

Aircraft Dynamics

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1 Introduction

This manuscript is mainly from *Miele, A., Flight Mechanics, Vol. 1: Theory of Flight Paths, Addison-Wesley, Reading, MA, 1962, pp. 42-50*. It is concerned with trajectories characterized by short ranges and/or velocities which are small with respect to the escape velocity. For the analysis of these trajectories, the Earth can be regarded as ideally flat and non-rotating, so that the general dynamical equation reduces to the simplified form represented by:

$$\mathbf{T} + \mathbf{A} + m\mathbf{g} = m\mathbf{a} = m\frac{d\mathbf{V}}{dt} \quad (1)$$

where \mathbf{T} is the thrust, \mathbf{A} the aerodynamic force, m the mass, \mathbf{g} the acceleration of gravity, \mathbf{a} the acceleration of the aircraft with respect to the Earth, and t the time. Furthermore, the symbol:

$$\mathbf{V} = \frac{d\mathbf{EO}}{dt} \quad (2)$$

denotes the velocity of the aircraft with respect to the Earth, \mathbf{EO} being the vector which joins point E on the surface of the Earth with the aircraft.

In the following sections, the scalar equation associated with the vectorial equations 1 and 2 are derived for the general case of paths in a three-dimensional space. These equations are then reduced to those relevant to flight in either a vertical plane or a horizontal plane. To do so, it is necessary to define several reference systems and derive rules relevant to the transformation of coordinates from one system to another.

2 Basic coordinate systems

The coordinate systems of interest for flight over a flat Earth are the following: the ground axes system $EXYZ$, the local horizon system $Ox_hy_hz_h$, the wind axes system $Ox_wy_wz_w$, and the body axes system $Ox_by_bz_b$, as shown in Fig. 1. These systems are now described with the assumption that the aircraft has a plane of symmetry.

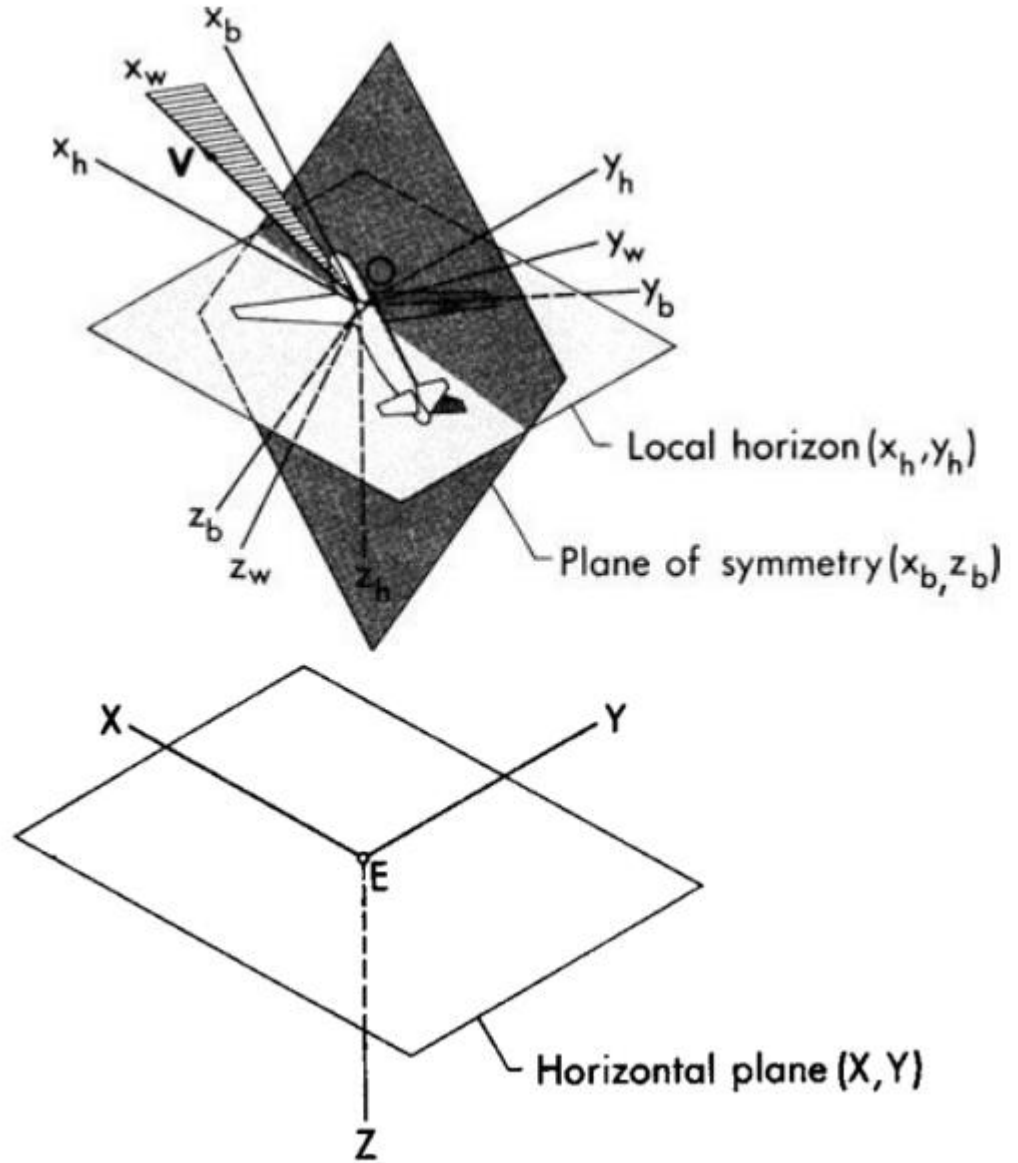


Figure 1: Coordinate systems for flight over a flat Earth

The *ground axes system* is fixed with respect to the Earth and is defined as follows: its origin E is a point on the Earth's surface; the Z-axis is vertical and positive downward; the X-axis and the Y-axis are contained in a horizontal plane and are directed in such a way that the trihedral $EXYZ$ is right-handed.

The *local horizon system* is defined as follows: its origin O is a point in the

plane of symmetry of the vehicle; its axes and the corresponding axes of the ground system are always parallel.

If the atmosphere is assumed to be at rest with respect to the Earth, the *wind axes system* is defined as follows: the x_w -axis is tangent to the flight path and is positive forward; the z_w -axis is perpendicular to the x_w -axis, contained in the plane of symmetry, and positive downward for the normal flight attitude of the aircraft; the y_w -axis is perpendicular to the $x_w z_w$ -plane and is directed in such a way that the trihedral $Ox_w y_w z_w$ is right-handed.

Finally, the *body axes system* is defined as follows: the x_b -axis is contained in the plane of symmetry and is positive forward; the z_b -axis is perpendicular to the x_b -axis, contained in the plane of symmetry, and positive downward for the normal flight attitude of the aircraft; the y_b -axis is perpendicular to the plane of symmetry and is directed in such a way that the trihedral $Ox_b y_b z_b$ is right-handed.

3 Angular relationships

In this section, the angular relationships between the different coordinate system are derived; more specifically, attention is focused on the following pairs: local horizon-ground axes, wind axes-local horizon, and body axes-wind axes.

3.1 Local horizon-ground axes

Since the local axes and the ground axes are always parallel, the following matrix relationship exists between the corresponding unit vectors:

$$\begin{bmatrix} i_h \\ j_h \\ k_h \end{bmatrix} = \begin{bmatrix} i_e \\ j_e \\ k_e \end{bmatrix} \quad (3)$$

where the subscript e refers to the ground system.

3.2 Wind axes-local horizon

The orientation of the wind axes with respect to the local horizon can be described in terms of three angular parameters. Although an infinite number of combinations of parameters can be imagined, the particular system which has become standard in aerodynamics is based on the three successive rotations of heading angle ψ , flight path angle θ and bank angle ϕ . To define these rotations, it is convenient to introduce two intermediate coordinate systems whose properties are as follows: the system $Ox_1 y_1 z_1$ is obtained from the local horizon system by means of a rotation ψ around the z_h -axis; the system $Px_2 y_2 z_2$ is obtained from $Ox_1 y_1 z_1$ by means of a rotation θ around the y_1 -axis; the wind axes system is obtained from $Ox_2 y_2 z_2$ by means of a rotation ϕ around the x_2 -axis.

These partial transformations are described by the matrix equations:

$$\begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_h \\ j_h \\ k_h \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} i_w \\ j_w \\ k_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix} \quad (6)$$

in which each scalar matrix is orthogonal. Consequently, after a matrix multiplication is performed, the relationship between the wind axes and the local horizon is given by:

$$\begin{bmatrix} i_w \\ j_w \\ k_w \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} i_h \\ j_h \\ k_h \end{bmatrix} \quad (7)$$

where the scalar matrix is also orthogonal.

3.3 Body axes-wind axes

Since the x_b, z_b, z_w -axis are contained in the plane of symmetry of the aircraft, only two angular coordinates, the sideslip angle σ and the angle of attack β , are necessary to determine the orientation of the body axes with respect to the wind axes. The system of rotations necessary to perform the transformation from the wind axes to the body axes is easily understood, if an intermediate coordinate system is introduced. Its properties are the following: the system $Ox_3y_3z_3$ is obtained from the wind axes system by means of a rotation σ around the z_w -axis; in turn, the body axes system is obtained from $Ox_3y_3z_3$ by means of a rotation β around the y_3 -axis. In matrix notation, these partial rotations are expressed by:

$$\begin{bmatrix} i_3 \\ j_3 \\ k_3 \end{bmatrix} = \begin{bmatrix} \cos\sigma & \sin\sigma & 0 \\ -\sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_w \\ j_w \\ k_w \end{bmatrix} \quad (8)$$

$$\begin{bmatrix} i_b \\ j_b \\ k_b \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} i_3 \\ j_3 \\ k_3 \end{bmatrix} \quad (9)$$

so that the relationship between the body axes and the wind axes becomes:

$$\begin{bmatrix} i_b \\ j_b \\ k_b \end{bmatrix} = \begin{bmatrix} \cos\beta\cos\sigma & \cos\beta\sin\sigma & -\sin\beta \\ -\sin\sigma & \cos\sigma & 0 \\ \sin\beta\cos\sigma & \sin\beta\sin\sigma & \cos\beta \end{bmatrix} \begin{bmatrix} i_w \\ j_w \\ k_w \end{bmatrix} \quad (10)$$

4 Evolutive velocity

In this section, the evolutive velocity, that is, the angular velocity of the wind axes with respect to the Earth axes, is calculated. To do so, consider the behavior of the aircraft between time instants t and $t + dt$, and denote the infinitesimal variations of the heading angle, the flight path angle, and the bank angle by $d\psi, d\theta, d\phi$. Since these infinitesimal scalar rotations occur around the z_h -axis, the y_1 -axis, and the x_2 -axis, respectively, the infinitesimal vectorial rotation is represented by

$$d\Omega_w = d\phi i_2 + d\theta j_1 + d\psi k_h \quad (11)$$

It can be rewritten by

$$\omega_w = \frac{d\Omega_w}{dt} = \dot{\phi} i_2 + \dot{\theta} j_1 + \dot{\psi} k_h \quad (12)$$

Owing to the fact that the previous unit vectors and those of the wind axes system are related by the matrix equation

$$\begin{bmatrix} i_2 \\ j_1 \\ k_h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} i_w \\ j_w \\ k_w \end{bmatrix} \quad (13)$$

the evolutive velocity can be rewritten in terms of its components on the wind axes as

$$\omega_w = p_w i_w + q_w j_w + r_w k_w \quad (14)$$

where

$$\begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & \sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (15)$$

5 Kinematic relationships

In this section, the scalar relationships corresponding to the vectorial equation 2 are derived. It is observed that, since the velocity is colinear with the x_w -axis, the left-hand side of equation 2 can be represented by:

$$\mathbf{V} = V i_w = V [\cos\theta\cos\psi i_h + \cos\theta\sin\psi j_h - \sin\theta k_h] \quad (16)$$

It is also observed that vector that the vector joining the origin of the ground system with the aircraft can be written in the form:

$$\mathbf{EO} = x i_h + y j_h + z k_h \quad (17)$$

the time derivative of this vector becomes:

$$\frac{d\mathbf{EO}}{dt} = \dot{x} i_h + \dot{y} j_h - \dot{z} k_h \quad (18)$$

Consequently, if equations 2, 16, 18 are combined, the following relationships are derived:

$$\begin{cases} \dot{x} = V \cos \theta \cos \psi \\ \dot{y} = V \cos \theta \sin \psi \\ \dot{z} = V \sin \theta \end{cases} \quad (19)$$

6 Dynamic relationships

In this section, the vectorial equation 1 is reduced to its equivalent scalar form. The method employed is analogous to that of the previous section and consists of determining the components of each vector on the wind axes.

First, in analogy with the sideslip angle and the angle of attack, the thrust sideslip angle v and the thrust angle of attack ϵ are introduced. These angles are the successive rotations to which the wind axes system must be subjected in order to turn the x_w -axis in a direction parallel to the thrust. Consequently, the thrust becomes:

$$\mathbf{T} = T[\cos v \cos \epsilon i_w + \cos \epsilon \sin v j_w - \sin \epsilon k_w] \quad (20)$$

Second, the aerodynamic force is written in terms of its components on the wind axes as:

$$\mathbf{A} = -[D i_w + Q j_w + L k_w] \quad (21)$$

where D is the drag, Q is the side force, and L the lift. Third, if it is noted that the acceleration of gravity has the same direction as the Z-axis, the following relation is obtained:

$$\mathbf{g} = g[-\sin \theta i_w + \sin \phi \cos \theta j_w + \cos \phi \cos \theta k_w] \quad (22)$$

Fourth, after calculating the time derivative of equation 16, one obtains the following expression for the acceleration of the aircraft relative to the Earth:

$$\frac{d\mathbf{V}}{dt} = \dot{v} i_w + v \frac{di_w}{dt} \quad (23)$$

where the dot sign denotes a derivative with respect to time. In consideration of Poisson's formulas, the time rate of change of the unit vector tangent to the flight path and the evolutive velocity are related by:

$$\frac{di_w}{dt} = \omega_w \times i_w = r_w j_w - q_w k_w \quad (24)$$

An important particular case occurs when the aircraft sideslip angle and the thrust sideslip angle are simultaneously zero, that is,

$$\sigma = v = 0 \quad (25)$$

Under this condition, the side force of a symmetric configuration is $Q = 0$. If the engine is fixed with respect to the aircraft, these equations must be completed by the relationship:

$$\epsilon - \beta = \text{Const} \quad (26)$$

In this manuscript, we assume $Const$ is zero.
the following relations are obtained:

$$\begin{cases} \mathbf{T} = T[\cos\beta i_b + \sin\beta \sin\phi j_b - \sin\beta \cos\phi k_b] \\ \mathbf{A} = -[Di_b + L\sin\phi j_b - L\cos\phi k_b] \\ \mathbf{g} = g[-\sin\theta i_b + \cos\theta k_b] \end{cases} \quad (27)$$

Consequently, the following scalar equations are derived:

$$\begin{cases} T\cos\beta - D - mg\sin\theta - m\dot{v} = 0 \\ T\sin\beta \sin\phi + L\sin\phi - mv\cos\theta\dot{\psi} = 0 \\ T\sin\beta \cos\phi + L\cos\phi - mg\cos\theta - mv\dot{\theta} = 0 \end{cases} \quad (28)$$