Research Article

Existence of Solutions for Nonlinear Impulsive Fractional Differential Equations via Common Fixed-Point Techniques in Complex Valued Fuzzy Metric Spaces

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The main purpose of this paper is to study the existence theorem for a common solution to a class of nonlinear three-point implicit boundary value problems of impulsive fractional differential equations. In this respect, we study the fuzzy version of some essential common fixed-point results from metric spaces in the newly introduced notion of complex valued fuzzy metric spaces. Also, we provide an illustrative example to demonstrate the validity of our derived results.

1. Introduction and Preliminaries

Zadeh initiated the concept of fuzzy sets in 1965 [1], which introduced a deep research activity leading to the improvement of attractive theory of fuzzy system. Afterwards, several researchers have contributed towards some basic significant results in fuzzy sets.

The notion of fuzzy metric was established by Kramosil and Michalek [2]. They generalized the concepts of probabilistic metric spaces to the fuzzy situation. George and Veeramani [3] amended the notion of fuzzy metric to derive a Hausdorff topology initiated by fuzzy metric. This obtained a milestone in the existence theory of fixed point in fuzzy metric spaces. Afterwards, a number of different generalizations appeared for the existence theory of fixed point in fuzzy metric. Garbiec [4] established the fuzzy version of Banach contraction principle in fuzzy metric spaces. For some necessary definitions, examples, and basic results, we refer to [5–7] and the references herein.

Fixed-point theory has a broad set of applications in modern mathematics. Banach contraction principle is the most basic and widely used technique in mathematical

analysis. Due to the constructive nature of Banach contraction principle, it is the most useful tool to solve several existence problems in mathematics. Several generalizations of the Banach contraction technique are made in many directions satisfying different types of contractive conditions and having many applications in mathematical disciplines [8, 9].

In the meanwhile, researchers realized that due to vector division, rational type contraction is not meaningful in cone metric spaces. Thus, many results cannot be extended to cone metric spaces. To overcome this problem, very recently, in 2011, Azam et al. [10] initiated a new setting of metric fixed-point theory which is known as complex valued metric spaces. Here, they considered the set of complex number instead of set of positive real numbers as a ground set endowing with a partial structure. The authors obtained fixed-point results satisfying rational contraction and discussed its applications in the said setting. Moreover, Rouzkard and Imdad [11] generalized the applications of complex valued metric spaces and wrote some beautiful remarks. Furthermore, Sintunavarat and Kumam obtained existence results of fixed point for single valued mappings

involving control functions instead of constants in contractive condition [12].

Very recently, Shukla et al. [13] introduced an innovative concept of complex valued fuzzy metric spaces where they defined several associated topological features for complex valued fuzzy metric spaces. Moreover, they established the fuzzy version of the well-celebrated Banach contraction principle in different directions and discussed its applications.

The theory of impulsive functional differential equations is emerging as an important area of investigation since such equations appear to represent a natural framework for mathematical modeling of many real processes and phenomena studied in optimal control, electronics, economics, and so on. To further study on impulsive functional differential equations, we refer the readers to [14, 15]. The researchers used

$$\begin{cases} {}^C_0D_{t_j}^\mu \mathbf{g}(t) = K\left(t, \mathbf{g}(t), {}^C_0D_{t_j}^\mu \mathbf{g}(t)\right), & t \in J, t \neq t_j, j = 1, 2, \dots, q, \\ \Delta \mathbf{g}(t_j) = \mathfrak{F}_j(\mathbf{g}(t)), \\ \Delta \mathbf{g}'(t_j) = \widetilde{\mathfrak{F}}_j(\mathbf{g}(t)), \\ \mathbf{g}(0) = -\mathbf{g}'(0), \\ \mathbf{g}(1) = -\mathbf{g}'(\rho), \end{cases} \quad \begin{matrix} j = 1, 2, \dots, q, \\ \rho \in (0, 1), \rho \neq t_j, j = 1, 2, \dots, q, \end{matrix} \quad (1)$$

where the notation ${}^C_0D_{t_j}^\mu$ stands for Caputo fractional derivative of order $\mu \in (1, 2]$, $J = [0, 1]$, and $K: J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function. Further, the nonlinear functions $\mathfrak{F}_j, \widetilde{\mathfrak{F}}_j: J \rightarrow \mathbb{R}$, are also continuous for $j = 1, 2, \dots, q$ and $\Delta \mathbf{g}(t_j) = \mathbf{g}(t_j^+) - \mathbf{g}(t_j^-)$ and $\Delta \mathbf{g}'(t_j) = \mathbf{g}'(t_j^+) - \mathbf{g}'(t_j^-)$, where $\mathbf{g}(t_j^+), \mathbf{g}(t_j^-)$ represent the right and left-hand limit of the function $\mathbf{g}(t)$, respectively. Also, $0 = t_0 < t_1 < t_2 < \dots < t_q < t_{q+1} = 1, q \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. Finally, we gave an illustrative example to make our results strong.

Throughout the paper, we have denoted the set of complex numbers by $\mathbb{C}$. Let $\mathbf{P} = \{(\mathbf{x}, \mathbf{y}): 0 \leq \mathbf{x} < \infty, 0 \leq \mathbf{y} < \infty\} \subset \mathbb{C}$. The elements $(0, 0), (1, 1) \in \mathbf{P}$ are denoted by θ and ℓ , respectively.

Define a partial ordering $<$ on $\mathbb{C}$ by $\xi_1 < \xi_2$ if $\xi_2 - \xi_1 \in \mathbf{P}$. The relations $\xi_1 < \xi_2$ and $\xi_1 \leq \xi_2$ indicate that $\text{Re}(\xi_1) \leq \text{Re}(\xi_2)$, $\text{Im}(\xi_1) \leq \text{Im}(\xi_2)$ and $\text{Re}(\xi_1) \leq \text{Re}(\xi_2)$, $\text{Im}(\xi_1) \leq \text{Im}(\xi_2)$, respectively. A sequence is monotonic with respect to $<$ if either $\xi_b < \xi_{b+1}$ or $\xi_{b+1} < \xi_b, \forall q \in \mathbb{N}$. Let the unit closed complex interval be denoted by $\mathbf{I} = \{(\mathbf{z}, \mathbf{y}): 0 \leq \mathbf{z} \leq 1, 0 \leq \mathbf{y} \leq 1\}$, the open unit complex interval be denoted by $\mathbf{I}_0 = \{(\mathbf{z}, \mathbf{y}): 0 \leq \mathbf{z} < 1, 0 \leq \mathbf{y} < 1\}$, and the set $\{(\mathbf{z}, \mathbf{y}): 0 < \mathbf{z} < \infty, 0 < \mathbf{y} < \infty\}$ be denoted by $\mathbf{P}_\theta$. Clearly, for $\xi, \mathbf{z} \in \mathbb{C}, \mathbf{z} < \xi$ if $\mathbf{z} - \xi \in \mathbf{P}_\theta$.

Let $\mathbb{D} \subset \mathbb{C}$. If there exists $\inf \mathbb{D}$ such that it is the lower bound of $\mathbb{D}$, that is $\inf \mathbb{D} < c, \forall c \in \mathbb{D}$ and $\mathbf{v} < \inf \mathbb{D}$ for every lower bound $\mathbf{v}$ of $\mathbb{D}$, then $\inf \mathbb{D}$ is called the greatest lower bound of $\mathbb{D}$. In the same way, we define $\sup \mathbb{D}$, (lub) the least upper bound of $\mathbb{D}$.

Definition 1 (see [13]). Let $\mathcal{Q} \subseteq X$ be a nonempty set. A complex fuzzy set $\mathcal{M}$ is characterized by a mapping $\mu_{\mathcal{Q}}(x)$

a set of different techniques to discuss the existence of solution of such models such as the homotopy perturbation method, Laplace transform method, Adomian decomposition method, and different types of approximation methods. Among such techniques, fixed-point theory is one of the main tools to investigate the analytical and numerical solution of mathematical models. Many researchers have utilized the existence results of fixed point to investigate the analytical solution of different types of differential and integral equations in different spaces. For instance, we refer to [16–24].

In our work, we extend fixed-point results under the general contractive condition in [25] to the setting of complex valued fuzzy metric spaces. Moreover, we studied a result of existence and uniqueness of the solution of non-linear impulsive fractional differential equations.

with domain $\mathcal{Q}$ and the range in the closed unit complex interval $\mathbf{I}$, which assigns each element $x \in \mathcal{Q}$, a grade of membership in $\mathcal{Q}$, and is thus of the form

$$\mu_{\mathcal{Q}}(x) = r_{\mathcal{Q}}(x)\ell, \quad (2)$$

where $r_{\mathcal{Q}}(x) \in [0, 1]$. The complex fuzzy set may be written as

$$\mathcal{M} = \{(x, \mu_{\mathcal{Q}}(x)) \mid x \in X\}. \quad (3)$$

Definition 2 (see [13]). A binary equation $*$: $\mathbf{I} \times \mathbf{I} \rightarrow \mathbf{I}$ is said to be complex valued t -norm if the following conditions hold:

- (1) $w_1 * w_2 = w_2 * w_1$
- (2) $w_1 * w_2 < w_3 * w_4$ whenever $w_1 < w_3, w_2 < w_4$
- (3) $w_1 * (w_2 * w_3) = (w_1 * w_2) * w_3$
- (4) $w * \theta = \theta, w * \ell = w$

for all $w, w_1, w_2, w_3, w_4 \in \mathbf{I}$.

Example 1. Let $*_a, *_b, *_c: \mathbf{I} \times \mathbf{I} \rightarrow \mathbf{I}$ be three binary operations defined, respectively, by

- (1) $o_1 *_a o_2 = \{e_1 e_2, c_1 c_2\},$ for all $o_1 = (e_1, c_1), o_2 = (e_2, c_2) \in \mathbf{I}$
- (2) $o_1 *_b o_2 = \{\min\{e_1, e_2\}, \min\{c_1, c_2\}\},$ for all $o_1 = (e_1, c_1), o_2 = (e_2, c_2) \in \mathbf{I}$
- (3) $o_1 *_c o_2 = \{\max\{e_1 + e_2 - 1, 0\}, \max\{c_1 + c_2 - 1, 0\}\},$ for all $o_1 = (e_1, c_1), o_2 = (e_2, c_2) \in \mathbf{I}$

Then, $*_a, *_b, *_c$ are complex valued t -norms.

Indeed if $\mathbf{I}_R = [0, 1]$ is the real unit closed interval and $*_a, *_b: \mathbf{I}_R \times \mathbf{I}_R \rightarrow \mathbf{I}_R$ are two t -norms, then $*$: $\mathbf{I} \times \mathbf{I} \rightarrow \mathbf{I}$ defined by

$$\begin{aligned} o_1 * o_2 &= (e_1 * {}_a e_2, c_1 * {}_b c_2), \\ \text{for all } o_1 &= (e_1, c_1), o_2 = (e_2, c_2) \in \mathbf{I}, \end{aligned} \quad (4)$$

is a complex valued t -norm.

Definition 3 (see [13]). Let $\mathcal{Q}$ be a nonempty set, $*$ be a continuous complex valued t -norm, and $\mathbf{M}$ be a complex fuzzy set on $\mathcal{Q} \times \mathcal{Q} \times \mathbf{P}_\theta \longrightarrow \mathbf{I}$ satisfying the following conditions:

- (1) $0 < \mathbf{M}(w, z, t)$
- (2) $\mathbf{M}(w, z, t) = \ell$ for every $t \in \mathbf{P}_\theta$ if and only if $w = z$
- (3) $\mathbf{M}(w, z, t) = \mathbf{M}(z, w, t)$
- (4) $\mathbf{M}(w, z, t) * \mathbf{M}(z, y, t') < \mathbf{M}(w, y, t + t')$
- (5) $\mathbf{M}(w, z, *) : \mathbf{P}_\theta \longrightarrow \mathbf{I}$ is continuous for all $w, z, y \in \mathcal{Q}$ and $t, t' \in \mathbf{P}_\theta$

Then, the triplet $(\mathcal{Q}, \mathbf{M}, *)$ is said to be a complex valued fuzzy metric space and $\mathbf{M}$ is called a complex valued fuzzy metric on $\mathcal{Q}$. The functions $\mathbf{M}(w, z, t)$ denote the degree of nearness and the degree of non-nearness between w and z with respect to the complex parameter t , respectively.

Example 2. Consider any metric space $(\mathcal{Q}, \sigma)$. Define $*$ by $o_1 * o_2 = (e_1 e_2, c_1 c_2)$, for all $o_1 = (e_1, c_1), o_2 = (e_2, c_2) \in \mathbf{I}$. Let the complex fuzzy set $\mathbf{M}$ defined be given by

$$\mathbf{M}(z, \xi, t) = \frac{e_1 + c_1}{e_1 + c_1 + \sigma(z, \xi)} \ell, \quad (5)$$

for all $z, \xi \in \mathbf{P}_\theta, t = (e, c)$. Then, $(\mathcal{Q}, \mathbf{M}, *)$ is a complex valued fuzzy metric space.

Indeed in above example, if $g : \mathbf{P}_\theta \longrightarrow (0, \infty)$ is continuous and nondecreasing function, that is, $o_1 < o_2$ implies that $g(o_1) \leq g(o_2)$, then $(\mathcal{Q}, \mathbf{M}, *)$ is a complex valued fuzzy metric space, where

$$\mathbf{M}(z, \xi, t) = \frac{g(t)}{g(t) + \sigma(z, \xi)} \ell. \quad (6)$$

Similarly, it is obvious that for the fuzzy set defined by

$$\mathbf{M}(z, \xi, t) = \left[\exp \left(\frac{\sigma(z, \xi)}{g(t)} \right) \right]^{-1} \ell, \quad (7)$$

$(\mathcal{Q}, \mathbf{M}, *)$ is a complex valued fuzzy metric space.

Definition 4 (see [13]). Let $(\mathcal{Q}, \mathbf{M}, *)$ be a complex valued fuzzy metric space. A sequence $\{z_n\} \in \mathcal{Q}$ converges to $z \in \mathcal{Q}$, if for each $\nu \in \mathbf{I}_0$ and $t \in \mathbf{P}_\theta$, there exists $k_0 \in \mathbf{N}$ with

$$\ell - \nu < \mathbf{M}(z_b, z, t), \quad \forall b > k_0. \quad (8)$$

Definition 5 (see [13]). Let $(\mathcal{Q}, \mathbf{M}, *)$ be a complex valued fuzzy metric space. A sequence $\{x_b\}$ in $\mathcal{Q}$ is known as a Cauchy sequence if

$$\lim_{b \longrightarrow \infty} \inf_{d > b} \mathbf{M}(x_b, x_d, t) = \ell, \quad \forall t \in \mathbf{P}_\theta. \quad (9)$$

The complex valued fuzzy metric space $(\mathcal{Q}, \mathbf{M}, *)$ is called complete if every Cauchy sequence is convergent in $\mathcal{Q}$.

Lemma 1 (see [13]). Let $(\mathcal{Q}, \mathbf{M}, *)$ be a complex valued fuzzy metric space. If $t, t' \in \mathbf{P}_\theta$ and $t < t'$, then $\mathbf{M}(z, \xi, t) < \mathbf{M}(z, \xi, t')$, $\forall z, \xi \in \mathcal{Q}$.

Lemma 2 (see [13]). Let $(\mathcal{Q}, \mathbf{M}, *)$ be complex valued fuzzy metric space. A sequence $\{z_b\}$ in $\mathcal{Q}$ converges to $v \in \mathcal{Q}$ if $\lim_{b \longrightarrow \infty} \mathbf{M}(z_b, v, t) = \ell$ holds $\forall t \in \mathbf{P}_\theta$.

Remark 1 (see [13]). Let $z_b \in \mathbf{P} \forall n \in \mathbf{N}$.

- (a) If the sequence $\{z_b\}$ is monotonic with respect to $<$ and there exist $\gamma, \eta \in \mathbf{P}$ with $\gamma < z_b < \eta, \forall b \in \mathbf{N}$, then there exists $z \in \mathbf{P}$ such that $\lim_{b \longrightarrow \infty} z_b = z$.
- (b) Although the partial ordering $<$ is not a linear order on $\mathbb{C}$, the pair $(\mathbb{C}, <)$ is a lattice.
- (c) If $\mathcal{Q} \subset \mathbb{C}$ and there exists $\gamma, \eta \in \mathbb{C}$ with $\gamma < s < \eta, \forall s \in \mathcal{Q}$, then $\inf \mathcal{Q}$ and $\sup \mathcal{Q}$ both exist.

Remark 2 (see [13]). Let $x_b, x'_b, w \in \mathbf{P}, \forall b \in \mathbf{N}$.

- (a) If $x_b < x'_b < \ell, \forall b \in \mathbf{N}$ and $\lim_{b \longrightarrow \infty} x_b = \ell$, then $\lim_{b \longrightarrow \infty} x'_b = \ell$.
- (b) If $x_b < w, \forall b \in \mathbf{N}$ and $\lim_{b \longrightarrow \infty} x_b = x$, then $x < w$.
- (c) If $w < x_b, \forall b \in \mathbf{N}$ and $\lim_{b \longrightarrow \infty} x_b = x$, then $w < x$.

2. Common Fixed-Point Results

Theorem 1. Let $(\mathcal{Q}, \mathbf{M}, *)$ be a complete complex valued fuzzy metric space and let $\mathbb{F}, \mathbb{E} : \mathcal{Q} \longrightarrow \mathcal{Q}$ be self-mappings, such that

$$(\ell - \mathbf{M}(\mathbb{F}z, \mathbb{E}\xi, t)) < \alpha(\ell - \Omega(z, \xi, t)), \quad (10)$$

for all $z \in \mathcal{Q}$, where $\alpha \in (0, 1)$.

$$\Omega(z, \xi, t) = \max \left\{ \mathbf{M}(z, \xi, t), \frac{\mathbf{M}(z, \mathbb{F}z, t) \mathbf{M}(\xi, \mathbb{E}\xi, t)}{\ell + \mathbf{M}(z, \xi, t)}, \frac{\mathbf{M}(\xi, \mathbb{F}z, t) \mathbf{M}(z, \mathbb{E}\xi, t)}{\ell + \mathbf{M}(z, \xi, t)} \right\}. \quad (11)$$

Then, the pair of mappings $(\mathbb{F}, \mathbb{E})$ has a unique common fixed point.

Proof. Let $z_0 \in \mathcal{Q}$. Define a sequence $\{z_b\}$ in $\mathcal{Q}$ by

$$\begin{aligned} \mathbf{z}_{2b+1} &= \mathbf{F}\mathbf{z}_{2b}, \\ \mathbf{z}_{2b+2} &= \mathbf{\Xi}\mathbf{z}_{2b+1}, \quad \text{for all } b \in \{0, 1, 2, \dots\}. \end{aligned} \quad (12)$$

By using (10), we have

$$\begin{aligned} (\ell - \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t})) &= (\ell - \mathbf{M}(\mathbf{F}\mathbf{z}_{2b}, \mathbf{\Xi}\mathbf{z}_{2b+1}, \mathbf{t})) \\ &< \alpha \left(\ell - \max \left\{ \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}), \frac{\mathbf{M}(\mathbf{z}_{2b}, \mathbf{F}\mathbf{z}_{2b}, \mathbf{t})\mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{\Xi}\mathbf{z}_{2b+1}, \mathbf{t})}{\ell + \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t})}, \frac{\mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{F}\mathbf{z}_{2b}, \mathbf{t})\mathbf{M}(\mathbf{z}_{2b}, \mathbf{\Xi}\mathbf{z}_{2b+1}, \mathbf{t})}{\ell + \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t})} \right\} \right) \\ &< \alpha \left(\ell - \max \left\{ \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}), \frac{\mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t})\mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t})}{\ell + \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t})}, \frac{\mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+1}, \mathbf{t})\mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+2}, \mathbf{t})}{\ell + \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t})} \right\} \right) \\ &< \alpha (\ell - \max\{\mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+2}, \mathbf{t})\}) \\ &< \ell - \max\{\mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+2}, \mathbf{t})\}, \end{aligned} \quad (13)$$

which yields

$$\mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t}) \geq \max\{\mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+2}, \mathbf{t})\}. \quad (14)$$

If $\max\{\mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+2}, \mathbf{t})\} = \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+2}, \mathbf{t})$, then

By Lemma 1, this leads to a contradiction; therefore, let

$$\begin{aligned} \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t}) &\geq \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+2}, \mathbf{t}) \\ &\geq \mathbf{M}\left(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \frac{\mathbf{t}}{2}\right) * \mathbf{M}\left(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \frac{\mathbf{t}}{2}\right). \end{aligned} \quad (15)$$

$$\max\{\mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+2}, \mathbf{t})\} = \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+2}, \mathbf{t}), \quad (16)$$

which implies that

$$\ell \geq \{\rho_{b+1}\} \geq \{\rho_b\} \geq \theta. \quad (18)$$

$$\mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t}) \geq \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}), \quad \text{for all } b = 0, 1, 2, \dots, \quad (17)$$

Hence, $\{\rho_b\}$ is monotonic sequence in $\mathbf{P}$, and using Remark 1 and (18), there exists $\ell' \in \mathbf{P}$ such that

$$\lim_{b \rightarrow \infty} \{\rho_b\} = \ell'. \quad (19)$$

Let $\mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}) = \rho_b$; we shall show that $\rho_b = \ell$. Therefore, by definition, we have

Inequality (13) suggests that

$$\begin{aligned} (1 - \alpha)\ell + \alpha \max\{\mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t}), \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+2}, \mathbf{t})\} &< \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t}) \\ \cdot (1 - \alpha)\ell + \alpha \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}) &< \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \mathbf{t}) (1 - \alpha)\ell + \alpha \rho_b < \rho_{b+1}. \end{aligned} \quad (20)$$

By using (19), we obtain

Since $\alpha \in [0, 1)$ and utilizing Remark 2, we must obtain $\ell' = \ell$. Thus,

$$(1 - \alpha)\ell < (1 - \alpha)\ell'. \quad (21) \quad \lim_{b \rightarrow \infty} \rho_b = \ell. \quad (22)$$

To show that $\{z_b\}$ is a Cauchy sequence, define

$$O_b = \{M(z_b, z_d, t) : d > b\} \subseteq I, \quad (23)$$

for $b \in \{0, 1, 2, \dots\}$ and fixed $t \in P_\theta$. Since $\theta < (z_b, z_d, t) < \ell$ for all $b \in \{0, 1, 2, \dots\}$, using Remark 1, we obtain that for all $b \in 0, 1, 2, \dots$ the infimum exists. For $d > b$, by (10), we have now for each positive integer d ,

$$M(z_b, z_{b+d}, t) \geq M(z_b, z_{b+1}, \frac{t}{d}) * M(z_{b+1}, z_{b+2}, \frac{t}{d}) * \dots * M(z_{b+d-1}, z_{b+d}, \frac{t}{d}). \quad (24)$$

It follows that

$$\lim_{b \rightarrow \infty} \inf_{b+d > b} M(z_b, z_{b+d}, t) \geq \ell * \ell * \dots * \ell = \ell. \quad (25)$$

Therefore,

$$\lim_{b \rightarrow \infty} \inf_{b+d > b} M(z_b, z_{b+d}, t) = \ell, \quad \text{for all } t \in P_0. \quad (26)$$

Therefore, from (26), we have showed that $\{z_b\}$ is a Cauchy sequence in $\mathcal{Q}$. Since $\mathcal{Q}$ is complete, by Lemma 2, there exists an element $\tau \in \mathcal{Q}$ such that

$$\lim_{b \rightarrow \infty} \inf_{b+d > b} M(z_b, \tau, t) = \ell, \quad \text{for all } t \in P_\theta. \quad (27)$$

For $b \in \mathbf{R}$ and for any $t \in P_\theta$, we obtain from (10) that

$$\begin{aligned} & \ell - M\left(F\tau, \Xi z_{2b+1}, \frac{t}{2}\right) \\ & < \alpha \left(\ell - \max \left\{ M\left(\tau, z_{2b+1}, \frac{t}{2}\right), \frac{M(\tau, F\tau, (t/2))M(z_{2b+1}, \Xi z_{2b+1}, (t/2))}{\ell + M(\tau, z_{2b+1}, (t/2))}, \frac{M(\tau, \Xi z_{2b+1}, (t/2))M(z_{2b+1}, F\tau, (t/2))}{\ell + M(\tau, z_{2b+1}, (t/2))} \right\} \right) \\ & < \alpha \left(\ell - \max \left\{ M\left(\tau, z_{2b+1}, \frac{t}{2}\right), \frac{M(\tau, F\tau, (t/2))M(z_{2b+1}, z_{2b+2}, (t/2))}{\ell + M(\tau, z_{2b+1}, (t/2))}, \frac{M(\tau, z_{2b+2}, (t/2))M(z_{2b+1}, F\tau, (t/2))}{\ell + M(\tau, z_{2b+1}, (t/2))} \right\} \right) \\ & < \ell - \max \left\{ M\left(\tau, z_{2b+1}, \frac{t}{2}\right), \frac{M(\tau, F\tau, (t/2))M(z_{2b+1}, z_{2b+2}, (t/2))}{\ell + M(\tau, z_{2b+1}, (t/2))}, \frac{M(\tau, z_{2b+2}, (t/2))M(z_{2b+1}, F\tau, (t/2))}{\ell + M(\tau, z_{2b+1}, (t/2))} \right\}, \end{aligned} \quad (28)$$

which implies that

$$M\left(F\tau, \Xi z_{2b+1}, \frac{t}{2}\right) \geq \max \left\{ M\left(\tau, z_{2b+1}, \frac{t}{2}\right), \frac{M(\tau, F\tau, (t/2))M(z_{2b+1}, z_{2b+2}, (t/2))}{\ell + M(\tau, z_{2b+1}, (t/2))}, \frac{M(\tau, z_{2b+2}, (t/2))M(z_{2b+1}, F\tau, (t/2))}{\ell + M(\tau, z_{2b+1}, (t/2))} \right\}. \quad (29)$$

Now, for any $t \in P_\theta$,

$$\begin{aligned} M(\tau, F\tau, t) & \geq M\left(\tau, z_{2b+2}, \frac{t}{2}\right) * M\left(z_{2b+2}, F\tau, \frac{t}{2}\right) \geq M\left(\tau, z_{2b+2}, \frac{t}{2}\right) * M\left(F\tau, \Xi z_{2b+1}, \frac{t}{2}\right) \\ & \geq M\left(\tau, z_{2b+2}, \frac{t}{2}\right) * \max \left\{ M\left(\tau, z_{2b+1}, \frac{t}{2}\right), \frac{M(\tau, F\tau, (t/2))M(z_{2b+1}, z_{2b+2}, (t/2))}{\ell + M(\tau, z_{2b+1}, (t/2))}, \frac{M(\tau, z_{2b+1}, (t/2))M(z_{2b+1}, F\tau, (t/2))}{\ell + M(\tau, z_{2b+1}, (t/2))} \right\}. \end{aligned} \quad (30)$$

By taking limit as $b \rightarrow \infty$ and using Remark 2 and (27), we have

$$M(\tau, F\tau, t) \geq \ell * \max \left\{ \ell, \frac{M(\tau, F\tau, (t/2))\ell}{\ell + \ell}, \frac{\ell M(z_{2b+1}, F\tau, (t/2))}{\ell + \ell} \right\} = \ell * \ell = \ell. \quad (31)$$

Thus, we obtain that $M(\tau, F\tau, t) = \ell$ for all $t \in P_\theta$, that is, $F\tau = \tau$. Similarly, it follows that $M(\tau, E\tau, t) = \ell$ and so $E\tau = \tau$. Hence, the pair (F, E) has a common fixed point.

Assume that v is any other common fixed point of (F, E) and there exists $t \in P_\theta$ with $M(\tau, v, t) \neq \ell$; then,

$$\begin{aligned} \ell - M(\tau, v, t) &= \ell - M(F\tau, Ev, t) \\ &< \alpha \left(\ell - \max \left\{ M(\tau, v, t), \frac{M(\tau, F\tau, t)M(v, Ev, t)}{\ell + M(\tau, v, t)}, \frac{M(\tau, Ev, t)M(v, F\tau, t)}{\ell + M(\tau, v, t)} \right\} \right) \\ &< \alpha(\ell - M(\tau, v, t)). \end{aligned} \quad (32)$$

This leads to a contradiction. Therefore, $M(\tau, v, t) = \ell$ for all $t \in P_\theta$, that is, $\tau = v$. $\square$

$z \in Q$, where t_b is a sequence in P_θ . Let $F, E: Q \rightarrow Q$ be self-mappings, such that

$$M(Fz, E\xi, \alpha t) \geq \Omega(z, \xi, t), \quad (33)$$

for all $z \in Q$, where $\alpha \in (0, 1)$.

Theorem 2. Let $(Q, M, *)$ be a complete complex valued fuzzy metric space with $\lim_{b \rightarrow \infty} \inf_{\xi \in Q} M(z, \xi, t_b) = \ell$ for all

$$\Omega(z, \xi, t) = \max \left\{ M(z, \xi, t), \frac{M(z, Fz, t)M(\xi, E\xi, t)}{\ell + M(z, \xi, t)}, \frac{M(\xi, Fz, t)M(z, E\xi, t)}{\ell + M(z, \xi, t)} \right\}. \quad (34)$$

Then, the pair of mappings (F, E) has a unique common fixed point.

$$z_{2b+1} = Fz_{2b} \text{ and } z_{2b+2} = Ez_{2b+1}, \text{ for all } b \in \{0, 1, 2, \dots\}. \quad (35)$$

Proof. Let z_0 be any arbitrary point in Q . Define a sequence $\{z_b\}$ in Q by

Case 1. When $z_{2b+1} \neq Fz_{2b}$, for $n = 0, 1, 2, \dots$ by using (33) with $z = z_{2b}$ and $\xi = z_{2b+1}$, we have

$$\begin{aligned} M(z_{2b+1}, z_{2b+2}, \alpha t) &= M(Fz_{2b}, Ez_{2b+1}, \alpha t) \\ &\geq \max \left\{ M(z_{2b}, z_{2b+1}, t), \frac{M(z_{2b}, Fz_{2b}, t)M(z_{2b+1}, Ez_{2b+1}, t)}{\ell + M(z_{2b}, z_{2b+1}, t)}, \frac{M(z_{2b+1}, Fz_{2b}, t)M(z_{2b}, Ez_{2b+1}, t)}{\ell + M(z_{2b}, z_{2b+1}, t)} \right\} \\ &\geq \max \left\{ M(z_{2b}, z_{2b+1}, t), \frac{M(z_{2b}, z_{2b+1}, t)M(z_{2b+1}, z_{2b+2}, t)}{\ell + M(z_{2b}, z_{2b+1}, t)}, \frac{M(z_{2b+1}, z_{2b+1}, t)M(z_{2b}, z_{2b+2}, t)}{\ell + M(z_{2b}, z_{2b+1}, t)} \right\} \\ &\geq \max \{ M(z_{2b}, z_{2b+1}, t), M(z_{2b+1}, z_{2b+2}, t), M(z_{2b}, z_{2b+2}, t) \}. \end{aligned} \quad (36)$$

Now suppose that if $\max \{ M(z_{2b}, z_{2b+1}, t), M(z_{2b+1}, z_{2b+2}, t), M(z_{2b}, z_{2b+2}, t) \} = M(z_{2b}, z_{2b+2}, t)$, then using (36), we have

$$\begin{aligned} M(z_{2b+1}, z_{2b+2}, \alpha t) &\geq M(z_{2b}, z_{2b+2}, t) \\ &\geq M(z_{2b}, z_{2b+1}, (1-\alpha)t) \\ &\quad * M(z_{2b+1}, z_{2b+2}, \alpha t). \end{aligned} \quad (37)$$

By Lemma 1, this leads to a contradiction; therefore, let

$$\max \{ M(z_{2b}, z_{2b+1}, t), M(z_{2b+1}, z_{2b+2}, t), M(z_{2b}, z_{2b+2}, t) \} = M(z_{2b}, z_{2b+1}, t), \quad (38)$$

and this implies that

$$\mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, \alpha \mathbf{t}) \geq \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2b+1}, \mathbf{t}), \quad \text{for all } b = 0, 1, 2, \dots \quad (39)$$

To show that $\{\mathbf{z}_n\}$ is a Cauchy sequence, define

$$O_b = \{\mathbf{M}(\mathbf{z}_b, \mathbf{z}_d, \mathbf{t}) : d > b\} \subseteq \mathbf{I}, \quad (40)$$

for $b \in \{0, 1, 2, \dots\}$ and fixed $\mathbf{t} \in \mathbf{P}_\theta$. Since $\theta < \mathbf{M}(\mathbf{z}_b, \mathbf{z}_d, \mathbf{t}) < \ell$ for all $b \in \{0, 1, 2, \dots\}$, using Remark 1, we obtain that for all $b \in \{0, 1, 2, \dots\}$ the infimum, $\inf O_b = \rho_b$ (say), exists. For $\mathbf{t} \in \mathbf{P}_\theta$ and $b, d \in \mathbf{N}$ with $d > b$, we obtain the following from (39) and Lemma 1:

$$\mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2d+2}, \alpha \mathbf{t}) \geq \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2d+1}, \mathbf{t}) \geq \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2d+1}, \alpha \mathbf{t}), \quad (41)$$

which yields

$$\mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2d+2}, \alpha \mathbf{t}) \geq \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2d+1}, \alpha \mathbf{t}). \quad (42)$$

Therefore, by definition, we have

$$\ell \geq \rho_{b+1} \geq \rho_b \geq \theta. \quad (43)$$

Hence, $\{\rho_b\}$ is monotonic sequence in $\mathbf{P}$, and using Remark 1 and (43), there exists $\ell' \in \mathbf{P}$ such that

$$\lim_{b \rightarrow \infty} \rho_b = \ell'. \quad (44)$$

Again from (39), we have the following for $\mathbf{t} \in \mathbf{P}$ and $b \in \mathbf{N}$:

$$\rho_{2b+1} = \inf_{d>b} \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2d+1}, \mathbf{t}) \geq \inf_{d>b} \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2d}, \frac{\mathbf{t}}{\alpha}). \quad (45)$$

Similarly, we get the following for $d > b$:

$$\begin{aligned} \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2d+1}, \mathbf{t}) &\geq \mathbf{M}(\mathbf{z}_{2b}, \mathbf{z}_{2d}, \frac{\mathbf{t}}{\alpha}) \geq \mathbf{M}(\mathbb{F}\mathbf{z}_{2b-1}, \mathbb{E}\mathbf{z}_{2d-1}, \frac{\mathbf{t}}{\alpha}) \geq \mathbf{M}(\mathbf{z}_{2b-1}, \mathbf{z}_{2d-1}, \frac{\mathbf{t}}{\alpha^2}) \\ &\geq \mathbf{M}(\mathbb{F}\mathbf{z}_{2b-2}, \mathbb{E}\mathbf{z}_{2d-2}, \frac{\mathbf{t}}{\alpha^2}) \geq \mathbf{M}(\mathbf{z}_{2b-2}, \mathbf{z}_{2d-2}, \frac{\mathbf{t}}{\alpha^3}) \geq \dots \geq \mathbf{M}(\mathbf{z}_0, \mathbf{z}_{2d-2b}, \frac{\mathbf{t}}{\alpha^{2b+1}}). \end{aligned} \quad (46)$$

Hence, for all $\mathbf{t} \in \mathbf{P}_\theta$ and $b \in \mathbf{N}$,

$$\rho_{2b+1} = \inf_{d>b} \mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2d+1}, \mathbf{t}) \geq \inf_{d>b} \mathbf{M}(\mathbf{z}_0, \mathbf{z}_{2d-2b}, \frac{\mathbf{t}}{\alpha^{2b+1}}) \geq \inf_{\xi \in \mathcal{Q}} \mathbf{M}(\mathbf{z}_0, \xi, \frac{\mathbf{t}}{\alpha^{2b+1}}). \quad (47)$$

Since $\lim_{b \rightarrow \infty} (\mathbf{t}/\alpha^{2b+1}) = \infty$, by (44) and by the hypothesis, we have

$$\ell_1 \geq \lim_{b \rightarrow \infty} \inf_{\xi \in \mathcal{Q}} \mathbf{M}(\mathbf{z}_0, \xi, \frac{\mathbf{t}}{\alpha^{2b+1}}) = \ell. \quad (48)$$

From (41) and (48), we obtain

$$\lim_{b \rightarrow \infty} \rho_b = \ell. \quad (49)$$

Hence, $\mathbf{z}_b$ is a Cauchy sequence in $\mathcal{Q}$. Since $\mathcal{Q}$ is complete and using Lemma 2, there exists $\mu \in \mathcal{Q}$ such that

$$\lim_{b \rightarrow \infty} \mathbf{M}(\mathbf{z}_b, \mu, \mathbf{t}) = \ell, \quad \text{for all } \mathbf{t} \in \mathbf{P}_\theta. \quad (50)$$

For any $\mathbf{t} \in \mathcal{Q}$, it follows from (33) that

$$\begin{aligned} \mathbf{M}(\mu, \mathbb{F}\mu, \mathbf{t}) &\geq \mathbf{M}(\mu, \mathbf{z}_{2b+2}, \frac{\mathbf{t}}{2}) * \mathbf{M}(\mathbf{z}_{2b+2}, \mathbb{F}\mu, \frac{\mathbf{t}}{2}) \\ &\geq \mathbf{M}(\mu, \mathbf{z}_{2b+2}, \frac{\mathbf{t}}{2}) * \mathbf{M}(\mathbb{F}\mu, \mathbb{E}\mathbf{z}_{2b+1}, \frac{\mathbf{t}}{2}) \\ &\geq \mathbf{M}(\mu, \mathbf{z}_{2b+2}, \frac{\mathbf{t}}{2}) * \max \left\{ \mathbf{M}(\mu, \mathbf{z}_{2b+1}, \frac{\mathbf{t}}{2}), \frac{\mathbf{M}(\mu, \mathbb{F}\mu, (\mathbf{t}/2))\mathbf{M}(\mathbf{z}_{2b+1}, \mathbf{z}_{2b+2}, (\mathbf{t}/2))}{\ell + \mathbf{M}(\mu, \mathbf{z}_{2b+1}, (\mathbf{t}/2))}, \frac{\mathbf{M}(\mu, \mathbf{z}_{2b+2}, (\mathbf{t}/2))\mathbf{M}(\mathbf{z}_{2b+1}, \mathbb{F}\mu, (\mathbf{t}/2))}{\ell + \mathbf{M}(\mu, \mathbf{z}_{2b+1}, (\mathbf{t}/2))} \right\}. \end{aligned} \quad (51)$$

By taking limit as $b \rightarrow \infty$ and using Remark 2 and (50), we obtain that $\mathbf{M}(\mu, \mathbb{F}\mu, \mathbf{t}) = \ell$ for all $\mathbf{t} \in \mathbf{P}_\theta$, that is, $\mathbb{F}\mu = \mu$. Similarly, it follows that $\mathbf{M}(\mu, \mathbb{E}\mu, \mathbf{t}) = \ell$, and so $\mathbb{E}\mu = \mu$.

Hence, the pair $(\mathbb{F}, \mathbb{E})$ has a common fixed point. Assume that v is any other common fixed point of $(\mathbb{F}, \mathbb{E})$ and there exists $\mathbf{t} \in \mathbf{P}_\theta$ with $\mathbf{M}(\mu, v, \mathbf{t}) \neq \ell$; then,

$$\begin{aligned}
\mathbf{M}(\mu, v, \mathbf{t}) &= \mathbf{M}(\mathbb{F}\mu, \Xi v, \mathbf{t}) \geq \mathbf{M}(\mu, v, \mathbf{t}) \\
&\geq \mathbf{M}\left(\mu, v, \frac{\mathbf{t}}{\alpha}\right) \geq \mathbf{M}\left(\mathbb{F}\mu, \Xi v, \frac{\mathbf{t}}{\alpha}\right) \\
&\geq \mathbf{M}\left(\mu, v, \frac{\mathbf{t}}{\alpha^2}\right) \geq \cdots \geq \mathbf{M}\left(\mu, v, \frac{\mathbf{t}}{\alpha^b}\right),
\end{aligned} \tag{52}$$

for all $b \in \mathbb{N}$. Applying $\lim_{b \rightarrow \infty} (\mathbf{t}/\alpha^b) = \infty$ and $\mathbf{M}(\mu, v, (\mathbf{t}/\alpha^b)) \geq \inf_{\xi \in Q} \mathbf{M}(\mu, \xi, (\mathbf{t}/\alpha^b))$, it follows that from the above inequality, we have $\mathbf{M}(\mu, v, \mathbf{t}) \geq \ell$. This leads to a contradiction. Therefore, $\mathbf{M}(\mu, v, \mathbf{t}) = \ell$ for all $\mathbf{t} \in \mathbf{P}_0$, that is, $\mu = v$.

Case 2. If $\mathbf{z}_{2b} = \mathbf{z}_{2b+1}$, $b = 0, 1, 2, \dots$. It implies that the sequence $\mathbf{z}_b$ is constant and so convergent. The rest of the proof can be completed on the steps of Case 1. This completes the proof.

Example 3. Let $\mathcal{X} = \{0\} \cup \{(1/b) : b \in \mathbb{N}\}$ with the metric δ defined by

$$\delta(w, z) = |w - z|, \quad \text{for all } w, z \in \mathcal{X}. \tag{53}$$

Define a t -norm “ $*$ ” by $a * e = \min\{a, e\}$ where $a, e \in \mathbf{I}$. Let $\mathbf{M}$ be the complex valued fuzzy set given by

$$\mathbf{M}(w, z, \mathbf{c}) = \frac{\mathbf{c}}{\mathbf{c} + \delta(w, z)} \ell, \tag{54}$$

for $\mathbf{c} \in \mathbf{P}_0$. Clearly, $(\mathcal{X}, \mathbf{M}, *)$ is a complex valued fuzzy metric space.

Certainly, for any sequence $\{\mathbf{c}_b\} \in \mathbf{P}_0$, $\mathbf{c}_b = (m_b, n_b)$ with $\lim_{b \rightarrow \infty} \mathbf{c}_b = \infty$ and for each fixed $w \in \mathcal{X}$, we have, $|w - z| \leq 1$, for all $z \in \{0, 1\}$, then

$$\begin{aligned}
\ell &\geq \inf_{z \in \mathcal{X}} (w, z, \mathbf{c}_b) \\
&= \inf_{z \in \mathcal{X}} \frac{m_b n_b}{m_b n_b + |w - z|} \ell = \inf_{z \in \mathcal{X}} \frac{1}{1 + (|w - z|/m_b n_b)} \ell \\
&= \frac{1}{1 + ((\sup_{z \in \mathcal{X}} |w - z|)/m_b n_b)} \ell \geq \frac{1}{1 + (|w - z|/m_b n_b)} \ell.
\end{aligned} \tag{55}$$

Therefore,

$$\ell \geq \lim_{b \rightarrow \infty} \inf_{z \in \mathcal{X}} (w, z, \mathbf{c}_b) \geq \lim_{b \rightarrow \infty} \frac{1}{1 + (|w - z|/m_b n_b)} \ell = \ell, \tag{56}$$

so that

$$\lim_{b \rightarrow \infty} \inf_{z \in \mathcal{X}} (w, z, \mathbf{c}_b) = \ell. \tag{57}$$

Define $\mathbb{F}, \Xi = (w/5)$. Note that with $\alpha = (2/7)$ and by routine calculation, one can easily verify that $\mathbb{F}, \Xi$ satisfy condition (33). Hence, all the assumptions of Theorem 2 are satisfied. Moreover, $w = 0$ remains fixed under $\mathbb{F}, \Xi$; therefore, they have a common fixed point.

Remark 3. In Theorem 1, the contraction condition (10) for the pair of self-mappings $(\mathbb{F}, \Xi)$ can be replaced by the following one, with analogous proof:

$$(\ell - \mathbf{M}(\mathbb{F}\mathbf{z}, \Xi\xi, \mathbf{t})) < \alpha(\mathbf{t})(\ell - \Omega(\mathbf{z}, \xi, \mathbf{t})), \tag{58}$$

for all $\mathbf{z} \in \mathcal{Q}$, where α is a real valued function $\alpha: \mathbf{P}_\theta \rightarrow [0, 1)$.

$$\Omega(\mathbf{z}, \xi, \mathbf{t}) = \max \left\{ \mathbf{M}(\mathbf{z}, \xi, \mathbf{t}), \frac{\mathbf{M}(\mathbf{z}, \mathbb{F}\mathbf{z}, \mathbf{t})\mathbf{M}(\xi, \Xi\xi, \mathbf{t})}{1 + \mathbf{M}(\mathbf{z}, \xi, \mathbf{t})}, \frac{\mathbf{M}(\xi, \mathbb{F}\mathbf{z}, \mathbf{t})\mathbf{M}(\mathbf{z}, \Xi\xi, \mathbf{t})}{1 + \mathbf{M}(\mathbf{z}, \xi, \mathbf{t})} \right\}. \tag{59}$$

Corollary 1. Let $(\mathcal{Q}, \mathbf{M}, *)$ be a complete complex valued fuzzy metric space and let $\mathbb{F}: \mathcal{Q} \rightarrow \mathcal{Q}$ be self-mapping, such that

$$(\ell - \mathbf{M}(\mathbb{F}\mathbf{z}, \mathbb{F}\xi, \mathbf{t})) < \alpha(\ell - \Omega(\mathbf{z}, \xi, \mathbf{t})), \tag{60}$$

for all $\mathbf{z} \in \mathcal{Q}$, where $\alpha \in (0, 1)$.

$$\Omega(\mathbf{z}, \xi, \mathbf{t}) = \max \left\{ \mathbf{M}(\mathbf{z}, \xi, \mathbf{t}), \frac{\mathbf{M}(\mathbf{z}, \mathbb{F}\mathbf{z}, \mathbf{t})\mathbf{M}(\xi, \mathbb{F}\xi, \mathbf{t})}{\ell + \mathbf{M}(\mathbf{z}, \xi, \mathbf{t})}, \frac{\mathbf{M}(\xi, \mathbb{F}\mathbf{z}, \mathbf{t})\mathbf{M}(\mathbf{z}, \mathbb{F}\xi, \mathbf{t})}{\ell + \mathbf{M}(\mathbf{z}, \xi, \mathbf{t})} \right\}. \tag{61}$$

Then, the mapping $\mathbb{F}$ has a unique fixed point.

Proof. The proof is immediate from Theorem 1 by putting $\mathbb{F} = \Xi$. $\square$

Corollary 2. Let $(\mathcal{Q}, \mathbf{M}, *)$ be a complete complex valued fuzzy metric space and let $\mathbb{F}: \mathcal{Q} \rightarrow \mathcal{Q}$ be self-mapping, such that

$$(\ell - \mathbf{M}(\mathbb{F}^b \mathbf{z}, \mathbb{F}^b \xi, \mathbf{t})) < \alpha(\ell - \Omega(\mathbf{z}, \xi, \mathbf{t})), \tag{62}$$

for all $\mathbf{z} \in \mathcal{Q}$, where $\alpha \in (0, 1)$.

$$\Omega(z, \xi, t) = \max \left\{ M(z, \xi, t), \frac{M(z, F^b z, t)M(\xi, F^b \xi, t)}{\ell + M(z, \xi, t)}, \frac{M(\xi, F^b z, t)M(z, F^b \xi, t)}{\ell + M(z, \xi, t)} \right\}. \quad (63)$$

Then, the mapping (F) has a unique fixed point.

$$F^b \tau = \tau. \quad (64)$$

So,

Proof. From Corollary 1, we obtain that there exists $\tau \in \mathcal{Q}$ such that

$$\begin{aligned} \ell - M(F\tau, \tau, t) &= \ell - M(F^b F\tau, F^b \tau, t) = \ell - M(F^b F\tau, F^b \tau, t) \\ &< \alpha \left(\ell - \max \left\{ M(F\tau, \tau, t), \frac{M(F\tau, F^b F\tau, t)M(\tau, F^b \tau, t)}{\ell + M(F\tau, \tau, t)}, \frac{M(\tau, F^b F\tau, t)M(F\tau, F^b \tau, t)}{\ell + M(F\tau, \tau, t)} \right\} \right) \\ &< \alpha \left(\ell - \max \left\{ M(F\tau, \tau, t), \frac{M(F\tau, F^b F\tau, t)M(\tau, F^b \tau, t)}{\ell + M(F\tau, \tau, t)}, \frac{M(\tau, F^b F\tau, t)M(F\tau, F^b \tau, t)}{\ell + M(F\tau, \tau, t)} \right\} \right) \\ &< \alpha(\ell - M(F\tau, \tau, t)) \\ &< \ell - \alpha M(F\tau, \tau, t). \end{aligned} \quad (65)$$

This yields

$$M(F\tau, \tau, t) \geq \alpha M(F\tau, \tau, t). \quad (66)$$

Since F^b satisfies all the conditions of Corollary 1, F^b has a unique common fixed $\tau \in \mathcal{Q}$. But $F^b F\tau = F\tau$ implies that $F\tau$ is another fixed point of F^b . As common fixed point is unique, we have $F\tau = \tau$. Since the fixed point of F is also a fixed point of F^b , the fixed point of F is unique. $\square$

3. Existence Theorem for Nonlinear Impulsive Fractional Differential Equations

In this section, we present a situation where our obtained results can be applied. Precisely, we study the existence of solution for a class of nonlinear three-point implicit boundary value problems of impulsive fractional differential equations (1). This problem is equivalent to the integral equation

$$\mathbf{g} = \begin{cases} \frac{1}{\Gamma(\mu)} \int_0^t (t-\varrho)^{\mu-1} K(\varrho, \mathbf{g}(\varrho), {}^C_0 D_{\varrho_j}^\mu \mathbf{g}(\varrho)) d\varrho + \mathcal{G}(1-t) \\ \frac{1}{\Gamma(\mu)} \int_{t_j}^t (t-\varrho)^{\mu-1} K(\varrho, \mathbf{g}(\varrho), {}^C_0 D_{\varrho_j}^\mu \mathbf{g}(\varrho)) d\varrho \\ + \frac{1}{\Gamma(\mu)} \sum_{j=1}^r \int_{t_{j-1}}^{t_j} (t-\varrho)^{\mu-1} K(\varrho, \mathbf{g}(\varrho), {}^C_0 D_{\varrho_j}^\mu \mathbf{g}(\varrho)) d\varrho \\ + \frac{1}{\Gamma(\mu-1)} \sum_{j=1}^r (t-t_j) \int_{t_{j-1}}^{t_j} (t_j-\varrho)^{\mu-2} K(\varrho, \mathbf{g}(\varrho), {}^C_0 D_{\varrho_j}^\mu \mathbf{g}(\varrho)) d\varrho \\ + \sum_{j=1}^r (t-t_j) \widetilde{\mathfrak{I}}_j(\mathbf{g}(t_j)) + \sum_{j=1}^r \mathfrak{I}_j(\mathbf{g}(t_j)) + \mathcal{G}(1-t), \quad t \in J_r, r = 1, 2, 3, \dots, q, \end{cases} \quad (67)$$

where

$$\begin{aligned}
\mathcal{G} = & \frac{1}{\Gamma(\mu)} \sum_{j=1}^{q+1} \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-1} K\left(\varrho, \mathbf{g}(\varrho), {}^C_0 D_{\varrho_j}^{\mu} \mathbf{g}(\varrho)\right) d\varrho \\
& + \frac{1}{\Gamma(\mu-1)} \sum_{j=1}^q (1-t_j) \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} K\left(\varrho, \mathbf{g}(\varrho), {}^C_0 D_{\varrho_j}^{\mu} \mathbf{g}(\varrho)\right) d\varrho \\
& + \frac{1}{\Gamma(\mu-1)} \int_{t_{j-1}}^{\rho} (\rho - \varrho)^{\mu-2} K\left(\varrho, \mathbf{g}(\varrho), {}^C_0 D_{\varrho_j}^{\mu} \mathbf{g}(\varrho)\right) d\varrho \\
& + \frac{1}{\Gamma(\mu-1)} \sum_{j=1}^q \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} K\left(\varrho, \mathbf{g}(\varrho), {}^C_0 D_{\varrho_j}^{\mu} \mathbf{g}(\varrho)\right) d\varrho \\
& + \sum_{j=1}^q (1-t_j) \widetilde{\mathfrak{F}}_j(\mathbf{g}(t_j)) + \sum_{j=1}^q \widetilde{\mathfrak{F}}_j(\mathbf{g}(t_j)) + \sum_{j=1}^q \mathfrak{F}_j(\mathbf{g}(t_j)).
\end{aligned} \tag{68}$$

For simplicity, we use the notation $\theta_{\mathbf{g}}(t) = K(t, \mathbf{g}(t), {}^C_0 D_{\varrho_j}^{\mu} \mathbf{g}(t))$.

If necessary, the reader can refer to [16] for a more detailed explanation of the background of the problem.

Here, we shall prove our result by establishing the existence of a common fixed point for a pair of integral operators $\mathfrak{F}_1, \mathfrak{F}_2$, defined on $C(J, \mathbb{R})$ as

$$\begin{aligned}
\mathfrak{F}_i(\mathbf{g})(t) = & \frac{1}{\Gamma(\mu)} \int_{t_j}^t (t - \varrho)^{\mu-1} \theta_i(\varrho) d\varrho + \frac{1}{\Gamma(\mu)} \sum_{0 \leq t_j \leq t} \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-1} \theta_i(\varrho) d\varrho \\
& + \frac{1}{\Gamma(\mu-1)} \sum_{0 \leq t_j \leq t} (t - t_j) \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} \theta_{i,\mathbf{g}}(\varrho) d\varrho + \sum_{0 \leq t_j \leq t} (t - t_j) \widetilde{\mathfrak{F}}_j(\mathbf{g}(t_j)) \\
& + \sum_{0 \leq t_j \leq t} (t - t_j) \mathfrak{F}_j(\mathbf{g}(t_j)) + (1-t) \left[\frac{1}{\Gamma(\mu)} \sum_{j=1}^{q+1} \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-1} \theta_{i,\mathbf{g}}(\varrho) d\varrho + \frac{1}{\Gamma(\mu-1)} \sum_{j=1}^q (1-t_j) \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} \theta_{i,\mathbf{g}}(\varrho) d\varrho \right. \\
& + \frac{1}{\Gamma(\mu-1)} \int_{t_{j-1}}^{\rho} (\rho - \varrho)^{\mu-2} \theta_{i,\mathbf{g}}(\varrho) d\varrho + \frac{1}{\Gamma(\mu-1)} \sum_{j=1}^q \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} \theta_{i,\mathbf{g}}(\varrho) d\varrho \\
& \left. + \sum_{j=1}^q (1-t_j) \widetilde{\mathfrak{F}}_j(\mathbf{g}(t_j)) + \sum_{j=1}^q \widetilde{\mathfrak{F}}_j(\mathbf{g}(t_j)) + \sum_{j=1}^q \mathfrak{F}_j(\mathbf{g}(t_j)) \right], \quad t \in J_r, i = 1, 2.
\end{aligned} \tag{69}$$

Theorem 3. Assume the following hypotheses are satisfied for $j = 1, 2, 3, \dots, q$.

- (i) (H_1) The nonlinear function $K_i: [0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous.
- (ii) (H_2) Let us have constants $\gamma, \delta \in (0, 1)$, which satisfy

$$\begin{aligned}
|K_1(t, \mathbf{g}, \overline{\mathbf{g}}) - K_2(t, f, \overline{f})| \leq & \gamma |\mathbf{g}(t) - f(t)| \\
& + \delta |\overline{\mathbf{g}}(t) - \overline{f}(t)|.
\end{aligned} \tag{70}$$

- (iii) (H_3) $|\mathfrak{F}_j(\mathbf{g}(t_j)) - \mathfrak{F}_j(\overline{\mathbf{g}}(t_j))| = h |\mathbf{g}(t_j) - \overline{\mathbf{g}}(t_j)|$, with $h > 0$.

- (iv) $(H_4) |\widetilde{\mathfrak{F}}_j(\mathbf{g}(t_j))t - n\widetilde{\mathfrak{F}}_j q(\overline{\mathbf{g}}(t_j))| = h^* |\mathbf{g}(t_j) - \overline{\mathbf{g}}(t_j)|$,
with $h^* > 0$.
- (v) $(H_5) \quad \Lambda = [(\gamma/1 - \delta)((2q + 3/\Gamma(q - 1)) + (3q + 1/\Gamma(q)) + q(2h + h^*))] \leq 1$ and

$$\mathbf{M}(\mathbf{g}, \overline{\mathbf{g}}, \mathbf{c}) = \max \left\{ \mathbf{M}(\mathbf{g}, \overline{\mathbf{g}}, \mathbf{c}), \frac{\mathbf{M}(\mathbf{g}, \mathfrak{F}_1 \mathbf{g}, \mathbf{c}) \mathbf{M}(\overline{\mathbf{g}}, t \mathfrak{F}_2 n \overline{\mathbf{g}} q, hc)}{\ell + \mathbf{M}(\mathbf{g}, \overline{\mathbf{g}}, \mathbf{c})}, \frac{\mathbf{M}(\overline{\mathbf{g}}, t \mathfrak{F}_1 n \mathbf{g} q, hc) \mathbf{M}(\mathbf{g}, \mathfrak{F}_2 \overline{\mathbf{g}}, \mathbf{c})}{\ell + \mathbf{M}(\mathbf{g}, \overline{\mathbf{g}}, \mathbf{c})} \right\}. \quad (71)$$

Then, the pair of nonlinear integral equations

$$\begin{aligned} \mathbf{g}(t) = & \frac{1}{\Gamma(\mu)} \int_{t_j}^t (t - \varrho)^{\mu-1} \theta_{i,\mathbf{g}}(\varrho) d\varrho + \frac{1}{\Gamma(\mu)} \sum_{0 \leq t_j \leq t} \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-1} \theta_{i,\mathbf{g}}(\varrho) d\varrho \\ & + \frac{1}{\Gamma(\mu-1)} \sum_{0 \leq t_j \leq t} (t - t_j) \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} \theta_{i,\mathbf{g}}(\varrho) d\varrho + \sum_{0 \leq t_j \leq t} (t - t_j) \widetilde{\mathfrak{F}}_j(\mathbf{g}(t_j)) \\ & + \sum_{0 \leq t_j \leq t} (t - t_j) \mathfrak{F}_j(\mathbf{g}(t_j)) + (1-t) \left[\frac{1}{\Gamma(\mu)} \sum_{j=1}^{q+1} \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-1} \theta_{i,\mathbf{g}}(\varrho) d\varrho + \frac{1}{\Gamma(\mu-1)} \sum_{j=1}^q (1-t_j) \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} \theta_{i,\mathbf{g}}(\varrho) d\varrho \right. \\ & + \frac{1}{\Gamma(\mu-1)} \int_{t_{j-1}}^{\rho} (\rho - \varrho)^{\mu-2} \theta_{i,\mathbf{g}}(\varrho) d\varrho + \frac{1}{\Gamma(\mu-1)} \sum_{j=1}^q \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} \theta_{i,\mathbf{g}}(\varrho) d\varrho \\ & \left. + \sum_{j=1}^q (1-t_j) \widetilde{\mathfrak{F}}_j(\mathbf{g}(t_j)) + \sum_{j=1}^q \widetilde{\mathfrak{F}}_j(\mathbf{g}(t_j)) + \sum_{j=1}^q \mathfrak{F}_j(\mathbf{g}(t_j)) \right], \quad t \in J, i = 1, 2. \end{aligned} \quad (72)$$

has a common solution in $C(J, \mathbb{R})$.

Proof. Consider $\Theta = C(J, \mathbb{R})$ with the metric

$$\sigma(\mathbf{g}, f) = \max_{t \in J} (|\mathbf{g}(t) - f(t)|). \quad (73)$$

Define t -norm as

$$\mathbf{c}_1 * \mathbf{c}_2 = \begin{cases} (a_1, b_1), & \text{if } (a_2, b_2) = \ell, \\ (a_2, b_2), & \text{if } (a_1, b_1) = \ell, \\ 0, & \text{otherwise,} \end{cases} \quad (74)$$

for $\mathbf{c}_1 = (a_1, b_1), \mathbf{c}_2 = (a_2, b_2) \in \mathbf{I}$ and $\mathbf{M}: \Theta \times \Theta \times P_\theta \longrightarrow \mathbf{I}$ defined by

$$\mathbf{M}(\mathbf{g}, f, \mathbf{c}) = \ell - \frac{\sigma(\mathbf{g}, f)}{1 + ab} \ell, \quad (75)$$

for $\mathbf{c} = (a, b) \in P_\theta$. It is obvious that $(\Theta, \mathbf{M}, *)$ is a complex valued fuzzy metric space. Let $\mathbf{g}, \overline{\mathbf{g}} \in \Theta$ and $t \in J$; then, one has

$$\begin{aligned}
|(\mathfrak{I}_1 \mathbf{g})(t) - (\mathfrak{I}_2 \bar{\mathbf{g}})(t)| &= \frac{1}{\Gamma(\mu)} \int_{t_j}^t (t - \varrho)^{\mu-1} |\theta_{1,\mathbf{g}}(\varrho) - \theta_{2,\bar{\mathbf{g}}}(\varrho)| d\varrho \\
&+ \frac{1}{\Gamma(\mu)} \sum_{0 \leq t_j \leq t} \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-1} |\theta_{1,\mathbf{g}}(\varrho) - \theta_{2,\bar{\mathbf{g}}}(\varrho)| d\varrho \\
&+ \frac{1}{\Gamma(\mu-1)} \sum_{0 \leq t_j \leq t} (t - t_j) \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} |\theta_{1,\mathbf{g}}(\varrho) - \theta_{2,\bar{\mathbf{g}}}(\varrho)| d\varrho \\
&+ \sum_{0 \leq t_j \leq t} (t - t_j) |\widetilde{\mathfrak{I}}_j(\mathbf{g}(t_j)) - \widetilde{\mathfrak{I}}_j(\bar{\mathbf{g}}(t_j))| + \sum_{0 \leq t_j \leq t} |\mathfrak{I}_j(\mathbf{g}(t_j)) - \mathfrak{I}_j(\bar{\mathbf{g}}(t_j))| \\
&+ \frac{1}{\Gamma(\mu)} \sum_{j=1}^{q+1} \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-1} |\theta_{1,\mathbf{g}}(\varrho) - \theta_{2,\bar{\mathbf{g}}}(\varrho)| d\varrho \\
&+ \frac{1}{\Gamma(\mu-1)} \sum_{j=1}^q (1 - t_j) \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} |\theta_{1,\mathbf{g}}(\varrho) - \theta_{2,\bar{\mathbf{g}}}(\varrho)| d\varrho \\
&+ \frac{1}{\Gamma(\mu-1)} \int_{t_{j-1}}^{\rho} (\rho - \varrho)^{\mu-2} |\theta_{1,\mathbf{g}}(\varrho) - \theta_{2,\bar{\mathbf{g}}}(\varrho)| d\varrho \\
&+ \frac{1}{\Gamma(\mu-1)} \sum_{j=1}^q \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} |\theta_{1,\mathbf{g}}(\varrho) - \theta_{2,\bar{\mathbf{g}}}(\varrho)| d\varrho \\
&+ \sum_{j=1}^q (1 - t_j) |\widetilde{\mathfrak{I}}_j(\mathbf{g}(t_j))t - n\widetilde{\mathfrak{I}}_j q(\bar{\mathbf{g}}(t_j))| + \sum_{j=1}^q |\widetilde{\mathfrak{I}}_j(\mathbf{g}(t_j))t - n\widetilde{\mathfrak{I}}_j q(\bar{\mathbf{g}}(t_j))| \\
&+ \sum_{j=1}^q |\mathfrak{I}_j(\mathbf{g}(t_j)) - \mathfrak{I}_j(\bar{\mathbf{g}}(t_j))|.
\end{aligned} \tag{76}$$

With the use of H_2 , one has

$$|\theta_{1,\mathbf{g}}(\varrho) - \theta_{2,\bar{\mathbf{g}}}(\varrho)| \leq \frac{\gamma}{1-\delta} |\mathbf{g} - \bar{\mathbf{g}}|. \tag{77}$$

Therefore, using (77) in (76), we obtain

$$\begin{aligned}
|(\mathfrak{I}_1 \mathbf{g}) - (\mathfrak{I}_2 \bar{\mathbf{g}})| &= \frac{\gamma |\mathbf{g} - \bar{\mathbf{g}}|}{\Gamma(\mu)(1-\delta)} \int_{t_j}^t (t - \varrho)^{\mu-1} d\varrho + \frac{2\gamma |\mathbf{g} - \bar{\mathbf{g}}|}{\Gamma(\mu)(1-\delta)} \sum_{j=1}^{q+1} \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-1} d\varrho \\
&+ \frac{3\gamma |\mathbf{g} - \bar{\mathbf{g}}|}{\Gamma(\mu-1)(1-\delta)} \sum_{j=1}^q \int_{t_{j-1}}^{t_j} (t_j - \varrho)^{\mu-2} d\varrho \\
&+ \frac{\gamma |\mathbf{g} - \bar{\mathbf{g}}|}{\Gamma(\mu-1)(1-\delta)} \int_{t_{j-1}}^{\rho} (\rho - \varrho)^{\mu-2} d\varrho \\
&+ 2qh|\mathbf{g} - \bar{\mathbf{g}}| + 3qh^*|\mathbf{g} - \bar{\mathbf{g}}| \leq \Lambda |\mathbf{g} - \bar{\mathbf{g}}|.
\end{aligned} \tag{78}$$

Now,

$$\begin{aligned}
 (\ell - \mathbf{M}(\mathfrak{I}_1(\mathbf{g}), \mathfrak{I}_2(\bar{\mathbf{g}}), \mathbf{c})) &= \left(\ell - \left(\ell - \frac{\sigma(\mathfrak{I}_1(\mathbf{g}), \mathfrak{I}_2(\bar{\mathbf{g}}))}{1+ab} \ell \right) \right) \\
 &= \left(\frac{\sigma(\mathfrak{I}_1(\mathbf{g}), \mathfrak{I}_2(\bar{\mathbf{g}}))}{1+ab} \ell * \ell \right) < \Lambda \left(\frac{\sigma(\mathbf{g}, \bar{\mathbf{g}})}{1+ab} \ell * \ell \right) \\
 &= \Lambda \left(\frac{\sigma(\mathbf{g}, \bar{\mathbf{g}})}{1+ab} \ell \right) < \Lambda \left(\ell - \left(\ell - \frac{\sigma(\mathbf{g}, \bar{\mathbf{g}})}{1+ab} \ell \right) \right) < \Lambda(\ell - \mathbf{M}(\mathbf{g}, \bar{\mathbf{g}}, \mathbf{c})) \\
 &= \Lambda \left(\ell - \max \left\{ \mathbf{M}(\mathbf{g}, \bar{\mathbf{g}}, \mathbf{c}), \frac{\mathbf{M}(\mathbf{g}, \mathfrak{I}_1 \mathbf{g}, \mathbf{c}) \mathbf{M}(\bar{\mathbf{g}}, t \mathfrak{I}_2 n \bar{\mathbf{g}} q, hc)}{1 + \mathbf{M}(\mathbf{g}, \bar{\mathbf{g}}, \mathbf{c})}, \frac{\mathbf{M}(\bar{\mathbf{g}}, t \mathfrak{I}_1 n g q, hc) \mathbf{M}(\mathbf{g}, \mathfrak{I}_2 \bar{\mathbf{g}}, \mathbf{c})}{1 + \mathbf{M}(\mathbf{g}, \bar{\mathbf{g}}, \mathbf{c})} \right\} \right).
 \end{aligned}
 \tag{79}$$

Thus, Theorem 1 applies to $\mathfrak{I}_1, \mathfrak{I}_2$, which have a common fixed point $\mathbf{g}^* \in \Theta$, that is, $\mathbf{g}^*$ is a common solution of (72).

As an immediate consequence of Theorem 3, in the case $\mathfrak{I}_1 = \mathfrak{I}_2$, we find that integral equation (67) has a solution in Θ , and hence the nonlinear implicit boundary value problems of impulsive fractional differential equations (1) have a solution. $\square$

4. Conclusion

By successful applications of our derived results, we have studied the existence and uniqueness of common solution to the proposed class of implicit impulsive differential equations. Further, some useful results were also obtained that ensure the generalization of some essential results from metric spaces to complex valued fuzzy metric spaces. We finally hope to study classes of fuzzy fractional differential problems in future works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally to the writing of this manuscript. All the authors read and approved the final version.

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