Research Article

Existence of Solutions for Nonlinear Impulsive Fractional Differential Equations via Common Fixed-Point Techniques in Complex Valued Fuzzy Metric Spaces

Humaira, Muhammad Sarwar, and Thabet Abdeljawad

1Department of Mathematics, University of Malakand Dir Lower, Chakdara, Khyber Pakhtunkhwa, Pakistan
2Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia
3Department of Medical Research, China Medical University, Taichung 40402, Taiwan
4Department of Computer Science and Information Engineering, Asia University, Taichung 40402, Taiwan

Correspondence should be addressed to Muhammad Sarwar; sarwarswati@gmail.com and Thabet Abdeljawad; tabdeljawad@psu.edu.sa

Received 22 May 2020; Accepted 14 August 2020; Published 21 October 2020

1. Introduction and Preliminaries

Zadeh initiated the concept of fuzzy sets in 1965 [1], which introduced a deep research activity leading to the improvement of attractive theory of fuzzy system. Afterwards, several researchers have contributed towards some basic significant results in fuzzy sets.

The notion of fuzzy metric was established by Kramosil and Michalek [2]. They generalized the concepts of probabilistic metric spaces to the fuzzy situation. George and Veeramani [3] amended the notion of fuzzy metric to derive a Hausdorff topology initiated by fuzzy metric. This obtained a milestone in the existence theory of fixed point in fuzzy metric spaces. Afterwards, a number of different generalizations appeared for the existence theory of fixed point in fuzzy metric. Garbice [4] established the fuzzy version of Banach contraction principle in fuzzy metric spaces. For some necessary definitions, examples, and basic results, we refer to [5–7] and the references herein.

Fixed-point theory has a broad set of applications in modern mathematics. Banach contraction principle is the most basic and widely used technique in mathematical analysis. Due to the constructive nature of Banach contraction principle, it is the most useful tool to solve several existence problems in mathematics. Several generalizations of the Banach contraction technique are made in many directions satisfying different types of contractive conditions and having many applications in mathematical disciplines [8, 9].

In the meanwhile, researchers realized that due to vector division, rational type contraction is not meaningful in cone metric spaces. Thus, many results cannot be extended to cone metric spaces. To overcome this problem, very recently, in 2011, Azam et al. [10] initiated a new setting of metric fixed-point theory which is known as complex valued metric spaces. Here, they considered the set of complex number instead of set of positive real numbers as a ground set endowing with a partial structure. The authors obtained fixed-point results satisfying rational contraction and discussed its applications in the said setting. Moreover, Rouzkard and Imdad [11] generalized the applications of complex valued metric spaces and wrote some beautiful remarks. Furthermore, Sintunavarat and Kumam obtained existence results of fixed point for single valued mappings.
involving control functions instead of constants in contractive condition [12].

Very recently, Shukla et al. [13] introduced an innovative concept of complex valued fuzzy metric spaces where they defined several associated topological features for complex valued fuzzy metric spaces. Moreover, they established the fuzzy version of the well-celebrated Banach contraction principle in different directions and discussed its applications.

The theory of impulsive functional differential equations is emerging as an important area of investigation since such equations appear to represent a natural framework for mathematical modeling of many real processes and phenomena studied in optimal control, electronics, economics, and so on. To further study on impulsive functional differential equations, we refer the readers to [14, 15]. The researchers used a set of different techniques to discuss the existence of solution of such models such as the homotopy perturbation method, Laplace transform method, Adomian decomposition method, and different types of approximation methods. Among such techniques, fixed-point theory is one of the main tools to investigate the analytical and numerical solution of mathematical models. Many researchers have utilized the existence results of fixed point to investigate the analytical solution of different types of differential and integral equations in different spaces. For instance, we refer to [16–24].

In our work, we extend fixed-point results under the general contractive condition in [25] to the setting of complex valued fuzzy metric spaces. Moreover, we studied a result of existence and uniqueness of the solution of non-linear impulsive fractional differential equations.

\[ \frac{C}{0}D_j^\mu g(t) = K\left( t, t_j g(t), \frac{C}{0}D_j^\mu g(t) \right), \]
\[ \Delta g(t_j) = \mathcal{F}_j(g(t)), \]
\[ \Delta g'(t_j) = \mathcal{F}_j(g(t)), \]
\[ g(0) = -g'(0), \]
\[ g(1) = -g'(1), \]

where the notation \( \frac{C}{0}D_j^\mu \) stands for Caputo fractional derivative of order \( \mu \in (1,2], J = [0,1], \) and \( K: J \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is a continuous function. Further, the nonlinear functions \( \mathcal{F}_j, \mathcal{F}_j: J \rightarrow \mathbb{R} \), are also continuous for \( j = 1, 2, \ldots, q \) and \( \Delta g(t_j) = g(t_j^+) - g(t_j^-) \) and \( \Delta g'(t_j) = g'(t_j^+) - g'(t_j^-) \), where \( g(t_j^+), g(t_j^-) \) represent the right and left-hand limit of the function \( g(t) \), respectively. Also, \( 0 = t_0 < t_1 < t_2 < \cdots < t_q < t_{q+1} = 1, q \in \mathbb{Z}^+ \), where \( \mathbb{Z}^+ \) is the set of positive integers. Finally, we gave an illustrative example to make our results strong.

Throughout the paper, we have denoted the set of complex numbers by \( \mathbb{C} \). Let \( \mathbb{P} = \{ (x, y) : 0 \leq x < \infty, 0 \leq y < \infty \} \subset \mathbb{C} \). The elements \((0,0), (1,1) \in \mathbb{P} \) are denoted by \( \theta \) and \( \ell \), respectively.

Define a partial ordering \( \prec \subset \mathbb{C} \) by \( \xi_1 \prec \xi_2 \) if \( \xi_2 - \xi_1 \in \mathbb{P} \). The relations \( \xi_1 \prec \xi_2 \) and \( \xi_1 \prec \xi_2 \) indicate that \( \text{Re}(\xi_1) \leq \text{Re}(\xi_2), \text{Im}(\xi_1) \leq \text{Im}(\xi_2) \) and \( \text{Re}(\xi_1) \leq \text{Re}(\xi_2), \text{Im}(\xi_1) \leq \text{Im}(\xi_2) \), respectively. A sequence is monotonic with respect to \( \prec \) if either \( \xi_0 \prec \xi_0+1 \) or \( \xi_0+1 \prec \xi_0 \) for all \( q \in \mathbb{N} \). Let the unit closed complex interval be denoted by \( I = \{ (x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1 \} \), the open unit complex interval be denoted by \( I_0 = \{ (x,y) : 0 < x < 1, 0 < y < 1 \} \), and the set \( \{ (x,y) : 0 < x < \infty, 0 < y < \infty \} \) be denoted by \( \mathbb{P}_B \). Clearly, for \( \xi, \xi \in \mathbb{C}, \xi < \xi \) if \( \xi - \xi \in \mathbb{P} \).

Let \( \mathbb{D} \subset \mathbb{C} \). If there exists inf\( \mathbb{D} \) such that it is the lower bound of \( \mathbb{D} \), that is inf\( \mathbb{D} < c, \forall c \in \mathbb{D} \) and \( \mathbb{D} < \text{infD} \) for every lower bound \( \mathbb{D} \) of \( \mathbb{D} \), then inf\( \mathbb{D} \) is called the greatest lower bound of \( \mathbb{D} \). In the same way, we define sup\( \mathbb{D} \), (lub) the least upper bound of \( \mathbb{D} \).

**Definition 1** (see [13]). Let \( \mathcal{E} \subset \mathcal{X} \) be a nonempty set. A complex fuzzy set \( \mathcal{M} \) is characterized by a mapping \( \mu_{\mathcal{E}}(x) \) with domain \( \mathcal{E} \) and the range in the closed unit complex interval \( I \), which assigns each element \( x \in \mathcal{E} \), a grade of membership in \( \mathcal{E} \), and is thus of the form

\[ \mu_{\mathcal{E}}(x) = r_{\mathcal{E}}(x), \]

where \( r_{\mathcal{E}}(x) \in [0,1] \). The complex fuzzy set may be written as

\[ \mathcal{M} = \{ (x, \mu_{\mathcal{E}}(x) | x \in \mathcal{X}) \}. \]

**Definition 2** (see [13]). A binary equation *: \( \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I} \) is said to be complex valued t-norm if the following conditions hold:

1. \( w_1 \ast w_2 = w_2 \ast w_1 \)
2. \( w_1 \ast w_3 \leq w_2 \ast w_4 \) whenever \( w_1 \leq w_3, w_2 \leq w_4 \)
3. \( w_1 \ast (w_2 \ast w_3) = (w_1 \ast w_2) \ast w_3 \)
4. \( w \ast \theta = w = \theta \ast w \)

for all \( w, w_1, w_2, w_3, w_4 \in \mathcal{I} \).

**Example 1**. Let \( a_\ast, b_\ast, c_\ast: \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I} \) be three binary operations defined, respectively, by

1. \( a_1 \ast a_2 = [e_1 e_2, c_1 c_2], \quad \text{for all } a_1 = (e_1, c_1), a_2 = (e_2, c_2) \in \mathcal{I} \)
2. \( a_1 \ast a_2 = [\min(e_1, e_2), \min(c_1, c_2)], \quad \text{for all } a_1 = (e_1, c_1), a_2 = (e_2, c_2) \in \mathcal{I} \)
3. \( a_1 \ast a_2 = [\max(e_1 + e_2 - 1, 0), \max(c_1 + c_2 - 1, 0)], \quad \text{for all } a_1 = (e_1, c_1), a_2 = (e_2, c_2) \in \mathcal{I} \)

Then, \( a_\ast, b_\ast, c_\ast \) are complex valued t-norms.

Indeed if \( I_R = [0,1] \) is the real unit closed interval and \( a_\ast, b_\ast, c_\ast: \mathcal{I}_R \times \mathcal{I}_R \rightarrow \mathcal{I}_R \) are two t-norms, then *: \( \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I} \) defined by
\[\alpha_1 \ast \alpha_2 = (e_1 \ast e_2, c_1 \ast c_2), \]
\[\forall \alpha_1 = (e_1, c_1), \alpha_2 = (e_2, c_2) \in I, \tag{4}\]
is a complex valued t-norm.

**Definition 3** (see [13]). Let \(Q\) be a nonempty set, \(\ast\) be a continuous complex valued t-norm, and \(M\) be a complex fuzzy set on \(Q \times Q \times P_\theta \rightarrow I\) satisfying the following conditions:

1. \[0 \times M(w, z, t) = \ell \text{ for every } t \in P_\theta \text{ if and only if } w = z\]
2. \[M(w, z, t) = M(z, w, t) \quad \forall t \in P_\theta\]
3. \[M(w, z, t) \ast M(z, y, t') < M(w, y, t + t') \quad \forall (w, z, t, y, t') \in Q \times Q \times P_\theta\]
4. \[M(w, z, t) \ast \sigma(z, \xi, t) < \ell, \quad \forall (w, z, t, \xi) \in Q \times Q \times P_\theta, \tag{5}\]

for all \(z, \xi \in P_\theta, t = (e, c)\). Then, \((Q, M, \ast)\) is said to be a complex valued fuzzy metric space and \(M\) is called a complex valued fuzzy metric on \(Q\). The functions \(M(w, z, t)\) denote the degree of nearness and the degree of non-nearness between \(w\) and \(z\) with respect to the complex parameter \(t\), respectively.

**Example 2.** Consider any metric space \((Q, \sigma)\). Define \(\ast\) by \(\alpha_1 \ast \alpha_2 = (e_1 e_2, c_1 c_2)\), for all \(\alpha_1 = (e_1, c_1), \alpha_2 = (e_2, c_2) \in I\). Let the complex fuzzy set \(M\) be defined as given by

\[M(z, \xi, t) = \frac{e_1 c_1 + \sigma(z, \xi)}{e_1 + c_1 + \sigma(z, \xi)}, \tag{6}\]

for all \(z, \xi \in P_\theta, t = (e, c)\). Then, \((Q, M, \ast)\) is a complex valued fuzzy metric space.

Indeed in above example, if \(g : P_\theta \rightarrow (0, \infty)\) is continuous and nondecreasing function, that is, \(\alpha_1 < \alpha_2\) implies that \(g(\alpha_1) < g(\alpha_2)\), then \((Q, M, \ast)\) is a complex valued fuzzy metric space, where

\[M(z, \xi, t) = \frac{g(t)}{g(t) + \alpha(z, \xi)} \ell. \tag{7}\]

Similarly, it is obvious that for the fuzzy set defined by

\[M(z, \xi, t) = \left[\exp\left(\frac{\sigma(z, \xi)}{g(t)}\right)\right]^{-1} \ell, \tag{7}\]

\((Q, M, \ast)\) is a complex valued fuzzy metric space.

**Definition 4** (see [13]). Let \((Q, M, \ast)\) be a complex valued fuzzy metric space. A sequence \(\{z_n\} \in Q\) converges to \(z \in Q\), if for each \(v \in I_0\) and \(t \in P_\theta\), there exists \(k_0 \in N\) with

\[\ell - \gamma \cdot M(z_0, z, t) < b > k_0. \tag{8}\]

**Definition 5** (see [13]). Let \((Q, M, \ast)\) be a complex valued fuzzy metric space. A sequence \(\{x_n\}\) in \(Q\) is known as a Cauchy sequence if

\[\lim_{b \to \infty} \inf_{d \to b} M(x_n, x_d, t) = \ell, \quad \forall t \in P_\theta. \tag{9}\]

The complex valued fuzzy metric space \((Q, M, \ast)\) is called complete if every Cauchy sequence is convergent in \(Q\).

**Lemma 1** (see [13]). Let \((Q, M, \ast)\) be a complex valued fuzzy metric space. If \(t, t' \in P_\theta\) and \(t < t'\), then \(M(z, \xi, t) < M(z, \xi, t')\), \(\forall z, \xi \in Q\).

**Lemma 2** (see [13]). Let \((Q, M, \ast)\) be complex valued fuzzy metric space. A sequence \(\{z_n\}\) in \(Q\) converges to \(v \in Q\) if \(\lim_{b \to \infty} M(z_n, v, t) = \ell\) holds \(\forall t \in P_\theta\).

**Remark 1** (see [13]). Let \(z_0 \in P_\theta \forall n \in N\).

(a) If the sequence \(\{z_n\}\) is monotonic with respect to \(\prec\) and there exist \(\gamma, \eta \in P\) with \(\gamma < z_b < \eta, \forall b \in N\), then there exists \(z \in P\) such that \(\lim_{b \to \infty} z_b = z\).
(b) Although the partial ordering \(\prec\) is not a linear order on \(C\), the pair \((C, \prec)\) is a lattice.
(c) If \(Q \subset C\) and there exists \(\gamma, \eta \in C\) with \(\gamma < s < \eta, \forall s \in Q\), then \(\inf Q\) and \(\sup Q\) both exist.

**Remark 2** (see [13]). Let \(x_0, x_0, w \in P, \forall b \in N\).

(a) If \(x_0 \prec x_0, \forall b \in N\) and \(\lim_{b \to \infty} x_b = \ell\), then \(\lim_{b \to \infty} x_0 = \ell\).
(b) If \(x_0 < w, \forall b \in N\) and \(\lim_{b \to \infty} x_b = x\), then \(x < w\).
(c) If \(w < x, \forall b \in N\) and \(\lim_{b \to \infty} x_b = x\), then \(w < x\).

### 2. Common Fixed-Point Results

**Theorem 1.** Let \((Q, M, \ast)\) be a complex valued fuzzy metric space and let \(\varphi, \Xi : Q \rightarrow Q\) be self-mappings, such that

\[\ell(\ell - M(\varphi z, \Xi z, t)) < a(\ell - \Omega(z, \xi, t)) \tag{10}\]

for all \(z \in Q\), where \(a \in (0, 1)\).

**Proof.** Let \(z_0 \in Q\). Define a sequence \(\{z_b\}\) in \(Q\) by

\[\Omega(z, \xi, t) = \max\left\{M(z, \xi, t), \frac{M(z, \varphi z, t)M(\Xi z, \xi, t)M(\varphi z, \xi, t)M(z, \Xi z, t)}{\ell + M(z, \xi, t)}, \frac{M(z, \varphi z, t)M(\Xi z, \xi, t)}{\ell + M(z, \xi, t)}\right\}. \tag{11}\]
\[ z_{2b+1} = Rz_{2b}, \]
\[ z_{2b+2} = z_{2b+1}, \quad \text{for all } b \in \{0, 1, 2, \ldots \}. \tag{12} \]

By using (10), we have

\[
(\ell - M(z_{2b+1}, z_{2b+2}, t)) = (\ell - M(Rz_{2b}, z_{2b+1}, t))
\]
\[
< \alpha \left( \ell - \max \left\{ M(z_{2b}, z_{2b+1}, t), \frac{M(z_{2b}, z_{2b+1}, t) M(z_{2b+1}, z_{2b+2}, t)}{\ell + M(z_{2b}, z_{2b+1}, t)} \right\} \right)
\]
\[
< \alpha \left( \ell - \max \left\{ M(z_{2b}, z_{2b+1}, t), \frac{M(z_{2b}, z_{2b+1}, t) M(z_{2b+1}, z_{2b+2}, t)}{\ell + M(z_{2b}, z_{2b+1}, t)} \right\} \right)
\]
\[
< \alpha \left( \ell - \max \left\{ M(z_{2b}, z_{2b+1}, t), M(z_{2b+1}, z_{2b+2}, t), M(z_{2b}, z_{2b+2}, t) \right\} \right)
\]
\[
< \ell - \max \left\{ M(z_{2b}, z_{2b+1}, t), M(z_{2b+1}, z_{2b+2}, t), M(z_{2b}, z_{2b+2}, t) \right\}, \tag{13}\]

which yields

\[
M(z_{2b+1}, z_{2b+2}, t) \geq \max \{M(z_{2b}, z_{2b+1}, t), M(z_{2b+1}, z_{2b+2}, t), M(z_{2b}, z_{2b+2}, t)\}. \tag{14}\]

By Lemma 1, this leads to a contradiction; therefore, let

\[
\max \{M(z_{2b}, z_{2b+1}, t), M(z_{2b+1}, z_{2b+2}, t), M(z_{2b}, z_{2b+2}, t)\} = M(z_{2b}, z_{2b+2}, t)
\]
\[
\Rightarrow M \left( z_{2b}, z_{2b+1}, t^2 \right) \cdot M \left( z_{2b+1}, z_{2b+2}, t^2 \right). \tag{15}\]

max\{M(z_{2b}, z_{2b+1}, t), M(z_{2b+1}, z_{2b+2}, t), M(z_{2b}, z_{2b+2}, t)\} = M(z_{2b}, z_{2b+2}, t), \tag{16}\]

which implies that

\[
M(z_{2b+1}, z_{2b+2}, t) \geq M(z_{2b}, z_{2b+1}, t), \quad \text{for all } b = 0, 1, 2, \ldots \tag{17}\]

Let \( M(z_{2b}, z_{2b+1}, t) = \rho_b \); we shall show that \( \rho_b = \ell \).

Therefore, by definition, we have

\[
(1 - \alpha)\ell + \alpha \max \{M(z_{2b}, z_{2b+1}, t), M(z_{2b+1}, z_{2b+2}, t), M(z_{2b}, z_{2b+2}, t)\} \cdot (1 - \alpha)\ell + \alpha M(z_{2b}, z_{2b+1}, t) < M(z_{2b+1}, z_{2b+2}, t) \tag{20}\]

By using (19), we obtain

\[
(1 - \alpha)\ell \times (1 - \alpha)\ell'. \tag{21}\]

Since \( \alpha \in [0, 1) \) and utilizing Remark 2, we must obtain

\[
\ell' = \ell. \tag{22}\]
To show that \([z_b]_0\) is a Cauchy sequence, define
\[
O_b = \{M(z_0, z_d, t) : d > b\} \subset \mathcal{I},
\] (23)
for \(b \in \{0, 1, 2, \ldots \}\) and fixed \(t \in P_0\). Since \(\theta < (z_b, q_d, t) < \ell\) for all \(b \in \{0, 1, 2, \ldots \}\), using Remark 1, we obtain that for all \(b \in 0, 1, 2, \ldots \) the infimum exists. For \(d > b\), by (10), we have now for each positive integer \(d\),

\[
M(z_0, z_{d+1}, t) \geq M(z_0, z_{d+1}, t) \cdot M(z_{d+1}, z_{d+2}, t) \cdots M(z_{d+1}, z_{d+1} - t).\] (24)

It follows that
\[
\lim_{b \to \infty} \inf_{b+d > b} M(z_0, z_{d+1}, t) = \ell \ast \ell \ast \cdots \ast \ell = \ell.
\] (25)

Therefore,
\[
\lim_{b \to \infty} \inf_{b+d > b} M(z_0, z_{d+1}, t) = \ell, \quad \text{for all } t \in P_0.
\] (26)

Therefore, from (26), we have showed that \([z_b]_0\) is a Cauchy sequence in \(\mathcal{I}\). Since \(\mathcal{I}\) is complete, by Lemma 2, there exists an element \(\tau \in \mathcal{I}\) such that
\[
\lim_{b \to \infty} \inf_{b+d > b} M(z_0, \tau, t) = \ell, \quad \text{for all } t \in P_0.
\] (27)

For \(b \in \mathbb{R}\) and for any \(t \in P_0\), we obtain from (10) that

\[
\ell - M\left(\tau, \frac{t}{2}\right) < a\left(\ell - \max\left\{M\left(\tau, z_{2b+1}, \frac{t}{2}\right), \frac{M\left(\tau, z_{2b+1}, \frac{t}{2}\right) + \ell + M\left(\tau, z_{2b+1}, \frac{t}{2}\right)}{2}\right\}\right)
\]

which implies that
\[
M\left(\tau, z_{2b+1}, \frac{t}{2}\right) > \max\left\{M\left(\tau, z_{2b+1}, \frac{t}{2}\right), \frac{M\left(\tau, z_{2b+1}, \frac{t}{2}\right) + \ell + M\left(\tau, z_{2b+1}, \frac{t}{2}\right)}{2}\right\}.
\] (29)

Now, for any \(t \in P_0\),

\[
M\left(\tau, z_{2b+2}, \frac{t}{2}\right) \geq M\left(\tau, z_{2b+2}, \frac{t}{2}\right) \cdot M\left(\tau, z_{2b+2}, \frac{t}{2}\right) \geq M\left(\tau, z_{2b+2}, \frac{t}{2}\right) \cdot M\left(\tau, z_{2b+2}, \frac{t}{2}\right) \cdot M\left(\tau, z_{2b+2}, \frac{t}{2}\right) \cdot M\left(\tau, z_{2b+2}, \frac{t}{2}\right) \cdot M\left(\tau, z_{2b+2}, \frac{t}{2}\right).
\] (30)

By taking limit as \(b \to \infty\) and using Remark 2 and (27), we have

\[
M(\tau, \tau, t) \geq \ell \ast \ell \ast \ell = \ell.
\] (31)
Thus, we obtain that $M(\tau, \tau, t) = \ell$ for all $t \in P_0$, that is, $\tau = \tau$. Similarly, it follows that $M(\tau, \tau, t) = \ell$ and so $\tau = \tau$. Hence, the pair $(\tau, \tau)$ has a common fixed point. Assume that $\nu$ is any other common fixed point of $(\tau, \Xi)$ and there exists $t \in P_0$ with $M(\nu, \tau, t) \neq \ell$; then,

\[
\ell - M(\tau, \nu, t) = \ell - M(\tau, \nu, \Xi, t)
\]

\[
< a \left( \ell - \max \left\{ M(\tau, \nu, t), \frac{M(\tau, \tau, t) M(\nu, \Xi, t) M(\nu, \tau, t)}{\ell + M(\tau, \nu, t)}, \frac{M(\tau, \Xi, t) M(\nu, \tau, t)}{\ell + M(\tau, \nu, t)} \right\} \right) \quad (32)
\]

This leads to a contradiction. Therefore, $M(\tau, \nu, t) = \ell$ for all $t \in P_0$, that is, $\tau = \nu$.

**Theorem 2.** Let $(\Theta, M, \star)$ be a complete complex valued fuzzy metric space with $\lim_{(\xi, t) \to \Theta} \inf M(\xi, \xi, t) = \ell$ for all $\xi \in \Theta$.

\[
\Omega(z, \xi, t) = \max \left\{ M(z, \xi, t), \frac{M(z, \tau, t) M(\xi, \Xi, t) M(z, \tau, t)}{\ell + M(z, \xi, t)}, \frac{M(\xi, \tau, t) M(z, \tau, t)}{\ell + M(z, \xi, t)} \right\}.
\]

Then, the pair of mappings $(\tau, \Xi)$ has a unique common fixed point.

**Proof.** Let $z_0$ be any arbitrary point in $\Theta$. Define a sequence $\{z_n\}$ in $\Theta$ by

\[
M(z_{2n+1}, z_{2n+2}, \alpha t) = M(\tau, \Xi, z_{2n+1}, \alpha t)
\]

\[
\geq \max \left\{ M(z_{2n}, z_{2n+1}, t), \frac{M(z_{2n}, \tau, t) M(z_{2n+1}, \Xi, t) M(z_{2n+1}, \tau, t)}{\ell + M(z_{2n}, z_{2n+1}, t)}, \frac{M(z_{2n+1}, \tau, t) M(z_{2n+1}, \Xi, t) M(z_{2n+1}, \tau, t)}{\ell + M(z_{2n}, z_{2n+1}, t)} \right\}
\]

\[
\geq \max \left\{ M(z_{2n}, z_{2n+1}, t), \frac{M(z_{2n}, z_{2n+1}, t) M(z_{2n+1}, z_{2n+2}, t) M(z_{2n+2}, z_{2n+1}, t)}{\ell + M(z_{2n}, z_{2n+1}, t)}, \frac{M(z_{2n+1}, z_{2n+2}, t) M(z_{2n+2}, z_{2n+1}, t)}{\ell + M(z_{2n}, z_{2n+1}, t)} \right\}
\]

\[
\geq \max \left\{ M(z_{2n}, z_{2n+1}, t), M(z_{2n+1}, z_{2n+2}, t), M(z_{2n}, z_{2n+2}, t) \right\}.
\]

Now suppose that if $\max M(z_{2n+1}, z_{2n+2}, t), M(z_{2n+1}, z_{2n+2}, t)) = M(z_{2n}, z_{2n+2}, t)$, then using (36), we have

\[
M(z_{2n+1}, z_{2n+2}, t) \geq M(z_{2n}, z_{2n+2}, t)
\]

\[
\geq M(z_{2n}, z_{2n+1}, (1 - \alpha)t) \quad (37)
\]

By Lemma 1, this leads to a contradiction; therefore, let

\[
\max M(z_{2n}, z_{2n+1}, t), M(z_{2n+1}, z_{2n+2}, t), M(z_{2n}, z_{2n+2}, t) = M(z_{2n}, z_{2n+1}, t),
\]

and this implies that

\[
z \in Q, \quad \text{where } t_q \text{ is a sequence in } P_0. \quad \text{Let } \Theta, \Xi : \Theta \to \Theta \text{ be self-mappings, such that}
\]

\[
M(\tau, \Xi, \xi, \alpha t) \geq \Omega(z, \xi, t),
\]

for all $z \in \Theta$, where $\alpha \in (0, 1)$.

**Case 1.** When $z_{2n+1} \neq \tau$, for $n = 0, 1, 2, \ldots$ by using (33) with $z = z_{2n}$ and $\xi = z_{2n+1}$, we have

\[
z_{2n+1} = \tau z_{2n} \text{ and } z_{2n+2} = \tau z_{2n+1}, \quad \text{for all } n \in \{0, 1, 2, \ldots\}.
\]

By Lemma 1, this leads to a contradiction; therefore, let

\[
\max M(z_{2n}, z_{2n+1}, t), M(z_{2n+1}, z_{2n+2}, t), M(z_{2n}, z_{2n+2}, t) = M(z_{2n}, z_{2n+1}, t),
\]

and this implies that
\[ M(z_{2b+1}, z_{2b+2}, at) \geq M(z_{2b}, z_{2b+1}, t), \quad \text{for all } b = 0, 1, 2, \ldots \]  
(39)

To show that \( \{z_b\} \) is a Cauchy sequence, define

\[ O_b = \{M(z_b, z_{b+d}, t) : d > b\} \subseteq I, \quad \text{(40)} \]

for \( b \in \{0, 1, 2, \ldots \} \) and fixed \( t \in P_b \). Since \( \theta(z_b, z_{d}, t) < \ell \) for all \( b \in \{0, 1, 2, \ldots \} \), using Remark 1, we obtain that for all \( b \in 0, 1, 2, \ldots \) the infimum, \( \inf O_b = \rho_b \) (say), exists. For \( t \in P_b \) and \( b, d \in \mathbb{N} \) with \( d > b \), we obtain the following from (39) and Lemma 1:

\[ M(z_{2b+1}, z_{2d+2}, at) \geq M(z_{2b}, z_{2d+1}, t) \geq M(z_{2b}, z_{2d+1}, at), \]  
(41)

which yields

\[ M(z_{2b+1}, z_{2d+1}, t) \geq M(z_{2b}, z_{2d+1}, t) \geq M(fz_{2b-1}, fz_{2d-1}, t) \geq M(z_{2b-1}, z_{2d-1}, t) \]
\[ \geq M(z_{2b-2}, z_{2d-2}, t) \geq \cdots \geq M(z_0, z_{2d-2b}, t) \]  
(46)

Hence, for all \( t \in P_\theta \) and \( b \in \mathbb{N} \),

\[ \rho_{2b+1} = \inf_{d > b} M(z_{2b+1}, z_{2d+1}, t) \geq \inf_{d > b} M(z_0, z_{2d-2b}, t) \]  
(47)

Since \( \lim_{b \to \infty} (t/\alpha^{2b+1}) = \infty \), by (44) and by the hypothesis, we have

\[ \ell' \geq \lim_{b \to \infty} \inf_{\ell \in \ell} M(z_{2d-2b}, \xi, \frac{t}{\alpha^{2b+1}}) = \ell'. \]  
(48)

From (41) and (48), we obtain

\[ M(\mu, f\mu, t) \geq M(\mu, z_{2b+1}, \frac{t}{2}) \geq M(z_{2b+1}, f\mu, \frac{t}{2}) \]
\[ \geq M(\mu, z_{2b+1}, t) \]  
(51)

by taking limit as \( b \to \infty \) and using Remark 2 and (50), we obtain that \( M(\mu, f\mu, t) = \ell \) for all \( t \in P_\theta \), that is, \( f\mu = \mu \). Similarly, it follows that \( M(\mu, \xi\mu, t) = \ell \), and so \( \xi\mu = \mu \).

Hence, the pair \((f, \Xi)\) has a common fixed point. Assume that \( \nu \) is any other common fixed point of \((f, \Xi)\) and there exists \( t \in P_\theta \) with \( M(\mu, \nu, t) \neq \ell \); then,
\[ M(\mu, v, t) = M(\mu, b, \Xi, v, t) \geq M(\mu, v, t) \]

\[ \geq M\left(\mu, \frac{t}{a}\right) \geq M\left(\mu, b, \Xi, v, \frac{t}{a}\right) \]  \hspace{1cm} (52)

\[ \geq M\left(\mu, \frac{t}{a}\right) \geq \cdots \geq M\left(\mu, v, \frac{t}{a}\right). \]

For all \( b \in \mathbb{N} \). Applying \( \lim_{t \to \infty} (t/a^b) = \infty \) and

\[ M(\mu, v, (t/a^b)) \geq \inf_{t \in \mathbb{Q}} M(\mu, \xi, (t/a^b)), \]

it follows that from the above inequality, we have \( M(\mu, v, t) \geq \ell \). This leads to a contradiction. Therefore, \( M(\mu, v, t) = \ell \) for all \( t \in \mathbb{P}_0 \), that is, \( \mu = v \).

**Case 2.** If \( z_{2b} = z_{2b+1} \) for all \( b \in \mathbb{N} \). It implies that the sequence \( z_b \) is constant and so convergent. The rest of the proof can be completed on the steps of Case 1. This completes the proof.

**Example 3.** Let \( \mathcal{X} = \{0\} \cup \{(1/b): b \in \mathbb{N}\} \) with the metric \( \delta \) defined by

\[ \delta(w, z) = |w - z|, \quad \text{for all } w, z \in \mathcal{X}. \]  \hspace{1cm} (53)

Define a \( t \)-norm “*” by \( a * e = \min\{a, e\} \) for any \( a, e \in \mathcal{X} \).

Let \( M \) be the complex valued fuzzy set given by

\[ M(w, z, c) = \frac{c}{c + \delta(w, z)} \ell, \]  \hspace{1cm} (54)

for \( c \in \mathbb{P}_0 \). Clearly, \((\mathcal{X}, M, *)\) is a complex valued fuzzy metric space.

Certainly, for any sequence \( \{c_b\} \in \mathbb{P}_0, c_b = (m_b, n_b) \) with \( \lim_{b \to \infty} c_b = \infty \) and for each fixed \( w \in \mathcal{X} \), we have, \( |w - z| \leq 1 \), for all \( z \in \{0, 1\} \), then

\[ \ell \geq \inf_{z \in \mathcal{X}} (w, z, c_b) \]

\[ = \inf_{z \in \mathcal{X}} \frac{m_b n_b}{m_b n_b + \|w - z\|} \ell = \inf_{z \in \mathcal{X}} \frac{1}{1 + (|w - z|/m_b n_b) \ell}. \]

\[ = \frac{1}{1 + (\sup_{z \in \mathcal{X}} |w - z|)/m_b n_b} \ell \geq \frac{1}{1 + (|w - z|/m_b n_b) \ell}. \]

Therefore,

\[ \ell \geq \lim_{b \to \infty} \inf_{z \in \mathcal{X}} (w, z, c_b) \geq \lim_{b \to \infty} \frac{1}{1 + (|w - z|/m_b n_b) \ell} \ell = \ell, \]  \hspace{1cm} (56)

so that

\[ \lim_{b \to \infty} \inf_{z \in \mathcal{X}} (w, z, c_b) = \ell. \]  \hspace{1cm} (57)

Define \( f, \Xi = (w/5) \). Note that with \( a = (2/7) \) and by routine calculation, one can easily verify that \( f, \Xi \) satisfy condition (33). Hence, all the assumptions of Theorem 2 are satisfied. Moreover, \( w = 0 \) remains fixed under \( f, \Xi \); therefore, they have a common fixed point.

**Remark 3.** In Theorem 1, the contraction condition (10) for the pair of self-mappings \( (f, \Xi) \) can be replaced by the following one, with analogous proof:

\[ (\ell - M(fz, \Xi, t)) < a(t)(\ell - \Omega(z, \xi, t)), \]  \hspace{1cm} (58)

for all \( z \in \mathcal{X} \), where \( a \) is a real valued function \( a: \mathbb{P}_0 \to [0, 1) \).

\[ \Omega(z, \xi, t) = \max\left\{ M(z, \xi, t), M(z, f\xi, t)M(\xi, \Xi, t), M(\xi, f\xi, t)M(z, \Xi, t) \right\} \]  \hspace{1cm} (59)

\[ = \frac{\ell - M(fz, \xi, t)}{\ell + M(z, \xi, t)} \leq a(\ell - \Omega(z, \xi, t)), \]  \hspace{1cm} (60)

for all \( z \in \mathcal{X} \), where \( a \in (0, 1) \).

\[ \Omega(z, \xi, t) = \max\left\{ M(z, \xi, t), M(z, f\xi, t)M(\xi, \Xi, t), M(\xi, f\xi, t)M(z, \Xi, t) \right\} \]  \hspace{1cm} (61)

Then, the mapping \( f \) has a unique fixed point.

**Proof.** The proof is immediate from Theorem 1 by putting \( f = \Xi \). \qed

**Corollary 2.** Let \((\mathcal{X}, M, *)\) be a complete complex valued fuzzy metric space and let \( f: \mathcal{X} \to \mathcal{X} \) be self-mapping, such that

\[ (\ell - M(bz, f^b \xi, t)) < a(\ell - \Omega(z, \xi, t)), \]  \hspace{1cm} (62)

for all \( z \in \mathcal{X} \), where \( a \in (0, 1) \).
Proof. From Corollary 1, we obtain that there exists \( \tau \in \Omega \) such that
\[
\Omega(z, \xi, t) = \max \left\{ M(z, \xi, t), \frac{M(z, f^b z, t)M(\xi, f^b \xi, t)}{\ell + M(z, \xi, t)} \right\}.
\] (63)

Then, the mapping \( f \) has a unique fixed point.

So,
\[
f^b \tau = \tau.
\] (64)

This yields
\[
M(f, \tau, t) = \alpha M(f, \tau, t).
\] (66)

Since \( f^b \) satisfies all the conditions of Corollary 1, \( f^b \) has a unique common fixed \( \tau \in \Omega \). But \( f^b f = f \tau \) implies that \( f \tau \) is another fixed point of \( f^b \). As common fixed point is unique, we have \( f \tau = \tau \). Since the fixed point of \( f \) is also a fixed point of \( f^b \), the fixed point of \( f \) is unique.

3. Existence Theorem for Nonlinear Impulsive Fractional Differential Equations

In this section, we present a situation where our obtained results can be applied. Precisely, we study the existence of solution for a class of nonlinear three-point implicit boundary value problems of impulsive fractional differential equations (1). This problem is equivalent to the integral equation

\[
g(t) = \begin{cases} 
\frac{1}{\Gamma(\mu)} \int_0^t (t-q)^{\mu-1} K(\varrho, g(\varrho), \int_0^\varrho D_\varrho^\mu g(\varrho)) \, d\varrho + \mathcal{G}(1-t) \\
\frac{1}{\Gamma(\mu)} \int_{t_j}^t (t-q)^{\mu-1} K(\varrho, g(\varrho), \int_0^\varrho D_\varrho^\mu g(\varrho)) \, d\varrho \\
+ \frac{1}{\Gamma(\mu)} \sum_{j=1}^r \int_{t_{j-1}}^{t_j} (t-q)^{\mu-1} K(\varrho, g(\varrho), \int_0^\varrho D_\varrho^\mu g(\varrho)) \, d\varrho \\
+ \frac{1}{\Gamma(\mu-1)} \sum_{j=1}^r (t-t_j) \int_{t_{j-1}}^{t_j} (t-j)^{\mu-2} K(\varrho, g(\varrho), \int_0^\varrho D_\varrho^\mu g(\varrho)) \, d\varrho \\
+ \sum_{j=1}^r (t-t_j) \mathcal{F}(g(t_j)) + \sum_{j=1}^r \mathcal{A}(g(t_j)) + \mathcal{G}(1-t), \quad t \in J_r, r = 1, 2, 3, \ldots, q,
\end{cases}
\] (67)

where
\[ \mathcal{G} = \frac{1}{\Gamma(\mu)} \sum_{j=1}^{q+1} \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-1} K\left(\varrho, g(\varrho), C D_{\varrho}^{\mu} g(\varrho)\right) d\varrho \]

\[ + \frac{1}{\Gamma(\mu - 1)} \sum_{j=1}^{q} (1 - t_j) \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} K\left(\varrho, g(\varrho), C D_{\varrho}^{\mu} g(\varrho)\right) d\varrho \]

\[ + \frac{1}{\Gamma(\mu - 1)} \int_{t_{j-1}}^{t_j} (\rho - \varrho)^{\mu-2} K\left(\varrho, g(\varrho), C D_{\varrho}^{\mu} g(\varrho)\right) d\varrho \]

\[ + \frac{1}{\Gamma(\mu - 1)} \sum_{j=1}^{q} \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} K\left(\varrho, g(\varrho), C D_{\varrho}^{\mu} g(\varrho)\right) d\varrho \]

\[ + \sum_{j=1}^{q} (1 - t_j) \mathfrak{A}_j(g(t_j)) + \sum_{j=1}^{q} \mathfrak{B}_j(g(t_j)) + \sum_{j=1}^{q} \mathfrak{C}_j(g(t_j)). \]

For simplicity, we use the notation \( \theta_{g}(t) = K(t, g(t), C D_{g}^{\mu} g(t)) \).

If necessary, the reader can refer to [16] for a more detailed explanation of the background of the problem.

Here, we shall prove our result by establishing the existence of a common fixed point for a pair of integral operators \( \mathfrak{A}_1, \mathfrak{A}_2 \), defined on \( C(I, \mathbb{R}) \) as

\[ \mathfrak{A}_j(g)(t) = \frac{1}{\Gamma(\mu)} \int_{t_j}^{t} (t - \varrho)^{\mu-1} \theta_{g}(\varrho) d\varrho + \frac{1}{\Gamma(\mu)} \sum_{0 \leq \varrho \leq t} \frac{1}{\Gamma(\mu)} \int_{t_{j-1}}^{t_{j}} (t_j - \varrho)^{\mu-1} \theta_{g}(\varrho) d\varrho \]

\[ + \frac{1}{\Gamma(\mu - 1)} \sum_{0 \leq \varrho \leq t} \left( t - \varrho \right) \int_{t_{j-1}}^{t_{j}} (t_j - \varrho)^{\mu-2} \theta_{g}(\varrho) d\varrho + \sum_{0 \leq \varrho \leq t} \left( t - \varrho \right) \mathfrak{B}_j(g(t_j)) \]

\[ + \sum_{0 \leq \varrho \leq t} \left( t - \varrho \right) \mathfrak{C}_j(g(t_j)) + \sum_{t_{j-1}}^{t_{j}} \int_{t_{j-1}}^{t_{j}} (t_j - \varrho)^{\mu-2} \theta_{g}(\varrho) d\varrho \]

\[ + \frac{1}{\Gamma(\mu - 1)} \int_{t_{j-1}}^{t_{j}} (\rho - \varrho)^{\mu-2} \theta_{g}(\varrho) d\varrho + \frac{1}{\Gamma(\mu - 1)} \sum_{t_{j-1}}^{t_{j}} \left( t_j - \varrho \right)^{\mu-2} \theta_{g}(\varrho) d\varrho \]

\[ + \sum_{t_{j-1}}^{t_{j}} \left( t_j - \varrho \right) \mathfrak{A}_j(g(t_j)) + \sum_{t_{j-1}}^{t_{j}} \mathfrak{B}_j(g(t_j)) + \sum_{t_{j-1}}^{t_{j}} \mathfrak{C}_j(g(t_j)), \quad t \in I_j, i = 1, 2. \]

**Theorem 3.** Assume the following hypotheses are satisfied for \( j = 1, 2, 3, \ldots, q \).

(i) \((H_1)\) The nonlinear function \( K_j : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous.

(ii) \((H_2)\) Let us have constants \( \gamma, \delta \in (0, 1) \), which satisfy

\[ |K_1(t, g, f) - K_2(t, f, f)| \leq \gamma |g(t) - f(t)| \]

\[ + \delta |g(t) - f(t)|. \]

(iii) \((H_3)\) \[ |\mathfrak{A}_j(g(t)) - \mathfrak{A}_j(f(t))| = h |g(t) - f(t)|, \] with \( h > 0 \).
Mathematical Problems in Engineering

(iv) \( (H_2) \) \( |\mathfrak{F}_j(g(t_j))g - n\mathfrak{F}_j(g(t_j))| = h^*|g(t_j) - \bar{g}(t_j)|, \) with \( h^* > 0. \)

(v) \( (H_3) \) \( \Lambda = [(\gamma/1-\delta)((2q + 3/\Gamma(q) + 1) + (3q + 1/\Gamma(q)) + q(2h + h^*)) \leq 1 \) and

\[
M(g, \bar{g}, c) = \max \left\{ M(g, \bar{g}, c), \frac{M(g, \mathfrak{F}_1 g, c)M(g, \mathfrak{F}_2 g, h, c)M(g, \mathfrak{F}_3 g, c)}{\ell + M(g, \bar{g}, c)}, \frac{M(g, \mathfrak{F}_1 g, c)M(g, \mathfrak{F}_2 g, h, c)M(g, \mathfrak{F}_3 g, c)}{\ell + M(g, \bar{g}, c)} \right\}. \tag{71}
\]

Then, the pair of nonlinear integral equations

\[
g(t) = \frac{1}{\Gamma(\mu)} \int_{t_j}^{t} (t - \varrho)^{\mu - 1} \theta_s(g) d\varrho + \frac{1}{\Gamma(\mu)} \sum_{0 \leq j, t_j} \int_{t_j}^{t_j} (t_j - \varrho)^{\mu - 1} \theta_s(g) d\varrho
\]

\[
\quad + \frac{1}{\Gamma(\mu - 1)} \sum_{0 \leq j, t_j} \int_{t_j}^{t_j} (t_j - \varrho)^{\mu - 2} \theta_s(g) d\varrho + \sum_{0 \leq j, t_j} (t_j - \varrho)^{\mu - 2} \theta_s(g(t_j))
\]

\[
\quad + \sum_{0 \leq j, t_j} (t - \varrho)\mathfrak{F}_j(g(t_j)) + (1 - t) \left[ \frac{1}{\Gamma(\mu)} \sum_{j=1}^{q+1} \int_{t_j}^{t} (t_j - \varrho)^{\mu - 1} \theta_s(g) d\varrho + \frac{1}{\Gamma(\mu - 1)} \sum_{j=1}^{q} (1 - t) \int_{t_j}^{t_j} (t_j - \varrho)^{\mu - 2} \theta_s(g) d\varrho \right]
\]

\[
\quad + \frac{1}{\Gamma(\mu - 1)} \int_{t_j}^{t} (t_j - \varrho)^{\mu - 2} \theta_s(g) d\varrho + \sum_{j=1}^{q} \mathfrak{F}_j(g(t_j)) \right] \quad t \in I, i = 1, 2. \tag{72}
\]

has a common solution in \( C(J, \mathbb{R}). \)

**Proof.** Consider \( \Theta = C(J, \mathbb{R}) \) with the metric

\[
\sigma(g, f) = \max_{t \in J} |g(t) - f(t)|. \tag{73}
\]

Define \( t \)-norm as

\[
c_1 \ast c_2 = \begin{cases} (a_1, b_1), & \text{if } (a_2, b_2) = \ell, \\ (a_2, b_2), & \text{if } (a_1, b_1) = \ell, \\ 0, & \text{otherwise}, \end{cases} \tag{74}
\]

for \( c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in \mathbb{I} \) and \( M : \Theta \times \Theta \times P_\mathbb{I} \rightarrow \mathbb{I} \) defined by

\[
M(g, f, c) = \ell - \frac{\sigma(g, f)}{1 + ab}, \tag{75}
\]

for \( c = (a, b) \in P_\mathbb{I}. \) It is obvious that \( (\Theta, M, \ast) \) is a complex valued fuzzy metric space. Let \( g, \bar{g} \in \Theta \) and \( t \in J; \) then, one has
\[
\|(\mathcal{G}_1 g) - (\mathcal{G}_2 \bar{g})\| = \frac{1}{\Gamma(\mu)} \int_{t_j}^{t} (t - \varrho)^{\mu - 1} \|\theta_{1, g}(\varrho) - \theta_{2, g}(\varrho)\| d\varrho
\]
Now,
\[
(\ell - M(\mathcal{F}_1(g), \mathcal{F}_2(g), c)) = \left( \ell - \left( \ell - \frac{\sigma(\mathcal{F}_1(g), \mathcal{F}_2(g))}{1 + ab} \ell \right) \right) \\
= \left( \frac{\sigma(\mathcal{F}_1(g), \mathcal{F}_2(g))}{1 + ab} \ell + \ell \right) < \Lambda \left( \frac{\sigma(g, g)}{1 + ab} \ell + \ell \right) \\
= \Lambda \left( \frac{\sigma(g, g)}{1 + ab} \ell < \Lambda \left( \ell - \left( \ell - \frac{\sigma(g, g)}{1 + ab} \ell \right) \right) \right) < \Lambda (\ell - M(g, g, c)) \\
= \Lambda \left( \ell - \max \left\{ M(g, g, c), \frac{M(g, g, c)M(g, \mathcal{F}_2(g), c)M(g, \mathcal{F}_2(g), c)}{1 + M(g, g, c)} \right\} \right).
\]

(79)

Thus, Theorem 1 applies to \( \mathcal{F}_1, \mathcal{F}_2 \), which have a common fixed point \( g^* \in \mathcal{F} \), that is, \( g^* \) is a common solution of (72).

As an immediate consequence of Theorem 3, in the case \( \mathcal{F}_1 = \mathcal{F}_2 \), we find that integral equation (67) has a solution in \( \mathcal{F} \), and hence the nonlinear implicit boundary value problems of impulsive fractional differential equations (1) have a solution.

4. Conclusion

By successful applications of our derived results, we have studied the existence and uniqueness of common solution to the proposed class of implicit impulsive differential equations. Further, some useful results were also obtained that ensure the generalization of some essential results from metric spaces to complex valued fuzzy metric spaces. We finally hope to study classes of fuzzy fractional differential problems in future works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All the authors contributed equally to the writing of this manuscript. All the authors read and approved the final version.

References


