B}_1 &= \left\{(t,x) \in [0,T] \times R; \nu^0 > 1\right\},\\
\mathscr{B}_2 &= \left\{(t,x) \in [0,T] \times R; 0 \le \nu^0 \le 1\right\},\\
\mathscr{B}_3 &= \left\{(t,x) \in [0,T] \times R; \nu^0 < 0\right\}.
\end{aligned} \tag{B.5}$$

For $(t,x) \in \mathscr{B}_1$, the supremum of equation (15) over ν is attained at $\nu^* = 1$. If $(t,x) \in \mathscr{B}_2$, equation (15) reaches its maximum at ν^0 given by equation (B.2). Similarly, the maximum of equation (15) over ν on $\mathscr{B}_3$ is attained at $\nu^* = 0$. The procedure of computation is similar to the case of variance premium principle. Therefore, we omit it here.

Through the above discussion, we can derive the optimal reinsurance strategy for each case listed in Tables 4 and 5. To proceed, we note that the supremum in equation (15) is attained at $\nu^* = 0$ for $\nu^0 < 0$ and $\nu^* = 1$ for $\nu^0 > 1$; moreover, from equation (B.2), $0 \le \nu^0 \le 1$ if and only if $t_4 \le t \le t_5$ (for $\alpha > \beta$) or $t_5 \le t \le t_4$ (for $\alpha < \beta$). With the procedure as demonstrated in the case of variance premium principle, as well as the continuity of the function $V(t,x)$, we can obtain the corresponding optimal value function under standard derivation premium principle which is summarized in Theorem 3.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

Danping Li gratefully acknowledges the financial support from the National Natural Science Foundation of China (nos. 11801179, 71771220, and 71701084), the "Chenguang Program," Shanghai, China (no. 18CG26), the Fundamental Research Funds for the Central Universities (no. 2019ECNU-HWFW028), and the 111 Project, China (no. B14019). Chaohai Shen gratefully acknowledges the financial support from the Shanghai Pujiang Program (no. 18PJC033).

References

[1] J. Cai and K. S. Tan, "Optimal retention for a stop-loss reinsurance under the VaR and CTE risk measures," *ASTIN Bulletin*, vol. 37, no. 1, pp. 93–112, 2007.

[2] K. Tan and C. Weng, "Enhancing insurer value using reinsurance and value-at-risk criterion," *The Geneva Risk and Insurance Review*, vol. 37, pp. 109–140, 2012.

[3] Y. Chi, "Optimal reinsurance under variance related premium principles," *Insurance: Mathematics and Economics*, vol. 51, no. 2, pp. 310–321, 2012.

[4] S. Román, A. M. Villegas, and J. G. Villegas, "An evolutionary strategy for multiobjective reinsurance optimization," *Journal of the Operational Research Society*, vol. 69, no. 10, pp. 1661–1677, 2018.

[5] H. Schmidli, "On minimizing the ruin probability by investment and reinsurance," *The Annals of Applied Probability*, vol. 12, no. 3, pp. 890–907, 2002.

[6] S. D. Promislow and V. R. Young, "Minimizing the probability of ruin when claims follow Brownian motion with drift," *North American Actuarial Journal*, vol. 9, no. 3, pp. 110–128, 2005.

[7] X. Liang and V. R. Young, "Minimizing the probability of ruin: optimal per-loss reinsurance," *Insurance: Mathematics and Economics*, vol. 82, pp. 181–190, 2018.

[8] N. Bäuerle, "Benchmark and mean-variance problems for insurers," *Mathematical Methods of Operations Research*, vol. 62, no. 1, pp. 159–165, 2005.

[9] Ł. Delong and R. Gerrard, "Mean-variance portfolio selection for a non-life insurance company," *Mathematical Methods of Operations Research*, vol. 66, no. 2, pp. 339–367, 2007.

[10] P. Chen and H. Yao, "Continuous-time mean-variance portfolio selection with no-shorting constraints and regime-switching," *Journal of Industrial & Management Optimization*, vol. 16, no. 2, pp. 531–551, 2020.

[11] D. Li, X. Rong, and H. Zhao, "Time-consistent reinsurance-investment strategy for an insurer and a reinsurer with mean-variance criterion under the CEV model," *Journal of Computational and Applied Mathematics*, vol. 283, pp. 142–162, 2015.

[12] X. Lin and Y. Qian, "Time-consistent mean-variance reinsurance-investment strategy for insurers under CEV model," *Scandinavian Actuarial Journal*, vol. 2016, no. 7, pp. 646–671, 2016.

[13] H. Wang, R. Wang, and J. Wei, "Time-consistent investment-proportional reinsurance strategy with random coefficients for mean-variance insurers," *Insurance: Mathematics and Economics*, vol. 85, pp. 104–114, 2019.

[14] J. Liu, K.-F. C. Yiu, T. K. Siu, and W.-K. Ching, "Optimal investment-reinsurance with dynamic risk constraint and regime switching," *Scandinavian Actuarial Journal*, vol. 2013, no. 4, pp. 263–285, 2013.

[15] Y. Huang, X. Yang, and J. Zhou, "Optimal investment and proportional reinsurance for a jump-diffusion risk model with constrained control variables," *Journal of Computational and Applied Mathematics*, vol. 296, pp. 443–461, 2016.

[16] X. Zheng, J. Zhou, and Z. Sun, "Robust optimal portfolio and proportional reinsurance for an insurer under a CEV model," *Insurance: Mathematics and Economics*, vol. 67, pp. 77–87, 2016.

[17] D. Li, Y. Zeng, and H. Yang, "Robust optimal excess-of-loss reinsurance and investment strategy for an insurer in a model with jumps," *Scandinavian Actuarial Journal*, vol. 2018, no. 2, pp. 145–171, 2018.

[18] X. Zhang and X. Zheng, "Optimal investment-reinsurance policy with stochastic interest and inflation rates," *Mathematical Problems in Engineering*, vol. 2019, Article ID 5176172, 14 pages, 2019.

[19] Y. Fang and Z. Qu, "Optimal combination of quota-share and stop-loss reinsurance treaties under the joint survival probability," *IMA Journal of Management Mathematics*, vol. 25, no. 1, pp. 89–103, 2014.

[20] J. Cai, Y. Fang, Z. Li, and G. E. Willmot, "Optimal reciprocal reinsurance treaties under the joint survival probability and

the joint profitable probability," *Journal of Risk and Insurance*, vol. 80, no. 1, pp. 145–168, 2013.

[21] X. Zeng and S. Luo, "Stochastic Pareto-optimal reinsurance policies," *Insurance: Mathematics and Economics*, vol. 53, no. 3, pp. 671–677, 2013.

[22] Y. Huang, Y. Ouyang, L. Tang, and J. Zhou, "Robust optimal investment and reinsurance problem for the product of the insurer's and the reinsurer's utilities," *Journal of Computational and Applied Mathematics*, vol. 344, pp. 532–552, 2018.

[23] D. Li, X. Rong, and H. Zhao, "Optimal reinsurance and investment problem for an insurer and a reinsurer with jump-diffusion risk process under the Heston model," *Computational and Applied Mathematics*, vol. 35, no. 2, pp. 533–557, 2016.

[24] D. Li, X. Rong, and H. Zhao, "The optimal investment problem for an insurer and a reinsurer under the constant elasticity of variance model," *IMA Journal of Management Mathematics*, vol. 27, no. 2, pp. 255–280, 2016.

[25] H. Zhao, C. Weng, Y. Shen, and Y. Zeng, "Time-consistent investment-reinsurance strategies towards joint interests of the insurer and the reinsurer under CEV models," *Science China Mathematics*, vol. 60, no. 2, pp. 317–344, 2017.

[26] Z. Liang and K. C. Yuen, "Optimal dynamic reinsurance with dependent risks: variance premium principle," *Scandinavian Actuarial Journal*, vol. 2016, no. 1, pp. 18–36, 2016.

[27] M. Taksar and X. Zeng, "Optimal non-proportional reinsurance control and stochastic differential games," *Insurance: Mathematics and Economics*, vol. 48, no. 1, pp. 64–71, 2011.

[28] W. Fleming and H. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, Berlin, Germany, 2nd edition, 2006.

[29] H. Gerber, *An Introduction to Mathematical Risk Theory, Series No. 8 of S.S. Huebner Foundation Monograph*, R. D. Irwin Inc., Homeward, IL, USA, 1979.

[30] X. Lin and P. Yang, "Optimal investment and reinsurance in a jump diffusion risk model," *The ANZIAM Journal*, vol. 52, no. 3, pp. 250–262, 2011.

[31] Y. Wen, "Optimal investment-reinsurance strategies for insurers with mean-reversion and mispricing under variance premium principle," *Applied Mathematics*, vol. 9, no. 7, pp. 806–820, 2018.