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Research Article

Combined Constrained Robust Least Squares Approach and Block-Pulse Functions Technique for Tracking Control Synthesis of Uncertain Bilinear Systems with Multiple Time-Delayed States under Bounded Input Control

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The present study tackles the tracking control problem for unstructured uncertain bilinear systems with multiple time-delayed states subject to control input constraints. First, a new method is introduced to design memory state feedback controllers with compensator gain based on the use of operational properties of block-pulse functions basis. The proposed technique permits transformation of the posed control problem into a constrained and robust optimization problem. The constrained robust least squares approach is then used for determination of the control gains. Second, new sufficient conditions are proposed for the practical stability analysis of the closed-loop system, where a domain of attraction is estimated. A real-world example, the headbox control of a paper machine, demonstrates the efficiency of the proposed method.

1. Introduction

Many physical systems existing in real life exhibit nonlinear behavior. For system analysis and control, an approximate model is practically used for the purposes of simplicity because an exact system model is too difficult to obtain or too complicated to handle. The class of bilinear systems, representing the particular nonlinear systems whose dynamics are jointly linear in state and input variables, was introduced in the control theory due to its simple structure and applicability in the 1960s. It is well known that bilinear structure can model nonlinear phenomena more accurately compared to linear structure [1]. Therefore, bilinear models have been found in various fields of research such as engineering, biology, and economics [1, 2].

A great deal of literature related to the stabilization control problems of such systems have been developed over the past decades. Among them, some results were concerned with continuous-time bilinear systems with only multiplicative control [3, 4]. For bilinear systems with both additive and multiplicative control inputs, there were some control designs, such as bang-bang control law with a nonlinear switching function [5], quadratic feedback control [6], and optimal control [7–9].

Since the exact system models are not always available, the model uncertainties may occur in the bilinear systems. Recently, some interesting works that used Lyapunov stability theory were devoted to the stabilization control problems for continuous-time bilinear systems affected by norm bounded uncertainties [10, 11]. The main obstacles of these design controls result firstly from the choice of the appropriate Lyapunov functional, which is always restricted to be of quadratic form, and secondly from the difficulty to test algorithmically the obtained nonnegativity conditions given in LMI formulation. Another common drawback of these methods is that they are limited to specific type of uncertainties, in which the system matrix uncertainty has to satisfy so-called "matching conditions."

Moreover, due to the transmission of information, natural properties of system elements, computation of variables, and so forth, time delays are often present in all actuations and measurements in practice. Their presence can degrade the performance of control system and even destabilize it [12, 13]. Thus, both time delays and uncertainties should be estimated when modeling an engineering system [14]. Hence, the stability analysis problem [15, 16] as well as the stabilization control problem [17] of continuous-time bilinear systems subject to time delays and uncertainties has received much attention in the past few years. Existing results are based on the use of Lyapunov–Krasovskii functional, even for other subclasses of nonlinear systems [18, 19]. The later technique used a weighting function in the form of a function of the current state [20]. There are two main drawbacks of these results, which can be cited as follows:

(i) The absence of a systematic way to construct a suitable Lyapunov–Krasovskii function, which is in most cases of quadratic form.

(ii) The difficulty to apply them to high-order systems with a high number of delay functions. This is due to the computational requirement increases.

On the other hand, it is well known that all control actuation devices are subject to magnitude and/or rate limits and this leads to degradation of the performance and even instability of closed-loop control systems [21]. Hard input constraints belong to the very important task in the controller design. At present, there are a few works in which the stabilization control problem of saturated bilinear systems without uncertainties and/or time delays has been studied [22–24]. All of these works have addressed only the case of discrete systems in which the nonlinear function is absorbed in a linear differential inclusion. If it exists, a polyhedral Lyapunov function is used to determine the control gains through an iterative procedure. The main drawback of polyhedral Lyapunov-based methods is that the computational burden required by their construction dramatically increases with the system dimension and the vertices number of the polytope of the considered quasi-linear models matrices, resulting from the approximation of the original bilinear model.

To the best of our knowledge, the setpoint tracking control problem for continuous-time bilinear systems under input saturation, affected by norm bounded unstructured uncertainties and multiple time delays in states, is not yet developed until now. This is mainly due to the difficulty of synthesizing a saturated tracking control law with the guaranteed performance and stability of the closed-loop system under the presence of uncertainties and time delays. It is for this reason that problems related to the tracking control synthesis for various nonlinear subclasses are intensively studied nowadays [25–28]. Recently, the setpoint tracking control problem for continuous-time bilinear system accompanied with unstructured uncertainties in matrices system was addressed in [29], where the authors propose an algebraic approach based on the use of piecewise orthogonal functions set as well as their operational matrices. This has allowed the conversion of the uncertain differential state equations to an uncertain system of algebraic ones. Then, resulting optimization problem is solved by means of robust least squares minimization, leading to the control law parameters.

In this article, we aim at extending the study presented in [29] to deal with the setpoint tracking control problem for a more complex subclass of continuous-time bilinear systems, mentioned above, and under the presence of actuator saturation. A new algebraic aspect framework is then presented. The whole development uses block-pulse functions as a tool of approximation as well as their operational matrices. Among all other piecewise constant basis functions, the block-pulse functions set proved to be the most fundamental, which has the advantage of reducing computational complexities and execution time [30–34].

The general idea of this work consists of equalizing the nondelayed state vector of the controlled system and the state vector of the reference model and thus the equalization of their projections on the considered orthogonal functions basis. Then, the application of the operational matrices jointly used with the Kronecker tensor product permits to obtain a constrained and robust optimization problem. Once the control gains are determined by solving the latter optimization problem in constrained robust least square sense, the practical stability of the closed-loop system is checked through simple conditions.

The main contributions in this work could be summarized as follows:

(1) Overcoming the drawbacks of the existing stabilization control methods based on the Lyapunov function approach or Lyapunov–Krasovskii function approach.

(2) Addressing the setpoint tracking control problem of unstructured uncertain bilinear systems with multiple time-delayed states subject to control input constraints, which have not been treated before.

(3) Deriving a memory state feedback control with feedforward gain using a new formulation based on the properties of block-pulse functions basis such as operational matrices jointly used with the Kronecker tensor product.

(4) Elaborating new sufficient conditions to check the practical stability of the close-loop system, where a domain of attraction is estimated.

This study is outlined in the following manner: the next section is dedicated for the description of the system under study and the clarification of the main objective of the work. In Section 3, the proposed approach of tracking control for unstructured uncertain bilinear systems with multiple time-delayed states under bounded input control is presented, leading to a constrained uncertain linear system of algebraic equations depending on the parameters of the feedback regulator, to be solved using the constrained robust least squares approach. In Section 4, we present new and simple sufficient conditions on robust trackability for this class of closed-loop dynamical systems. Some analytical methods and a Gronwall–Bellman inequality are employed to derive these practical stability conditions. Simulation results are presented in the final section to illustrate the effectiveness and performance of the proposed technique when applied on a paper-making machine.

Notation. Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, while $\mathbb{R}^{n\times m}$ refers to the set of all real matrices with n rows and m columns. I_n denotes the identity matrix of size $n\times n$. A^T represents the transpose of the matrix A. The adopted vector norm is the Euclidean norm and the matrix norm is the corresponding induced norm.

2. System Description and Control Objective

Consider the uncertain bilinear system with known fixed multitime delays in states, described by the following state equations:

$$\begin{cases} \dot{x}(t) = (A_0 + \Delta A_0)x(t) + \sum_{l=1}^{s}(D_{l0} + \Delta D_{l0})x(t-\tau_l) + \sum_{i=1}^{m}(A_i + \Delta A_i)x(t)u_i(t) + \sum_{l=1}^{s}\sum_{i=1}^{m}(D_{li} + \Delta D_{li})x(t-\tau_l)u_i(t) + Bu(t), \\ y(t) = Cx(t), \\ x(t) = \zeta(t), \quad \text{for } t \in [-\overline{\tau}, 0], \end{cases} \tag{1}$$

where $u(t) = \begin{bmatrix} u_1(t) & \dots & u_m(t) \end{bmatrix}^T \in \mathbb{R}^m$ is the input vector, $x(t) \in \mathbb{R}^n$ is the nondelayed state vector, $x(t-\tau_l) \in \mathbb{R}^n$ is the delayed state vector with τ_l denoted time delay, and $y(t) \in \mathbb{R}^p$ is the output vector.

The continuous vector valued function $\zeta(t)$ denotes the initial data, where $\overline{\tau} = \max \tau_l$ for each $l \in \{1,\dots,s\}$.

In equation (1), for each $i \in \{0,\dots,m\}$ and for each $l \in \{1,\dots,s\}$, ΔA_i and ΔD_{li} are unknown but bounded matrices.

The state model in (1) can be rewritten as follows:

$$\begin{cases} \dot{x}(t) = (A_0 + \Delta A_0)x(t) + \sum_{l=1}^{s}(D_{l0} + \Delta D_{l0})x(t-\tau_l) + (\mathscr{F} + \Delta\mathscr{F})(u(t)\otimes x(t)) + \sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)(u(t)\otimes x(t-\tau_l)) + Bu(t), \\ y(t) = Cx(t), \\ x(t) = \zeta(t), \quad \text{for } t \in [-\overline{\tau}, 0], \end{cases} \tag{2}$$

with

$$\begin{aligned} \mathscr{F} &= [A_1,\dots,A_m], \\ \Delta\mathscr{F} &= [\Delta A_1,\dots,\Delta A_m], \end{aligned} \tag{3}$$

and, for each $l \in \{1,\dots,s\}$,

$$\begin{aligned} \mathscr{H}_l &= [D_{l1},\dots,D_{lm}], \\ \Delta\mathscr{H}_l &= [\Delta D_{l1},\dots,\Delta D_{lm}], \end{aligned} \tag{4}$$

where $\otimes$ is the symbol of the Kronecker product [35].

Assumption 1. (i) System (2) is locally controllable around $x_0 = \zeta(0)$ and its n state components are all physically measurable.

(ii) The uncertainty matrices are bounded as follows:

$$\begin{aligned} \|\Delta A_0\| &\le \gamma_1, \\ \|\Delta\mathscr{F}\| &\le \gamma_2, \end{aligned} \tag{5}$$

$$\begin{aligned} \|\Delta D_{l0}\| &\le \eta_l, \\ \|\Delta\mathscr{H}_l\| &\le \overline{\eta}_l, \end{aligned} \tag{6}$$

where, for each $l \in \{1,\dots,s\}$, γ_1, γ_2, η_l, and $\overline{\eta}_l$ are given positive reals.

Strategy of Control. The main objective of the framework is the synthesis of a memory state feedback control with compensator gain, given by

$$u(t) = \overline{N}y_c(t) - Kx(t) - \sum_{l=1}^{s}\overline{K}_l x(t-\tau_l), \tag{7}$$

where $\overline{N} \in \mathbb{R}^{m\times p}$, $K \in \mathbb{R}^{m\times n}$, $\overline{K}_l \in \mathbb{R}^{m\times n}$, and $y_c(t) \in \mathbb{R}^p$ is the reference input vector.

Each element of the actuator vector $u_i(t)$ should satisfy the following constraint:

$$-\overline{u}_i \le u_i(t) \le \overline{u}_i \text{ with } 0 \prec \overline{u}_i, \tag{8}$$

which is equivalent to

$$-u_{\max} \le u(t) \le u_{\max}, \tag{9}$$

where $u_{\max} = [\,\overline{u}_1 \ \ldots \ \overline{u}_m\,]^T$ is given positive reals vector.

The constrained inputs, denoted by $\operatorname{sat}(u_i(t))$, are saturating functions and are defined for each $i \in \{1,\ldots,m\}$ as follows:

$$\operatorname{sat}(u_i(t)) = \begin{cases} \overline{u}_i, & \text{if } u_i(t) \succ \overline{u}_i, \\ u_i(t), & \text{if } -\overline{u}_i \le u_i(t) \le \overline{u}_i, \\ -\overline{u}_i, & \text{if } u_i(t) \prec -\overline{u}_i, \end{cases} \tag{10}$$

with

$$\operatorname{sat}(u(t)) = [\,\operatorname{sat}(u_1(t)) \ \ldots \ \operatorname{sat}(u_m(t))\,]^T. \tag{11}$$

Control Objective. Under input control saturation $\operatorname{sat}(u(t))$, the closed-loop system is modeled by the following state equations:

$$\begin{cases} \dot{x}(t) = (A_0 + \Delta A_0)x(t) + \sum_{l=1}^{s}(D_{l0} + \Delta D_{l0})x(t-\tau_l) + (\mathscr{F} + \Delta\mathscr{F})(\operatorname{sat}(u(t)) \otimes x(t)) \\ +\sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)(sat(u(t)) \otimes x(t-\tau_l)) + B\operatorname{sat}(u(t)), \\ y(t) = Cx(t), \\ x(t) = \zeta(t), \quad \text{for } t \in [-\overline{\tau}, 0]. \end{cases} \tag{12}$$

The main goal of the proposed control strategy is to force the controlled system (12) to reproduce sharply the dynamical behavior of a linear reference model and therefore respond to desired performances. Such reference model is described by the following state equations:

$$\begin{cases} \dot{x}_r(t) = Ex_r(t) + Fy_c(t), \\ y_r(t) = Gx_r(t), \end{cases} \tag{13}$$

where $x_r(t) \in \mathbb{R}^n$ and $y_r(t) \in \mathbb{R}^p$.

Remark 1. The structure of the control strategy adopted $u(t)$ is perfectly adequate to the structure of the considered state model (12). This is justified by the following facts:

(i) The compensator gain $\overline{N}$ is designed to eliminate the steady-state error for a constant or step input vector:

$$e(\infty) = y_c(\infty) - y(\infty). \tag{14}$$

(ii) The control gain K is designed to ensure the stability performance of the following closed-loop uncertain bilinear system under input control saturation $u_a(t)$:

$$\begin{aligned} \dot{x}(t) = {} & (A_0 + \Delta A_0)x(t) + (\mathscr{F} + \Delta\mathscr{F})(\operatorname{sat}(u_a(t)) \\ & \otimes x(t)) + B\operatorname{sat}(u_a(t)) \end{aligned}, \tag{15}$$

with

$$u_a(t) = -Kx(t). \tag{16}$$

(iii) The control gains $\overline{K}_l$ for each $l \in \{1,\ldots,s\}$ are designed to ensure the stability performance of the following closed-loop uncertain bilinear multiple time delays system under input control saturation $u_b(t)$:

$$\begin{aligned} \dot{x}(t) = {} & \sum_{l=1}^{s}(D_{l0} + \Delta D_{l0})x(t-\tau_l) + \sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l) \\ & \cdot (\operatorname{sat}(u_b(t)) \otimes x(t-\tau_l)) + B\operatorname{sat}(u_b(t)), \end{aligned} \tag{17}$$

with

$$u_b(t) = -\sum_{l=1}^{s}\overline{K}_l x(t-\tau_l). \tag{18}$$

3. Proposed Uncertain Bilinear Time Delays System Tracking Control Approach

3.1. Control Approach Development. From relations (2) and (7), state equation could be written as follows:

$$\begin{aligned}\dot{x}(t) &= (A_0+\Delta A_0)x(t)\\ &\quad+\sum_{l=1}^{s}(D_{l0}+\Delta D_{l0})x(t-\tau_l)+(\mathscr{F}+\Delta\mathscr{F})(\overline{N}\otimes I_n)(y_c(t)\otimes x(t))+B\overline{N}y_c(t)-(\mathscr{F}+\Delta\mathscr{F})\\ &\quad\cdot(K\otimes I_n)(x(t)\otimes x(t))-BKx(t)\\ &\quad-\sum_{l=1}^{s}(\mathscr{F}+\Delta\mathscr{F})(\overline{K}_l\otimes I_n)(x(t-\tau_l)\otimes x(t))\\ &\quad-\sum_{l=1}^{s}B\overline{K}_l x(t-\tau_l)+\sum_{l=1}^{s}(\mathscr{H}_l+\Delta\mathscr{H}_l)(\overline{N}\otimes I_n)(y_c(t)\otimes x(t-\tau_l))\\ &\quad-\sum_{l=1}^{s}(\mathscr{H}_l+\Delta\mathscr{H}_l)(K\otimes I_n)(x(t)\otimes x(t-\tau_l))-\sum_{j=1}^{s}\sum_{l=1}^{s}(\mathscr{H}_l+\Delta\mathscr{H}_l)(\overline{K}_j\otimes I_n)(x(t-\tau_j)\otimes x(t-\tau_l)).\end{aligned}\tag{19}$$

The integration of equation (19) with respect to t over the time interval $[0,T]$ leads to

$$\begin{aligned}x(t)-x(0) &= (A_0+\Delta A_0)\int_0^t x(\sigma)\mathrm{d}\sigma\\ &\quad+\sum_{l=1}^{s}(D_{l0}+\Delta D_{l0})\int_0^t x(\sigma-\tau_l)\mathrm{d}\sigma+(\mathscr{F}+\Delta\mathscr{F})(\overline{N}\otimes I_n)\int_0^t(y_c(\sigma)\otimes x(\sigma))\mathrm{d}\sigma\\ &\quad+B\overline{N}\int_0^t y_c(\sigma)\mathrm{d}\sigma-(\mathscr{F}+\Delta\mathscr{F})(K\otimes I_n)\int_0^t(x(\sigma)\otimes x(\sigma))\mathrm{d}\sigma-\mathrm{BK}\int_0^t x(\sigma)\mathrm{d}\sigma\\ &\quad-\sum_{l=1}^{s}(\mathscr{F}+\Delta\mathscr{F})(\overline{K}_l\otimes I_n)\int_0^t(x(\sigma-\tau_l)\otimes x(\sigma))\mathrm{d}\sigma-\sum_{l=1}^{s}B\overline{K}_l\int_0^t x(\sigma-\tau_l)\mathrm{d}\sigma\\ &\quad+\sum_{l=1}^{s}(\mathscr{H}_l+\Delta\mathscr{H}_l)(\overline{N}\otimes I_n)\int_0^t(y_c(\sigma)\otimes x(\sigma-\tau_l))\mathrm{d}\sigma\\ &\quad-\sum_{l=1}^{s}(\mathscr{H}_l+\Delta\mathscr{H}_l)(K\otimes I_n)\int_0^t(x(\sigma)\otimes x(\sigma-\tau_l))\mathrm{d}\sigma\\ &\quad-\sum_{j=1}^{s}\sum_{l=1}^{s}(\mathscr{H}_l+\Delta\mathscr{H}_l)(\overline{K}_j\otimes I_n)\int_0^t(x(\sigma-\tau_j)\otimes x(\sigma-\tau_l))\mathrm{d}\sigma.\end{aligned}\tag{20}$$

The expansion of the fixed reference input vector $y_c(t)$ over the basis of block-pulse functions $S_N(t)$ truncated to an order N can be written as

$$y_c(t)=Y_{cN}S_N(t),\tag{21}$$

where Y_{cN} denote reference input coefficients resulting from the scalar product (A.4).

We underline that the main idea consists of equalizing the nondelayed state vector of the controlled bilinear system and the state vector of the reference model. That is to say,

$$x(t)=x_r(t)=X_{rN}S_N(t),\tag{22}$$

where X_{rN} denotes the state coefficients of the reference model, which are computed from the scalar product (A.4).

Therefore, for each $l\in\{1,\ldots,s\}$, the delayed state vector of the controlled system $x(t-\tau_l)$ is expressed as follows:

$$x(t-\tau_l)=\begin{cases}\zeta(t-\tau_l), & \text{for } 0\le t\le\tau_l,\\ x(t-\tau_l)=x_r(t-\tau_l), & \text{for } \tau_l<t\le T,\end{cases}\tag{23}$$

and then the block-pulse series approximation of $x(t-\tau_l)$ is given:

$$x(t-\tau_l)=X^{*}_{rN}(\tau_l)S_N(t),\tag{24}$$

where $X^{*}_{rN}(\tau_l)$ denote the delayed state coefficients, which are computed from the scalar product (A.9).

Based on the operational matrix of integration (see equation (A.14) in Appendix A), the Kronecker product terms in equation (20) can be also written as follows:

$$
\begin{aligned}
y_c(t)\otimes x(t) &= \left(\left(Y_{cN}S_N(t)\right)\otimes\left(X_{rN}S_N(t)\right)\right) = \left(Y_{cN}\otimes X_{rN}\right)\left(S_N(t)\otimes S_N(t)\right)\\
&= \left(Y_{cN}\otimes X_{rN}\right)M_N S_N(t),\\
x(t)\otimes x(t) &= \left(\left(X_{rN}S_N(t)\right)\otimes\left(X_{rN}S_N(t)\right)\right) = \left(X_{rN}\otimes X_{rN}\right)\left(S_N(t)\otimes S_N(t)\right)\\
&= \left(X_{rN}\otimes X_{rN}\right)M_N S_N(t),\\
x(t-\tau_l)\otimes x(t) &= \left(\left(X^*_{rN}(\tau_l)S_N(t)\right)\otimes\left(X_{rN}S_N(t)\right)\right)\\
&= \left(X^*_{rN}(\tau_l)\otimes X_{rN}\right)\left(S_N(t)\otimes S_N(t)\right) = \left(X^*_{rN}(\tau_l)\otimes X_{rN}\right)M_N S_N(t),\\
y_c(t)\otimes x(t-\tau_l) &= \left(\left(Y_{cN}S_N(t)\right)\otimes\left(X^*_{rN}(\tau_l)S_N(t)\right)\right)\\
&= \left(Y_{cN}\otimes X^*_{rN}(\tau_l)\right)\left(S_N(t)\otimes S_N(t)\right) = \left(Y_{cN}\otimes X^*_{rN}(\tau_l)\right)M_N S_N(t),\\
x(t)\otimes x(t-\tau_l) &= \left(\left(X_{rN}S_N(t)\right)\otimes\left(X^*_{rN}(\tau_l)S_N(t)\right)\right)\\
&= \left(X_{rN}\otimes X^*_{rN}(\tau_l)\right)\left(S_N(t)\otimes S_N(t)\right) = \left(X_{rN}\otimes X^*_{rN}(\tau_l)\right)M_N S_N(t),\\
x(t-\tau_j)\otimes x(t-\tau_l) &= \left(\left(X^*_{rN}(\tau_j)S_N(t)\right)\otimes\left(X^*_{rN}(\tau_l)S_N(t)\right)\right)\\
&= \left(X^*_{rN}(\tau_j)\otimes X^*_{rN}(\tau_l)\right)\left(S_N(t)\otimes S_N(t)\right) = \left(X^*_{rN}(\tau_j)\otimes X^*_{rN}(\tau_l)\right)M_N S_N(t).
\end{aligned}
\tag{25}
$$

Then, the expansion of equation (20) over the considered block-pulse functions basis yields

$$
\begin{aligned}
X_{rN}S_N(t) - X_{0N}S_N(t) &= \left(A_0+\Delta A_0\right)X_{rN}\int_0^t S_N(\sigma)\mathrm{d}\sigma\\
&\quad+\sum_{l=1}^{s}\left(D_{l0}+\Delta D_{l0}\right)X^*_{rN}(\tau_l)\int_0^t S_N(\sigma)\mathrm{d}\sigma\\
&\quad+\left(\mathcal{F}+\Delta\mathcal{F}\right)\left(\overline{N}\otimes I_n\right)\left(Y_{cN}\otimes X_{rN}\right)M_N\int_0^t S_N(\sigma)\mathrm{d}\sigma + B\overline{N}Y_{cN}\int_0^t S_N(\sigma)\mathrm{d}\sigma\\
&\quad-\left(\mathcal{F}+\Delta\mathcal{F}\right)\left(K\otimes I_n\right)\left(X_{rN}\otimes X_{rN}\right)M_N\int_0^t S_N(\sigma)\mathrm{d}\sigma - \mathrm{BKX}_{rN}\int_0^t S_N(\sigma)\mathrm{d}\sigma\\
&\quad-\sum_{l=1}^{s}\left(\mathcal{F}+\Delta\mathcal{F}\right)\left(\overline{K}_l\otimes I_n\right)\left(X^*_{rN}(\tau_l)\otimes X_{rN}\right)M_N\int_0^t S_N(\sigma)\mathrm{d}\sigma\\
&\quad-\sum_{l=1}^{s}B\overline{K}_l X^*_{rN}(\tau_l)\int_0^t S_N(\sigma)\mathrm{d}\sigma\\
&\quad+\sum_{l=1}^{s}\left(\mathcal{H}_l+\Delta\mathcal{H}_l\right)\left(\overline{N}\otimes I_n\right)\left(Y_{cN}\otimes X^*_{rN}(\tau_l)\right)M_N\int_0^t S_N(\sigma)\mathrm{d}\sigma\\
&\quad-\sum_{l=1}^{s}\left(\mathcal{H}_l+\Delta\mathcal{H}_l\right)\left(K\otimes I_n\right)\left(X_{rN}\otimes X^*_{rN}(\tau_l)\right)M_N\int_0^t S_N(\sigma)\mathrm{d}\sigma\\
&\quad-\sum_{j=1}^{s}\sum_{l=1}^{s}\left(\mathcal{H}_l+\Delta\mathcal{H}_l\right)\left(\overline{K}_j\otimes I_n\right)\left(X^*_{rN}(\tau_j)\otimes X^*_{rN}(\tau_l)\right)M_N\int_0^t S_N(\sigma)\mathrm{d}\sigma.
\end{aligned}
\tag{26}
$$

The use of the integration operational matrix P_N defined by relation (A.11) yields

$$
\begin{aligned}
X_{rN}S_N(t) - X_{0N}S_N(t) = {} & (A_0 + \Delta A_0)X_{rN}P_N S_N(t) \\
& + \sum_{l=1}^{s}(D_{l0} + \Delta D_{l0})X_{rN}^{*}(\tau_l)P_N S_N(t) \\
& + (\mathscr{F} + \Delta\mathscr{F})(\overline{N}\otimes I_n)(Y_{cN}\otimes X_{rN})M_N P_N S_N(t) \\
& - (\mathscr{F} + \Delta\mathscr{F})(K\otimes I_n)(X_{rN}\otimes X_{rN})M_N P_N S_N(t) \\
& - \sum_{l=1}^{s}(\mathscr{F} + \Delta\mathscr{F})(\overline{K}_l\otimes I_n)(X_{rN}^{*}(\tau_l)\otimes X_{rN})M_N P_N S_N(t) \\
& + B\overline{N}Y_{cN}P_N S_N(t) - \mathrm{BKX}_{rN}P_N S_N(t) - \sum_{l=1}^{s}B\overline{K}_l X_{rN}^{*}(\tau_l)P_N S_N(t) \\
& + \sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)(\overline{N}\otimes I_n)(Y_{cN}\otimes X_{rN}^{*}(\tau_l))M_N P_N S_N(t) \\
& - \sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)(K\otimes I_n)(X_{rN}\otimes X_{rN}^{*}(\tau_l))M_N P_N S_N(t) \\
& - \sum_{j=1}^{s}\sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)\left(\overline{K}_j\otimes I_n\right)\left(X_{rN}^{*}\left(\tau_j\right)\otimes X_{rN}^{*}(\tau_l)\right)M_N P_N S_N(t).
\end{aligned}
\tag{27}
$$

Simplifying the vector $S_N(t)$ in both sides of (27) gives

$$
\begin{aligned}
X_{rN} - X_{0N} = {} & (A_0 + \Delta A_0)X_{rN}P_N + \sum_{l=1}^{s}(D_{l0} + \Delta D_{l0})X_{rN}^{*}(\tau_l)P_N \\
& + (\mathscr{F} + \Delta\mathscr{F})(\overline{N}\otimes I_n)(Y_{cN}\otimes X_{rN})M_N P_N \\
& - (\mathscr{F} + \Delta\mathscr{F})(K\otimes I_n)(X_{rN}\otimes X_{rN})M_N P_N \\
& - \sum_{l=1}^{s}(\mathscr{F} + \Delta\mathscr{F})(\overline{K}_l\otimes I_n)(X_{rN}^{*}(\tau_l)\otimes X_{rN})M_N P_N \\
& + B\overline{N}Y_{cN}P_N - \mathrm{BKX}_{rN}P_N - \sum_{l=1}^{s}B\overline{K}_l X_{rN}^{*}(\tau_l)P_N \\
& + \sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)(\overline{N}\otimes I_n)(Y_{cN}\otimes X_{rN}^{*}(\tau_l))M_N P_N \\
& - \sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)(K\otimes I_n)(X_{rN}\otimes X_{rN}^{*}(\tau_l))M_N P_N \\
& - \sum_{j=1}^{s}\sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)\left(\overline{K}_j\otimes I_n\right)\left(X_{rN}^{*}\left(\tau_j\right)\otimes X_{rN}^{*}(\tau_l)\right)M_N P_N.
\end{aligned}
\tag{28}
$$

The use of *vec* operator (see Appendix B) leads to

$$\begin{aligned}
\operatorname{vec}(X_{rN}) - \operatorname{vec}(X_{0N}) = {} & \left(P_N^T \otimes A_0\right)\operatorname{vec}(X_{rN}) + \left(P_N^T \otimes \Delta A_0\right)\operatorname{vec}(X_{rN}) \\
& + \sum_{l=1}^{s}\left(P_N^T \otimes D_{l0}\right)\operatorname{vec}(X_{rN}^*(\tau_l)) + \sum_{l=1}^{s}\left(P_N^T \otimes \Delta D_{l0}\right)\operatorname{vec}(X_{rN}^*(\tau_l)) \\
& + \left(((Y_{cN} \otimes X_{rN})M_N P_N)^T \otimes \mathscr{F}\right)\operatorname{vec}(\overline{N} \otimes I_n) \\
& - \left(((X_{rN} \otimes X_{rN})M_N P_N)^T \otimes \mathscr{F}\right)\operatorname{vec}(K \otimes I_n) \\
& + \left(((Y_{cN} \otimes X_{rN})M_N P_N)^T \otimes \Delta\mathscr{F}\right)\operatorname{vec}(\overline{N} \otimes I_n) \\
& - \left(((X_{rN} \otimes X_{rN})M_N P_N)^T \otimes \Delta\mathscr{F}\right)\operatorname{vec}(K \otimes I_n) \\
& - \sum_{l=1}^{s}\left(((X_{rN}^*(\tau_l) \otimes X_{rN})M_N P_N)^T \otimes \mathscr{F}\right)\operatorname{vec}(\overline{K}_l \otimes I_n) \\
& - \sum_{l=1}^{s}\left(((X_{rN}^*(\tau_l) \otimes X_{rN})M_N P_N)^T \otimes \Delta\mathscr{F}\right)\operatorname{vec}(\overline{K}_l \otimes I_n) \\
& + \left((Y_{cN} P_N)^T \otimes B\right)\operatorname{vec}(\overline{N}) - \left((X_{rN} P_N)^T \otimes B\right)\operatorname{vec}(K) \\
& - \sum_{l=1}^{s}\left((X_{rN}^*(\tau_l) P_N)^T \otimes B\right)\operatorname{vec}(\overline{K}_l) \\
& + \sum_{l=1}^{s}\left(((Y_{cN} \otimes X_{rN}^*(\tau_l))M_N P_N)^T \otimes \mathscr{H}_l\right)\operatorname{vec}(\overline{N} \otimes I_n) \\
& - \sum_{l=1}^{s}\left(((X_{rN} \otimes X_{rN}^*(\tau_l))M_N P_N)^T \otimes \mathscr{H}_l\right)\operatorname{vec}(K \otimes I_n) \\
& + \sum_{l=1}^{s}\left(((Y_{cN} \otimes X_{rN}^*(\tau_l))M_N P_N)^T \otimes \Delta\mathscr{H}_l\right)\operatorname{vec}(\overline{N} \otimes I_n) \\
& - \sum_{l=1}^{s}\left(((X_{rN} \otimes X_{rN}^*(\tau_l))M_N P_N)^T \otimes \Delta\mathscr{H}_l\right)\operatorname{vec}(K \otimes I_n) \\
& - \sum_{j=1}^{s}\sum_{l=1}^{s}\left(\left(\left(X_{rN}^*(\tau_j) \otimes X_{rN}^*(\tau_l)\right)M_N P_N\right)^T \otimes \mathscr{H}_l\right)\operatorname{vec}\left(\overline{K}_j \otimes I_n\right) \\
& - \sum_{j=1}^{s}\sum_{l=1}^{s}\left(\left(\left(X_{rN}^*(\tau_j) \otimes X_{rN}^*(\tau_l)\right)M_N P_N\right)^T \otimes \Delta\mathscr{H}_l\right)\operatorname{vec}\left(\overline{K}_j \otimes I_n\right).
\end{aligned} \tag{29}$$

Based on Property 2 (see Appendix B), it comes out that

$$
\begin{aligned}
\operatorname{vec}(X_{rN}) - \operatorname{vec}(X_{0N}) = {} & \left(P_N^T \otimes A_0\right)\operatorname{vec}(X_{rN}) \\
& + \left(P_N^T \otimes \Delta A_0\right)\operatorname{vec}(X_{rN}) + \sum_{l=1}^{s}\left(P_N^T \otimes D_{l0}\right)\operatorname{vec}(X_{rN}^*(\tau_l)) \\
& + \sum_{l=1}^{s}\left(P_N^T \otimes \Delta D_{l0}\right)\operatorname{vec}(X_{rN}^*(\tau_l)) \\
& + \left(((Y_{cN} \otimes X_{rN})M_N P_N)^T \otimes \mathscr{F}\right)\Pi_{m,p}(I_n)\operatorname{vec}(\overline{N}) \\
& - \left(((X_{rN} \otimes X_{rN})M_N P_N)^T \otimes \mathscr{F}\right)\Pi_{m,n}(I_n)\operatorname{vec}(K) \\
& + \left(((Y_{cN} \otimes X_{rN})M_N P_N)^T \otimes \Delta\mathscr{F}\right)\Pi_{m,p}(I_n)\operatorname{vec}(\overline{N}) \\
& - \left(((X_{rN} \otimes X_{rN})M_N P_N)^T \otimes \Delta\mathscr{F}\right)\Pi_{m,n}(I_n)\operatorname{vec}(K) \\
& - \sum_{l=1}^{s}\left(((X_{rN}^*(\tau_l) \otimes X_{rN})M_N P_N)^T \otimes \mathscr{F}\right)\Pi_{m,n}(I_n)\operatorname{vec}(\overline{K}_l) \\
& - \sum_{l=1}^{s}\left(((X_{rN}^*(\tau_l) \otimes X_{rN})M_N P_N)^T \otimes \Delta\mathscr{F}\right)\Pi_{m,n}(I_n)\operatorname{vec}(\overline{K}_l) \\
& + \left((Y_{cN}P_N)^T \otimes B\right)\operatorname{vec}(\overline{N}) - \left((X_{rN}P_N)^T \otimes B\right)\operatorname{vec}(K) \\
& - \sum_{l=1}^{s}\left((X_{rN}^*(\tau_l)P_N)^T \otimes B\right)\operatorname{vec}(\overline{K}_l) \\
& + \sum_{l=1}^{s}\left(((Y_{cN} \otimes X_{rN}^*(\tau_l))M_N P_N)^T \otimes \mathscr{H}_l\right)\Pi_{m,p}(I_n)\operatorname{vec}(\overline{N}) \\
& - \sum_{l=1}^{s}\left(((X_{rN} \otimes X_{rN}^*(\tau_l))M_N P_N)^T \otimes \mathscr{H}_l\right)\Pi_{m,n}(I_n)\operatorname{vec}(K) \\
& + \sum_{l=1}^{s}\left(((Y_{cN} \otimes X_{rN}^*(\tau_l))M_N P_N)^T \otimes \Delta\mathscr{H}_l\right)\Pi_{m,p}(I_n)\operatorname{vec}(\overline{N}) \\
& - \sum_{l=1}^{s}\left(((X_{rN} \otimes X_{rN}^*(\tau_l))M_N P_N)^T \otimes \Delta\mathscr{H}_l\right)\Pi_{m,n}(I_n)\operatorname{vec}(K) \\
& - \sum_{j=1}^{s}\sum_{l=1}^{s}\left(\left(\left(X_{rN}^*(\tau_j) \otimes X_{rN}^*(\tau_l)\right)M_N P_N\right)^T \otimes \mathscr{H}_l\right)\Pi_{m,n}(I_n)\operatorname{vec}\left(\overline{K}_j\right) \\
& - \sum_{j=1}^{s}\sum_{l=1}^{s}\left(\left(\left(X_{rN}^*(\tau_j) \otimes X_{rN}^*(\tau_l)\right)M_N P_N\right)^T \otimes \Delta\mathscr{H}_l\right)\Pi_{m,n}(I_n)\operatorname{vec}\left(\overline{K}_j\right),
\end{aligned}
\tag{30}
$$

which is equivalent to

$$\beta+\Delta\beta=\alpha_1\operatorname{vec}(\overline{N})+\alpha_2\operatorname{vec}(K)+\sum_{l=1}^{s}\aleph_l\operatorname{vec}(\overline{K}_l)+\sum_{j=1}^{s}\sum_{l=1}^{s}\vartheta_{jl}\operatorname{vec}(\overline{K}_j)+\Delta\alpha_1\operatorname{vec}(\overline{N})+\Delta\alpha_2\operatorname{vec}(K)+\sum_{l=1}^{s}\Delta\aleph_l\operatorname{vec}(\overline{K}_l)+\sum_{j=1}^{s}\sum_{l=1}^{s}\Delta\vartheta_{jl}\operatorname{vec}(\overline{K}_j), \tag{31}$$

where

$$\begin{aligned}
\beta &= \operatorname{vec}(X_{rN})-\operatorname{vec}(X_{0N})-\left(P_N^T\otimes A_0\right)\operatorname{vec}(X_{rN})-\sum_{l=1}^{s}\left(P_N^T\otimes D_{l0}\right)\operatorname{vec}(X_{rN}^*(\tau_l)),\\
\Delta\beta &= -\left(P_N^T\otimes\Delta A_0\right)\operatorname{vec}(X_{rN})-\sum_{l=1}^{s}\left(P_N^T\otimes\Delta D_{l0}\right)\operatorname{vec}(X_{rN}^*(\tau_l)),\\
\alpha_1 &= \left(((Y_{cN}\otimes X_{rN})M_NP_N)^T\otimes\mathscr{F}\right)\Pi_{m,p}(I_n)+\left((Y_{cN}P_N)^T\otimes B\right)+\sum_{l=1}^{s}\left(((Y_{cN}\otimes X_{rN}^*(\tau_l))M_NP_N)^T\otimes\mathscr{H}_l\right)\Pi_{m,p}(I_n),\\
\alpha_2 &= -\left(((X_{rN}\otimes X_{rN})M_NP_N)^T\otimes\mathscr{F}\right)\Pi_{m,n}(I_n)-\left((X_{rN}P_N)^T\otimes B\right)-\sum_{l=1}^{s}\left(((X_{rN}\otimes X_{rN}^*(\tau_l))M_NP_N)^T\otimes\mathscr{H}_l\right)\Pi_{m,n}(I_n),\\
\Delta\alpha_1 &= \left(((Y_{cN}\otimes X_{rN})M_NP_N)^T\otimes\Delta\mathscr{F}\right)\Pi_{m,p}(I_n)+\sum_{l=1}^{s}\left(((Y_{cN}\otimes X_{rN}^*(\tau_l))M_NP_N)^T\otimes\Delta\mathscr{H}_l\right)\Pi_{m,p}(I_n),\\
\Delta\alpha_2 &= -\left(((X_{rN}\otimes X_{rN})M_NP_N)^T\otimes\Delta\mathscr{F}\right)\Pi_{m,n}(I_n)-\sum_{l=1}^{s}\left(((X_{rN}\otimes X_{rN}^*(\tau_l))M_NP_N)^T\otimes\Delta\mathscr{H}_l\right)\Pi_{m,n}(I_n),
\end{aligned} \tag{32}$$

and, for each $j,l\in\{1,\ldots,s\}$,

$$\begin{aligned}
\aleph_l &= -\left(((X_{rN}^*(\tau_l)\otimes X_{rN})M_NP_N)^T\otimes\mathscr{F}\right)\Pi_{m,n}(I_n)-\left((X_{rN}^*(\tau_l)P_N)^T\otimes B\right),\\
\vartheta_{jl} &= -\left(\left(\left(X_{rN}^*(\tau_j)\otimes X_{rN}^*(\tau_l)\right)M_NP_N\right)^T\otimes\mathscr{H}_l\right)\Pi_{m,n}(I_n),\\
\Delta\aleph_l &= -\left(((X_{rN}^*(\tau_l)\otimes X_{rN})M_NP_N)^T\otimes\Delta\mathscr{F}\right)\Pi_{m,n}(I_n),\\
\Delta\vartheta_{jl} &= -\left(\left(\left(X_{rN}^*(\tau_j)\otimes X_{rN}^*(\tau_l)\right)M_NP_N\right)^T\otimes\Delta\mathscr{H}_l\right)\Pi_{m,n}(I_n).
\end{aligned} \tag{33}$$

The uncertainty vector $\Delta\beta$ becomes

$$\Delta\beta=(I_N\otimes\Delta A_0)\Gamma_1+\sum_{l=1}^{s}(I_N\otimes\Delta D_{l0})\overline{\Gamma}_l, \tag{34}$$

with

$$\Gamma_1=-\left(P_N^T\otimes I_n\right)\operatorname{vec}(X_{rN}), \tag{35}$$

and, for each $l\in\{1,\ldots,s\}$,

$$\overline{\Gamma}_l=-\left(P_N^T\otimes I_n\right)\operatorname{vec}(X_{rN}^*(\tau_l)). \tag{36}$$

Furthermore, the uncertainty matrices $\Delta\alpha_1$ and $\Delta\alpha_2$ can be written as follows:

$$\Delta\alpha_1=(I_N\otimes\Delta\mathscr{F})\Upsilon_1+\sum_{l=1}^{s}(I_N\otimes\Delta\mathscr{H}_l)\overline{\Upsilon}_l, \tag{37}$$

with

$$\Upsilon_1=\left(((Y_{cN}\otimes X_{rN})M_NP_N)^T\otimes I_{nm}\right)\Pi_{m,p}(I_n), \tag{38}$$

and, for each $l\in\{1,\ldots,s\}$,

$$\overline{\Upsilon}_l = \left(\left((Y_{cN} \otimes X_{rN}^*(\tau_l)) M_N P_N \right)^T \otimes I_{nm} \right) \Pi_{m,p}(I_n),$$
$$\Delta\alpha_2 = (I_N \otimes \Delta\mathscr{F})\Omega_1 + \sum_{l=1}^{s} (I_N \otimes \Delta\mathscr{H}_l)\overline{\Omega}_l, \tag{39}$$

with

$$\Omega_1 = -\left(\left((X_{rN} \otimes X_{rN}) M_N P_N \right)^T \otimes I_{nm} \right) \Pi_{m,n}(I_n), \tag{40}$$

and, for each $l \in \{1, \ldots, s\}$,

$$\overline{\Omega}_l = -\left(\left((X_{rN} \otimes X_{rN}^*(\tau_l)) M_N P_N \right)^T \otimes I_{nm} \right) \Pi_{m,n}(I_n). \tag{41}$$

Likewise, for each $j, l \in \{1, \ldots, s\}$, the uncertainty matrices $\Delta\aleph_l$ and $\Delta\vartheta_{jl}$ can be written as follows:

$$\Delta\aleph_l = (I_N \otimes \Delta\mathscr{F})\Psi_l, \tag{42}$$

with

$$\Psi_l = -\left(\left((X_{rN}^*(\tau_l) \otimes X_{rN}) M_N P_N \right)^T \otimes I_{nm} \right) \Pi_{m,n}(I_n),$$
$$\Delta\vartheta_{jl} = (I_N \otimes \Delta\mathscr{H}_l)\Phi_{jl}, \tag{43}$$

with

$$\Phi_{jl} = -\left(\left(\left(X_{rN}^*(\tau_j) \otimes X_{rN}^*(\tau_l)\right) M_N P_N \right)^T \otimes I_{nm} \right) \Pi_{m,n}(I_n). \tag{44}$$

Then, relation (31) becomes

$$\begin{aligned}
\beta + (I_N \otimes \Delta A_0)\Gamma_1 + \sum_{l=1}^{s} (I_N \otimes \Delta D_{l0})\overline{\Gamma}_l = {} & \alpha_1 \mathrm{vec}(\overline{N}) + \alpha_2 \mathrm{vec}(K) \\
& + \sum_{l=1}^{s} \aleph_l \mathrm{vec}(\overline{K}_l) + \sum_{j=1}^{s}\sum_{l=1}^{s} \vartheta_{jl} \mathrm{vec}(\overline{K}_j) + (I_N \otimes \Delta\mathscr{F})\Upsilon_1 \mathrm{vec}(\overline{N}) \\
& + \sum_{l=1}^{s} (I_N \otimes \Delta\mathscr{H}_l)\overline{\Upsilon}_l \mathrm{vec}(\overline{N}) + (I_N \otimes \Delta\mathscr{F})\Omega_1 \mathrm{vec}(K) \\
& + \sum_{l=1}^{s} (I_N \otimes \Delta\mathscr{H}_l)\overline{\Omega}_l \mathrm{vec}(K) \\
& + \sum_{l=1}^{s} (I_N \otimes \Delta\mathscr{F})\Psi_l \mathrm{vec}(\overline{K}_l) + \sum_{j=1}^{s}\sum_{l=1}^{s} (I_N \otimes \Delta\mathscr{H}_l)\Phi_{jl} \mathrm{vec}(\overline{K}_j).
\end{aligned} \tag{45}$$

Let

$$\begin{aligned}
\overline{\alpha} &= [\, \alpha_1 \ \ \alpha_2 \,], \\
\overline{\aleph} &= [\, \aleph_1 \ \ \aleph_2 \ \ \ldots \ \ \aleph_s \,], \\
\varepsilon_l &= [\, \vartheta_{1l} \ \ \vartheta_{2l} \ \ \ldots \ \ \vartheta_{sl} \,], \\
\overline{\varpi}_l &= [\, \Phi_{1l} \ \ \Phi_{2l} \ \ \ldots \ \ \Phi_{sl} \,], \\
\varpi &= [\, \Psi_1 \ \ \Psi_2 \ \ \ldots \ \ \Psi_s \,], \\
\wp &= [\, \Upsilon_1 \ \ \Omega_1 \,], \\
\overline{\wp}_l &= [\, \overline{\Upsilon}_l \ \ \overline{\Omega}_l \,], \\
\theta_1 &= \begin{bmatrix} \mathrm{vec}(\overline{N}) \\ \mathrm{vec}(K) \end{bmatrix}, \\
\theta_2 &= \begin{bmatrix} \mathrm{vec}(\overline{K}_1) \\ \mathrm{vec}(\overline{K}_2) \\ \vdots \\ \mathrm{vec}(\overline{K}_s) \end{bmatrix}.
\end{aligned} \tag{46}$$

It comes out that

$$\begin{aligned}
\beta + (I_N \otimes \Delta A_0)\Gamma_1 + \sum_{l=1}^{s} (I_N \otimes \Delta D_{l0})\overline{\Gamma}_l = {} & \overline{\alpha}\theta_1 + \overline{\aleph}\theta_2 \\
& + \sum_{l=1}^{s} \varepsilon_l \theta_2 + (I_N \otimes \Delta\mathscr{F})\wp\theta_1 + \sum_{l=1}^{s} (I_N \otimes \Delta\mathscr{H}_l)\overline{\wp}_l\theta_1 + (I_N \otimes \Delta\mathscr{F})\varpi\theta_2 \\
& + \sum_{l=1}^{s} (I_N \otimes \Delta\mathscr{H}_l)\overline{\varpi}_l\theta_2.
\end{aligned} \tag{47}$$

Hence, it would be interesting to formulate this problem under the following unstructured linear system of algebraic equations:

$$(A + \Delta A)\theta = B + \Delta B, \tag{48}$$

where

$$\begin{aligned} A &= \left[\bar{\alpha} \left(\overline{\aleph} + \sum_{l=1}^{s} \varepsilon_l \right) \right], \\ \theta &= \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \\ B &= \beta, \\ \Delta A &= (I_N \otimes \Delta \mathscr{F}) [\, \wp \quad \mho \,] + \sum_{l=1}^{s} (I_N \otimes \Delta \mathscr{H}_l) [\, \overline{\wp}_l \quad \overline{\mho}_l \,], \\ \Delta B &= (I_N \otimes \Delta A_0) \Gamma_1 + \sum_{l=1}^{s} (I_N \otimes \Delta D_{l0}) \overline{\Gamma}_l. \end{aligned} \tag{49}$$

Furthermore, the control law $u(t)$ has to respect upper and lower bounds (9), that is,

$$\begin{aligned} -u_{\max} \le u(t) \le u_{\max} \Leftrightarrow -u_{\max} \le \overline{N} y_c(t) - Kx(t) \\ - \sum_{l=1}^{s} \overline{K}_l x(t - \tau_l) \le u_{\max}, \end{aligned} \tag{50}$$

and, by taking into account relation (22), it results that

$$-u_{\max} \le \overline{N} y_c(t) - Kx_r(t) - \sum_{l=1}^{s} \overline{K}_l x_r(t - \tau_l) \le u_{\max}, \tag{51}$$

and then the expansion of (51) over the considered block-pulse functions basis yields

$$\begin{aligned} -U_{\max,N} S_N(t) \le \left(\overline{N} Y_{cN} - KX_{rN} - \sum_{l=1}^{s} \overline{K}_l X_{rN}^{*}(\tau_l) \right) S_N(t) \\ \le U_{\max,N} S_N(t), \end{aligned} \tag{52}$$

where $U_{\max,N}$ denote the upper bound coefficients, which are computed from the scalar product (A.4).

Simplifying the vector $S_N(t)$ in both sides of (52) and using the *vec* operator gives

$$-\mathrm{vec}\left(U_{\max,N}\right) \le \overline{C}\theta \le \mathrm{vec}\left(U_{\max,N}\right), \tag{53}$$

where

$$\overline{C} = \left[\left(Y_{cN}^T \otimes I_m \right) - \left(X_{rN}^T \otimes I_m \right) \mathscr{L} \right], \tag{54}$$

with

$$\mathscr{L} = -\left[\left(\left(X_{rN}^{*}(\tau_1) \right)^T \otimes I_m \right) \ \ldots \ \left(\left(X_{rN}^{*}(\tau_s) \right)^T \otimes I_m \right) \right]. \tag{55}$$

Finally, the posed control problem is reduced to the following constrained uncertain linear system, that depending on the vector of the controller parameters to be tuned:

$$(A + \Delta A)\theta = B + \Delta B, \tag{56a}$$

subject to

$$-\mathrm{vec}\left(U_{\max,N}\right) \le \overline{C}\theta \le \mathrm{vec}\left(U_{\max,N}\right). \tag{56b}$$

3.2. Resolution. From (5) and (6), it comes out that

$$\begin{aligned} \|\Delta A\| \le \delta_A &= \gamma_2 \left\| [\, \wp \ \ \mho \,] \right\| + \sum_{l=1}^{s} \overline{\eta}_l \left\| [\, \overline{\wp}_l \ \ \overline{\mho}_l \,] \right\|, \\ \|\Delta B\| \le \delta_B &= \gamma_1 \left\| \Gamma_1 \right\| + \sum_{l=1}^{s} \eta_l \left\| \overline{\Gamma}_l \right\|. \end{aligned} \tag{57}$$

The problem of finding a solution to the obtained problem in (56a) and (56b) can be solved in the constrained robust least squares sense by the formulation as an optimization problem as follows [36]:

$$\min_{\theta \in \mathbb{R}^{m(p+(1+s)n)}} \max_{\|\Delta A\| \le \delta_A, \|\Delta B\| \le \delta_B} \|(A + \Delta A)\theta - (B + \Delta B)\|, \tag{58a}$$

subject to

$$-\mathrm{vec}\left(U_{\max,N}\right) \le \overline{C}\theta \le \mathrm{vec}\left(U_{\max,N}\right). \tag{58b}$$

From Lemma 1 (see Appendix C), the problem in (58a) and (58b) is reduced to the following minimization problem:

$$\min_{\theta \in \mathbb{R}^{m(p+(1+s)n)}} \|A\theta - B\| + \delta_A \|\theta\| + \delta_B, \tag{59a}$$

subject to

$$-\mathrm{vec}\left(U_{\max,N}\right) \le \overline{C}\theta \le \mathrm{vec}\left(U_{\max,N}\right). \tag{59b}$$

The latter problem can be rewritten as a second-order cone programming problem [37]:

$$\begin{aligned} &min \quad \alpha \quad subject \quad to \\ &\quad \|\mathrm{A}\theta - \mathrm{B}\| \le \alpha - \tau \\ &\quad \delta_{\mathrm{A}} \|\theta\| \le \tau \\ &\quad -vec\left(U_{max,N}\right) \le \overline{C}\theta \le vec\left(U_{max,N}\right) \end{aligned} \tag{60}$$

with the variables $\theta \in \mathbb{R}^{m(p+(1+s)n)}$, $\alpha \geq 0$, and $\tau \geq 0$.

4. Practical Stability Analysis and Attraction Domain Estimation

Once control parameters $\overline{N}$ and K and for each $l \in \{1, \ldots, s\}\overline{K}_l$ are determined by solving equation (60) for constant reference input $y_c(t) = \delta \in \mathbb{R}^p$ and for given positive real γ_1, γ_2, η_l, and $\overline{\eta}_l$, we propose to analyze the controlled system.

Now, let us define the following matrices:

$$M_0 = A_0 - \frac{1}{2}\mathrm{BK} + \frac{1}{2}\mathscr{F}\left(\overline{N} \otimes I_n\right)\left(\delta \otimes I_n\right), \quad (61)$$

$$\Delta M_0 = \Delta A_0 + \frac{1}{2}\Delta\mathscr{F}\left(\overline{N} \otimes I_n\right)\left(\delta \otimes I_n\right), \quad (62)$$

and, for each $l \in \{1, \ldots, s\}$,

$$M_l = D_{l0} - \frac{1}{2}B\overline{K}_l + \frac{1}{2}\mathscr{H}_l\left(\overline{N} \otimes I_n\right)\left(\delta \otimes I_n\right), \quad (63)$$

$$\Delta M_l = \Delta D_{l0} + \frac{1}{2}\Delta\mathscr{H}_l\left(\overline{N} \otimes I_n\right)\left(\delta \otimes I_n\right). \quad (64)$$

Furthermore, let us define the following positive reals, for each $l \in \{1, \ldots, s\}$:

$$\begin{aligned} \overline{\beta}_0 &= \gamma_1 + \frac{1}{2}\gamma_2 \|\overline{N}\| \|\delta\|, \\ \overline{\beta}_l &= \eta_l + \frac{1}{2}\overline{\eta}_l \|\overline{N}\| \|\delta\|, \end{aligned} \quad (65)$$

$$\begin{aligned} Q_1 &= \overline{\beta}_0 + \sum_{l=1}^{s}\left(\left\|M_l\right\| + \overline{\beta}_l\right) + \frac{1}{2}\left(\|\mathscr{F}\| + \gamma_2\right)\|\overline{N}\delta\| \\ &\quad + \frac{1}{2}\|\overline{N}\delta\|\sum_{l=1}^{s}\left(\left\|\mathscr{H}_l\right\| + \overline{\eta}_l\right), \\ Q_2 &= \frac{1}{2}\left(\|\mathscr{F}\| + \gamma_2\right)\left\|\left(K \otimes I_n\right)\right\| + \frac{1}{2}\left(\|\mathscr{F}\| + \gamma_2\right)\sum_{l=1}^{s}\left\|\left(\overline{K}_l \otimes I_n\right)\right\| \\ &\quad + \frac{1}{2}\left\|\left(K \otimes I_n\right)\right\|\sum_{l=1}^{s}\left(\left\|\mathscr{H}_l\right\| + \overline{\eta}_l\right) \\ &\quad + \frac{1}{2}\sum_{l=1}^{s}\sum_{j=1}^{s}\left(\left\|\mathscr{H}_l\right\| + \overline{\eta}_l\right)\left\|\left(\overline{K}_j \otimes I_n\right)\right\| \\ &\quad + \frac{1}{2}\left(\|\mathscr{F}\| + \gamma_2\right)\left(\|K\| + \sum_{l=1}^{s}\left\|\overline{K}_l\right\|\right) \\ &\quad + \frac{1}{2}\|K\|\sum_{l=1}^{s}\left(\left\|\mathscr{H}_l\right\| + \overline{\eta}_l\right) \\ &\quad + \frac{1}{2}\sum_{l=1}^{s}\sum_{j=1}^{s}\left(\left\|\mathscr{H}_l\right\| + \overline{\eta}_l\right)\left\|\overline{K}_j\right\|, \\ Q_3 &= \frac{1}{2}\|B\overline{N}\delta\| + \frac{1}{2}\|B\|\left\|u_{\max}\right\|. \end{aligned} \quad (66)$$

Definition 1. System (12) is said to be practically stable, if there exist R_0 and r, with $0 < R_0 < r$, such that [38]

$$\|x(0)\| < R_0 \Rightarrow \|x(t)\| < r, \quad \forall t \geq 0. \quad (67)$$

Theorem 1. *The closed-loop system (12) is practically stable if all eigenvalues of matrix M_0 have a strictly negative real part and if*

$$\frac{\omega}{\lambda} + Q_1 < 0, \quad (68)$$

and if

$$\|x(0)\| < -\frac{1}{\lambda Q_2}\left(\frac{\omega}{\lambda} + Q_1\right), \quad (69)$$

where $\omega < 0$ and $\lambda > 0$ are given scalars satisfying

$$\left\|e^{M_0 t}\right\| \leq \lambda e^{\omega t}, \quad \forall t \geq 0. \quad (70)$$

Proof. We propose to prove the existence of a region of initial conditions ensuring the practical stability of closed-loop system (12). This region is assumed to be a ball centered in the origin and of radius R_0; that is,

$$B\left(0, R_0\right) = \left\{x(0) \in \mathbb{R}^n, \quad \|x(0)\| < R_0\right\}. \quad (71)$$

From (12) and by taking into account the fact that $\mathrm{sat}(u(t)) = \mathrm{sat}(u(t)) - (1/2)u(t) + (1/2)u(t)$, the state equation of the closed-loop system can be written as follows:

$$\begin{aligned} \dot{x}(t) &= \left(A_0 + \Delta A_0\right)x(t) + \sum_{l=1}^{s}\left(D_{l0} + \Delta D_{l0}\right)x\left(t - \tau_l\right) \\ &\quad + \left(\mathscr{F} + \Delta\mathscr{F}\right)\left(\left(\mathrm{sat}(u(t)) - \frac{1}{2}u(t)\right) \otimes x(t)\right) \\ &\quad + \frac{1}{2}\left(\mathscr{F} + \Delta\mathscr{F}\right)\left(u(t) \otimes x(t)\right) \\ &\quad + \sum_{l=1}^{s}\left(\mathscr{H}_l + \Delta\mathscr{H}_l\right)\left(\left(\mathrm{sat}(u(t)) - \frac{1}{2}u(t)\right) \otimes x\left(t - \tau_l\right)\right) \\ &\quad + \frac{1}{2}\sum_{l=1}^{s}\left(\mathscr{H}_l + \Delta\mathscr{H}_l\right)\left(u(t) \otimes x\left(t - \tau_l\right)\right) \\ &\quad + B\left(\mathrm{sat}(u(t)) - \frac{1}{2}u(t)\right) + \frac{1}{2}Bu(t). \end{aligned} \quad (72)$$

By taking into account relation (7) and equations (61)–(64), it comes out that

$$\begin{aligned}
\dot{x}(t) - M_0 x(t) = {} & \Delta M_0 x(t) + \sum_{l=1}^{s} (M_l + \Delta M_l) x(t-\tau_l) \\
& - \frac{1}{2} (\mathscr{F} + \Delta\mathscr{F}) (K \otimes I_n) (x(t) \otimes x(t)) + \frac{1}{2} B\overline{N}\delta \\
& - \frac{1}{2} \sum_{l=1}^{s} (\mathscr{F} + \Delta\mathscr{F}) (\overline{K}_l \otimes I_n) (x(t-\tau_l) \otimes x(t)) \\
& - \frac{1}{2} \sum_{l=1}^{s} (\mathscr{H}_l + \Delta\mathscr{H}_l) (K \otimes I_n) (x(t) \otimes x(t-\tau_l)) \\
& - \frac{1}{2} \sum_{j=1}^{s} \sum_{l=1}^{s} (\mathscr{H}_l + \Delta\mathscr{H}_l) (\overline{K}_j \otimes I_n) \left(x\left(t-\tau_j\right) \otimes x(t-\tau_l) \right) \\
& + (\mathscr{F} + \Delta\mathscr{F}) \left(\left(\operatorname{sat}(u(t)) - \frac{1}{2} u(t) \right) \otimes x(t) \right) \\
& + \sum_{l=1}^{s} (\mathscr{H}_l + \Delta\mathscr{H}_l) \left(\left(\operatorname{sat}(u(t)) - \frac{1}{2} u(t) \right) \otimes x(t-\tau_l) \right) + B\left(\operatorname{sat}(u(t)) - \frac{1}{2} u(t) \right).
\end{aligned} \tag{73}$$

Suppose that $x(t)$ is a solution of equation (73) with $x_0 = \zeta(0)$; then, by taking into account the fact that $M_0 e^{-M_0 t} = e^{-M_0 t} M_0$ which may be shown directly from the series definition, we can write

$$\begin{aligned}
\frac{d}{dt}\left(e^{-M_0 t} x(t) \right) &= -M_0 e^{-M_0 t} x(t) + e^{-M_0 t} \dot{x}(t) = e^{-M_0 t} (\dot{x}(t) - M_0 x(t)) \\
&= e^{-M_0 t} \left(\Delta M_0 x(t) + \sum_{l=1}^{s} (M_l + \Delta M_l) x(t-\tau_l) \right) \\
&\quad - \frac{1}{2} e^{-M_0 t} \left((\mathscr{F} + \Delta\mathscr{F}) (K \otimes I_n) (x(t) \otimes x(t)) - B\overline{N}\delta \right) \\
&\quad - \frac{1}{2} e^{-M_0 t} \left(\sum_{l=1}^{s} (\mathscr{F} + \Delta\mathscr{F}) (\overline{K}_l \otimes I_n) (x(t-\tau_l) \otimes x(t)) \right) \\
&\quad - \frac{1}{2} e^{-M_0 t} \left(\sum_{l=1}^{s} (\mathscr{H}_l + \Delta\mathscr{H}_l) (K \otimes I_n) (x(t) \otimes x(t-\tau_l)) \right) \\
&\quad - \frac{1}{2} e^{-M_0 t} \left(\sum_{j=1}^{s} \sum_{l=1}^{s} (\mathscr{H}_l + \Delta\mathscr{H}_l) (\overline{K}_j \otimes I_n) \left(x\left(t-\tau_j\right) \otimes x(t-\tau_l) \right) \right) \\
&\quad + e^{-M_0 t} \left((\mathscr{F} + \Delta\mathscr{F}) \left(\left(\operatorname{sat}(u(t)) - \frac{1}{2} u(t) \right) \otimes x(t) \right) \right) \\
&\quad + e^{-M_0 t} \left(\sum_{l=1}^{s} (\mathscr{H}_l + \Delta\mathscr{H}_l) \left(\left(\operatorname{sat}(u(t)) - \frac{1}{2} u(t) \right) \otimes x(t-\tau_l) \right) \right) \\
&\quad + e^{-M_0 t} \left(B\left(\operatorname{sat}(u(t)) - \frac{1}{2} u(t) \right) \right),
\end{aligned} \tag{74}$$

and then the integration of the last equation from 0 to t gives

$$\begin{aligned}
e^{-M_0 t}\dot{x}(t) - x_0 = {} & \int_0^t e^{-M_0\sigma}\left(\Delta M_0 x(\sigma) + \sum_{l=1}^{s}(M_l + \Delta M_l)x(\sigma - \tau_l)\right)\mathrm{d}\sigma \\
& -\frac{1}{2}\int_0^t e^{-M_0\sigma}\left((\mathscr{F} + \Delta\mathscr{F})(K \otimes I_n)(x(\sigma)\otimes x(\sigma)) - B\overline{N}\delta\right)\mathrm{d}\sigma \\
& -\frac{1}{2}\int_0^t e^{-M_0\sigma}\left(\sum_{l=1}^{s}(\mathscr{F} + \Delta\mathscr{F})(\overline{K}_l \otimes I_n)(x(\sigma - \tau_l)\otimes x(\sigma))\right)\mathrm{d}\sigma \\
& -\frac{1}{2}\int_0^t e^{-M_0\sigma}\left(\sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)(K \otimes I_n)(x(\sigma)\otimes x(\sigma - \tau_l))\right)\mathrm{d}\sigma \\
& -\frac{1}{2}\int_0^t e^{-M_0\sigma}\left(\sum_{j=1}^{s}\sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)(\overline{K}_j \otimes I_n)\left(x\left(\sigma - \tau_j\right)\otimes x(\sigma - \tau_l)\right)\right)\mathrm{d}\sigma \\
& + \int_0^t e^{-M_0\sigma}\left((\mathscr{F} + \Delta\mathscr{F})\left(\left(\operatorname{sat}(u(\sigma)) - \frac{1}{2}u(\sigma)\right)\otimes x(\sigma)\right)\right)\mathrm{d}\sigma \\
& + \int_0^t e^{-M_0\sigma}\left(\sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)\left(\left(\operatorname{sat}(u(\sigma)) - \frac{1}{2}u(\sigma)\right)\otimes x(\sigma - \tau_l)\right)\right)\mathrm{d}\sigma \\
& + \int_0^t e^{-M_0\sigma}\left(B\left(\operatorname{sat}(u(\sigma)) - \frac{1}{2}u(\sigma)\right)\right)\mathrm{d}\sigma,
\end{aligned} \tag{75}$$

which is equivalent to

$$\begin{aligned}
\dot{x}(t) = e^{M_0 t}x_0 & + \int_0^t e^{M_0(t-\sigma)}\left(\Delta M_0 x(\sigma) + \sum_{l=1}^{s}(M_l + \Delta M_l)x(\sigma - \tau_l)\right)\mathrm{d}\sigma \\
& -\frac{1}{2}\int_0^t e^{M_0(t-\sigma)}\left((\mathscr{F} + \Delta\mathscr{F})(K \otimes I_n)(x(\sigma)\otimes x(\sigma)) - B\overline{N}\delta\right)\mathrm{d}\sigma \\
& -\frac{1}{2}\int_0^t e^{M_0(t-\sigma)}\left(\sum_{l=1}^{s}(\mathscr{F} + \Delta\mathscr{F})(\overline{K}_l \otimes I_n)(x(\sigma - \tau_l)\otimes x(\sigma))\right)\mathrm{d}\sigma \\
& -\frac{1}{2}\int_0^t e^{M_0(t-\sigma)}\left(\sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)(K \otimes I_n)(x(\sigma)\otimes x(\sigma - \tau_l))\right)\mathrm{d}\sigma \\
& -\frac{1}{2}\int_0^t e^{M_0(t-\sigma)}\left(\sum_{j=1}^{s}\sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)(\overline{K}_j \otimes I_n)\left(x\left(\sigma - \tau_j\right)\otimes x(\sigma - \tau_l)\right)\right)\mathrm{d}\sigma \\
& + \int_0^t e^{-M_0(t-\sigma)}\left((\mathscr{F} + \Delta\mathscr{F})\left(\left(\operatorname{sat}(u(\sigma)) - \frac{1}{2}u(\sigma)\right)\otimes x(\sigma)\right)\right)\mathrm{d}\sigma \\
& + \int_0^t e^{-M_0(t-\sigma)}\left(\sum_{l=1}^{s}(\mathscr{H}_l + \Delta\mathscr{H}_l)\left(\left(\operatorname{sat}(u(\sigma)) - \frac{1}{2}u(\sigma)\right)\otimes x(\sigma - \tau_l)\right)\right)\mathrm{d}\sigma \\
& + \int_0^t e^{-M_0(t-\sigma)}\left(B\left(\operatorname{sat}(u(\sigma)) - \frac{1}{2}u(\sigma)\right)\right)\mathrm{d}\sigma.
\end{aligned} \tag{76}$$

If the obtained gain matrices $\overline{N}$, K, and $\overline{K}_l$ for each $l \in \{1, \ldots, s\}$ verify that all eigenvalues of matrix M_0 have a strictly negative real part such that relation (70) holds on, then $\|x(t)\|$ can be bounded as

$$\begin{aligned}
\|x(t)\| \le{}& \lambda e^{\omega t}\|x_0\| + \lambda e^{\omega t}\|\Delta M_0\| \int_0^t e^{-\omega\sigma}\|x(\sigma)\| \mathrm{d}\sigma \\
&+ \lambda e^{\omega t}\sum_{l=1}^{s}\|(M_l + \Delta M_l)\| \int_0^t e^{-\omega\sigma}\|x(\sigma - \tau_l)\| \mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda e^{\omega t}\|(\mathscr{F} + \Delta\mathscr{F})\| \|(K \otimes I_n)\| \int_0^t e^{-\omega\sigma}\|(x(\sigma) \otimes x(\sigma))\| \mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda e^{\omega t}\|B\overline{N}\delta\| \int_0^t e^{-\omega\sigma} \mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda e^{\omega t}\|(\mathscr{F} + \Delta\mathscr{F})\|\sum_{l=1}^{s}\|(\overline{K}_l \otimes I_n)\| \int_0^t e^{-\omega\sigma}\|(x(\sigma - \tau_l) \otimes x(\sigma))\| \mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda e^{\omega t}\|(K \otimes I_n)\|\sum_{l=1}^{s}\|(\mathscr{H}_l + \Delta\mathscr{H}_l)\| \int_0^t e^{-\omega\sigma}\|(x(\sigma) \otimes x(\sigma - \tau_l))\| \mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda e^{\omega t}\sum_{l=1}^{s}\sum_{j=1}^{s}\|(\mathscr{H}_l + \Delta\mathscr{H}_l)\| \|(\overline{K}_j \otimes I_n)\| \int_0^t e^{-\omega\sigma}\left\|\left(x\left(\sigma - \tau_j\right) \otimes x(\sigma - \tau_l)\right)\right\| \mathrm{d}\sigma \\
&+ \lambda e^{\omega t}\|(\mathscr{F} + \Delta\mathscr{F})\| \int_0^t e^{-\omega\sigma}\left\|\left(\left(\mathrm{sat}(u(\sigma)) - \frac{1}{2}u(\sigma)\right) \otimes x(\sigma)\right)\right\| \mathrm{d}\sigma \\
&+ \lambda e^{\omega t}\sum_{l=1}^{s}\|(\mathscr{H}_l + \Delta\mathscr{H}_l)\| \int_0^t e^{-\omega\sigma}\left\|\left(\left(\mathrm{sat}(u(\sigma)) - \frac{1}{2}u(\sigma)\right) \otimes x(\sigma - \tau_l)\right)\right\| \mathrm{d}\sigma \\
&+ \lambda e^{\omega t}\|B\| \int_0^t e^{-\omega\sigma}\left\|\left(\mathrm{sat}(u(\sigma)) - \frac{1}{2}u(\sigma)\right)\right\| \mathrm{d}\sigma.
\end{aligned} \tag{77}$$

It has been shown in [39, 40] that

$$\left\|\mathrm{sat}(u(t)) - \frac{1}{2}u(t)\right\| \le \frac{1}{2}\|u(t)\|, \tag{78}$$

and then, by taking into account relation (9), the use of the following matrix norm property, for each $l, j \in \{1, \ldots, s\}$,

$$\begin{gathered}
\left\|\left(\left(\mathrm{sat}(u(t)) - \frac{1}{2}u(t)\right) \otimes x(t)\right)\right\| \le \left\|\left(\mathrm{sat}(u(t)) - \frac{1}{2}u(t)\right)\right\| \|x(t)\| \le \\
\frac{1}{2}\|u(t)\|\|x(t)\| \le \frac{1}{2}\|\overline{N} \\
\left\|\left(\left(\mathrm{sat}(u(t)) - \frac{1}{2}u(t)\right) \otimes x(t - \tau_l)\right)\right\| \le \left\|\left(\mathrm{sat}(u(t)) - \frac{1}{2}u(t)\right)\right\| \|x(t - \tau_l)\| \le \\
\frac{1}{2}\|u(t)\|\|x(t - \tau_l)\| \le \frac{1}{2}\|\overline{N} \\
\left\|\left(\mathrm{sat}(u(t)) - \frac{1}{2}u(t)\right)\right\| \le \frac{1}{2}\|u(t)\| \le \frac{1}{2}\|u_{\max}\|,
\end{gathered} \tag{79}$$

leads to

$$\begin{aligned}
e^{-\omega t}\|x(t)\| \le{}& \lambda\|x_0\| + \lambda\|\Delta M_0\| \int_0^t e^{-\omega\sigma}\|x(\sigma)\|\mathrm{d}\sigma \\
&+ \lambda\sum_{l=1}^{s}\|(M_l + \Delta M_l)\| \int_0^t e^{-\omega\sigma}\|x(\sigma-\tau_l)\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\|(\mathscr{F}+\Delta\mathscr{F})\|\|(K\otimes I_n)\| \int_0^t e^{-\omega\sigma}\|(x(\sigma)\otimes x(\sigma))\|\mathrm{d}\sigma + \frac{1}{2}\lambda\|B\overline{N}\delta\| \int_0^t e^{-\omega\sigma}\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\|(\mathscr{F}+\Delta\mathscr{F})\|\sum_{l=1}^{s}\|(\overline{K}_l\otimes I_n)\| \int_0^t e^{-\omega\sigma}\|(x(\sigma-\tau_l)\otimes x(\sigma))\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\|(K\otimes I_n)\|\sum_{l=1}^{s}\|(\mathscr{H}_l+\Delta\mathscr{H}_l)\| \int_0^t e^{-\omega\sigma}\|(x(\sigma)\otimes x(\sigma-\tau_l))\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\sum_{l=1}^{s}\sum_{j=1}^{s}\|(\mathscr{H}_l+\Delta\mathscr{H}_l)\|\|(\overline{K}_j\otimes I_n)\| \int_0^t e^{-\omega\sigma}\left\|\left(x(\sigma-\tau_j)\otimes x(\sigma-\tau_l)\right)\right\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\|(\mathscr{F}+\Delta\mathscr{F})\| \int_0^t e^{-\omega\sigma}\left(\|\overline{N}\delta\|\|x(\sigma)\| + \|K\|\|x(\sigma)\|^2\right)\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\|(\mathscr{F}+\Delta\mathscr{F})\|\sum_{l=1}^{s}\|\overline{K}_l\| \int_0^t e^{-\omega\sigma}\|x(\sigma-\tau_l)\|\|x(\sigma)\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\sum_{l=1}^{s}\|(\mathscr{H}_l+\Delta\mathscr{H}_l)\| \int_0^t e^{-\omega\sigma}\left(\|\overline{N}\delta\|\|x(\sigma-\tau_l)\| + \|K\|\|x(\sigma)\|\|x(\sigma-\tau_l)\|\right)\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\sum_{l=1}^{s}\sum_{j=1}^{s}\|(\mathscr{H}_l+\Delta\mathscr{H}_l)\|\|\overline{K}_j\| \int_0^t e^{-\omega\sigma}\left\|x(\sigma-\tau_j)\right\|\|x(\sigma-\tau_l)\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\|B\|\|u_{\max}\| \int_0^t e^{-\omega\sigma}\mathrm{d}\sigma.
\end{aligned} \tag{80}$$

From relation (65), it comes out that

$$\begin{aligned}
e^{-\omega t}\|x(t)\| \leq{}& \lambda\|x_0\| + \lambda\overline{\beta}_0 \int_0^t e^{-\omega\sigma}\|x(\sigma)\|\mathrm{d}\sigma \\
&+ \lambda\sum_{l=1}^{s}\left(\|M_l\| + \overline{\beta}_l\right) \int_0^t e^{-\omega\sigma}\|x(\sigma-\tau_l)\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda(\|\mathscr{F}\| + \gamma_2)\|(K\otimes I_n)\| \int_0^t e^{-\omega\sigma}\|(x(\sigma)\otimes x(\sigma))\|\mathrm{d}\sigma + \frac{1}{2}\lambda\|B\overline{N}\delta\|\int_0^t e^{-\omega\sigma}\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda(\|\mathscr{F}\| + \gamma_2)\sum_{l=1}^{s}\|(\overline{K}_l\otimes I_n)\| \int_0^t e^{-\omega\sigma}\|(x(\sigma-\tau_l)\otimes x(\sigma))\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\|(K\otimes I_n)\|\sum_{l=1}^{s}\left(\|\mathscr{H}_l\| + \overline{\eta}_l\right) \int_0^t e^{-\omega\sigma}\|(x(\sigma)\otimes x(\sigma-\tau_l))\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\sum_{l=1}^{s}\sum_{j=1}^{s}\left(\|\mathscr{H}_l\| + \overline{\eta}_l\right)\|(\overline{K}_j\otimes I_n)\| \int_0^t e^{-\omega\sigma}\|(x(\sigma-\tau_j)\otimes x(\sigma-\tau_l))\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda(\|\mathscr{F}\| + \gamma_2) \int_0^t e^{-\omega\sigma}(\|\overline{N}\delta\|\|x(\sigma)\| + \|K\|\|x(\sigma)\|^2)\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda(\|\mathscr{F}\| + \gamma_2)\sum_{l=1}^{s}\|\overline{K}_l\| \int_0^t e^{-\omega\sigma}\|x(\sigma-\tau_l)\|\|x(\sigma)\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\sum_{l=1}^{s}\left(\|\mathscr{H}_l\| + \overline{\eta}_l\right) \int_0^t e^{-\omega\sigma}(\|\overline{N}\delta\|\|x(\sigma-\tau_l)\| + \|K\|\|x(\sigma)\|\|x(\sigma-\tau_l)\|)\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\sum_{l=1}^{s}\sum_{j=1}^{s}\left(\|\mathscr{H}_l\| + \overline{\eta}_l\right)\|\overline{K}_j\| \int_0^t e^{-\omega\sigma}\|x(\sigma-\tau_j)\|\|x(\sigma-\tau_l)\|\mathrm{d}\sigma \\
&+ \frac{1}{2}\lambda\|B\|\|u_{\max}\| \int_0^t e^{-\omega\sigma}\mathrm{d}\sigma.
\end{aligned} \tag{81}$$

Let us assume that

$$\|x(t)\| < R. \tag{82}$$

Then, using the following matrix norm property, for each $l, j \in \{1, \ldots, s\}$,

$$\begin{aligned}
\|x(t) \otimes x(t)\| &< R\|x(t)\|, \\
\left\|x(t-\tau_l) \otimes x(t)\right\| &< R\|x(t)\|, \\
\left\|x(t) \otimes x(t-\tau_l)\right\| &< R\|x(t)\|, \\
\left\|x\left(t-\tau_j\right) \otimes x(t-\tau_l)\right\| &< R\left\|x(t-\tau_l)\right\|,
\end{aligned} \tag{83}$$

and inequality (81) could be written as

$$\begin{aligned}
e^{-\omega t}\|x(t)\| \le{}& \lambda\left\|x_0\right\| + \lambda\overline{\beta}_0 \int_0^t e^{-\omega\sigma}\|x(\sigma)\| d\sigma \\
&+ \lambda\sum_{l=1}^{s}\left(\left\|M_l\right\| + \overline{\beta}_l\right) \int_0^t e^{-\omega\sigma}\left\|x(\sigma-\tau_l)\right\| d\sigma \\
&+ \frac{1}{2}\lambda\left(\|\mathscr{F}\| + \gamma_2\right)\left\|(K \otimes I_n)\right\| R \int_0^t e^{-\omega\sigma}\|x(\sigma)\| d\sigma + \frac{1}{2}\lambda\|B\overline{N}\delta\| \int_0^t e^{-\omega\sigma} d\sigma \\
&+ \frac{1}{2}\lambda\left(\|\mathscr{F}\| + \gamma_2\right)\sum_{l=1}^{s}\left\|(\overline{K}_l \otimes I_n)\right\| R \int_0^t e^{-\omega\sigma}\|x(\sigma)\| d\sigma \\
&+ \frac{1}{2}\lambda\left\|(K \otimes I_n)\right\|\sum_{l=1}^{s}\left(\left\|\mathscr{H}_l\right\| + \overline{\eta}_l\right) R \int_0^t e^{-\omega\sigma}\|x(\sigma)\| d\sigma \\
&+ \frac{1}{2}\lambda\sum_{l=1}^{s}\sum_{j=1}^{s}\left(\left\|\mathscr{H}_l\right\| + \overline{\eta}_l\right)\left\|\left(\overline{K}_j \otimes I_n\right)\right\| R \int_0^t e^{-\omega\sigma}\left\|x(\sigma-\tau_l)\right\| d\sigma \\
&+ \frac{1}{2}\lambda\left(\|\mathscr{F}\| + \gamma_2\right)\|\overline{N}\delta\| \int_0^t e^{-\omega\sigma}\|x(\sigma)\| d\sigma \\
&+ \frac{1}{2}\lambda\left(\|\mathscr{F}\| + \gamma_2\right)\left(\|K\| + \sum_{l=1}^{s}\left\|\overline{K}_l\right\|\right) R \int_0^t e^{-\omega\sigma}\|x(\sigma)\| d\sigma \\
&+ \frac{1}{2}\lambda\|\overline{N}\delta\|\sum_{l=1}^{s}\left(\left\|\mathscr{H}_l\right\| + \overline{\eta}_l\right) \int_0^t e^{-\omega\sigma}\left\|x(\sigma-\tau_l)\right\| d\sigma \\
&+ \frac{1}{2}\lambda\|K\|\sum_{l=1}^{s}\left(\left\|\mathscr{H}_l\right\| + \overline{\eta}_l\right) R \int_0^t e^{-\omega\sigma}\|x(\sigma)\| d\sigma \\
&+ \frac{1}{2}\lambda\sum_{l=1}^{s}\sum_{j=1}^{s}\left(\left\|\mathscr{H}_l\right\| + \overline{\eta}_l\right)\left\|\overline{K}_j\right\| R \int_0^t e^{-\omega\sigma}\left\|x(\sigma-\tau_l)\right\| d\sigma \\
&+ \frac{1}{2}\lambda\|B\|\left\|u_{\max}\right\| \int_0^t e^{-\omega\sigma} d\sigma.
\end{aligned} \tag{84}$$

Let

$$X(t) = \left\{\sup_{\rho\in[t-\bar{\tau},t]} \|x(\rho)\|\right\} e^{-\omega t}, \quad \forall t \geq 0, \tag{85}$$

and then, by taking into account the fact that $\forall t \geq 0$ and $l \in \{1, \ldots, s\}$

$$\begin{aligned} \|x(t)\| &\leq \sup_{\rho\in[t-\bar{\tau},t]} \|x(\rho)\|, \\ \left\|x(t-\tau_l)\right\| &\leq \sup_{\rho\in[t-\bar{\tau},t]} \|x(\rho)\|, \end{aligned} \tag{86}$$

it comes out that

$$\begin{aligned} e^{-\omega t}\|x(t)\| \quad &\leq \lambda\left\|x_0\right\| + \lambda Q_1 \int_0^t X(\sigma)\mathrm{d}\sigma + \lambda Q_2 R \\ &\cdot \int_0^t X(\sigma)\mathrm{d}\sigma + \lambda Q_3 \int_0^t e^{-\omega\sigma}\mathrm{d}\sigma \end{aligned}, \tag{87}$$

where Q_1, Q_2, and Q_3 are given in relation (66).

Let us define $S(t)$ as the right-hand member of inequality (84):

$$S(t) = \lambda\left\|x_0\right\| + \lambda\left(Q_1 + Q_2 R\right) \int_0^t X(\sigma)\mathrm{d}\sigma + \lambda Q_3 \int_0^t e^{-\omega\sigma}\mathrm{d}\sigma. \tag{88}$$

It is clear that $S(t)$ is a nondecreasing function in terms of the definition of $X(t)$; then, from inequality (87), we can have

$$\forall t \geq 0, \quad X(t) < S(t). \tag{89}$$

The derivation of function $S(t)$ leads to

$$\frac{dS(t)}{\mathrm{d}t} = \lambda\left(Q_1 + Q_2 R\right) X(t) + \lambda Q_3 e^{-\omega t}, \tag{90}$$

and the following inequality can be deduced from (87):

$$\frac{dS(t)}{\mathrm{d}t} < \lambda\left(Q_1 + Q_2 R\right) S(t) + \lambda Q_3 e^{-\omega t}. \tag{91}$$

Now, using Lemma 2 (see Appendix C), the integration of inequality (91) on the time interval $[0, t]$ leads to

$$S(t) < e^{\int_0^t \lambda\left(Q_1 + Q_2 R\right)\mathrm{d}\tau}\left[S(0) + \int_0^t \lambda Q_3 e^{-\omega\tau} e^{-\int_0^\tau \lambda\left(Q_1 + Q_2 R\right)\mathrm{d}\nu} d\tau\right], \tag{92}$$

where

$$S(0) = \lambda\left\|x_0\right\|. \tag{93}$$

Inequality (87) and expression (92) imply

$$\begin{aligned} e^{-\omega t}\|x(t)\| &< \lambda\left\|x_0\right\| e^{\lambda\left(Q_1+Q_2 R\right)t} \\ &+ \frac{\lambda Q_3}{\lambda\left(Q_1 + Q_2 R\right) + \omega} e^{\lambda\left(Q_1+Q_2 R\right)t} \\ &\cdot \left(1 - e^{-\left(\lambda\left(Q_1+Q_2 R\right)+\omega\right)t}\right), \end{aligned} \tag{94}$$

which is equivalent to

$$\begin{aligned} \|x(t)\| \quad &< \lambda\left\|x_0\right\| e^{\left(\lambda\left(Q_1+Q_2 R\right)+\omega\right)t} + \frac{\lambda Q_3}{\lambda\left(Q_1 + Q_2 R\right) + \omega} \\ &\cdot \left(e^{\left(\lambda\left(Q_1+Q_2 R\right)+\omega\right)t} - 1\right) \end{aligned}. \tag{95}$$

If condition (68) holds on, then, for

$$R < -\frac{1}{Q_2}\left(\frac{\omega}{\lambda} + Q_1\right), \tag{96}$$

it results that

$$\lambda\left(Q_1 + Q_2 R\right) + \omega < 0, \tag{97}$$

which permits deducing the following inequality ensuring the boundedness of solution $x(t)$:

$$\forall t \geq 0, \quad \|x(t)\| < \lambda\left\|x_0\right\| - \frac{\lambda Q_3}{\lambda\left(Q_1 + Q_2 R\right) + \omega}. \tag{98}$$

Now, to ensure the hypothesis given by inequality (82), for all $t \geq 0$, it suffices to have

$$\lambda\left\|x_0\right\| < -\frac{1}{Q_2}\left(\frac{\omega}{\lambda} + Q_1\right). \tag{99}$$

So, from inequality (98) and condition (99), it follows that

$$\begin{aligned} \left\|x_0\right\| < R_0 &= -\frac{1}{\lambda Q_2}\left(\frac{\omega}{\lambda} + Q_1\right) \Rightarrow \|x(t)\| < r \\ &= \lambda R_0 - \frac{\lambda Q_3}{\lambda\left(Q_1 + Q_2 R\right) + \omega}. \end{aligned} \tag{100}$$

Hence, the closed-loop system (12) is practically stable. □

5. Application to a Paper-Making Machine

A precision paper-making machine at a paper mill, which produces the super-thin condenser paper, can be described concisely in Figure 1.

The thick stock from pulp workshop is pumped into a mixing box where it is mixed with chemicals and white water; then the mixture is filled into the headbox through a filter in which the dregs in stock are removed. The next step is to place the stock onto the forming wire and to remove most of the water to form paper. The paper sheet goes through the press part and dryer section to remove the remaining water and mill paper and subsequently to accomplish the process of production.

As we know, the headbox system is very important in paper-making process. The level and consistency of stock in the headbox are the main factors affecting production quality. In general, we take the flow rate of white water and stock going into the mixing box as control variables to control the level and consistency of stock in headbox. Since there exists strong interaction between level control system and consistency control system [41], it is important to recall here that all published researches based on the certain and undelay bilinear modeling of paper-making machine have studied only the control problems without input constraints [41–45].

In this paper, an unstructured uncertain bilinear model with multiple time-delayed states is used to describe the dynamic behavior of the paper-making machine. Such mathematical modeling leads to complex model, which can be turned out to be difficult to apply for the synthesis of a performance controller. Nevertheless, the adopted control strategy is applied in order to guarantee the robust trackability of both level and consistency control systems for a constant reference input vector under the presence of constraints on the input vector. Thus, those features constitute the major superiority of the proposed control law compared to existing results.

5.1. System Modeling. For the mixing tank, according to the mass balance law, we have [42, 43]

$$\frac{dH_1}{dt} = \frac{1}{A_1}\left(G_p + G_w - G_1\right), \tag{101}$$

$$\frac{dN_1}{dt} = \frac{G_pN_p + G_wN_w - (H_1/R_1)N_1 - N_{10}\left(G_p + G_w - (H_1/R_1)\right)}{A_1H_{10} + G_p + G_w - (H_1/R_1)}, \tag{102}$$

where G_p, N_p are flow rate and consistency of the stock from pulp workshop, respectively; G_w, N_w are flow rate and consistency of the white water, respectively; G_1, N_1 are flow rate and consistency of the mixing stock out of mixing tank, respectively; A_1, H_1 are cross-sectional area and liquid level of the mixing tank, respectively; H_{10} is the steady state of liquid level H_1; N_{10} is the steady state of consistency of the mixing stock out N_1; and R_1 is the flow resistance of mixing tank.

Using steady-state operation conditions,

$$\begin{aligned} G_{10} &= G_{p0} + G_{w0}, \\ G_{10}N_{10} &= G_{p0}N_{p0} + G_{w0}N_{w0}, \end{aligned} \tag{103}$$

and approximate relationship is

$$\partial G_1 = \frac{\partial H_1}{R_1}, \tag{104}$$

and then, by taking the deviation of equations (101) and (102), we get

$$\frac{dH_1}{dt} = \frac{1}{A_1}\left(G_p + G_w - \frac{H_1}{R_1}\right), \tag{105}$$

$$\begin{aligned} \frac{dN_1}{dt} = {} & \frac{1}{A_1H_{10}}\left(\left(N_{p0} - N_{10}\right)G_p + (N_{w0} - N_{10})G_w - \frac{H_{10}}{R_1}N_1\right) \\ & + \frac{H_{10}}{(A_1H_{10})^2R_1}\left(G_pN_1 + G_wN_1\right) \\ & + \frac{1}{(A_1H_{10})^2R_1}\left(\left(N_{p0} - N_{10}\right)G_pH_1 \right. \\ & \left. + (N_{w0} - N_{10})G_wH_1\right). \end{aligned} \tag{106}$$

Similarly, for the headbox, we have

$$\frac{dH_2}{dt} = \frac{1}{A_2R_1}H_1 - \frac{1}{A_2R_2}H_2, \tag{107}$$

$$\begin{aligned} \frac{dN_2}{dt} = {} & \frac{H_{10}}{A_2H_{20}R_1}N_1 + \frac{N_{10} - N_{20}}{A_2H_{20}R_1}H_1 - \frac{N_{20} - N_{10}}{A_2H_{20}R_1}H_2 \\ & + \frac{1}{A_2R_2}N_2, \end{aligned} \tag{108}$$

where H_2, N_2 are the level and consistency of the stock in headbox, respectively; A_2 is the cross-sectional area of the headbox; and R_2 is the flow resistance of the headbox.

Substituting all steady-state data

$$\begin{array}{ll} G_{p0} = 5.28T/h & N_{p0} = 1.015\% \\ G_{w0} = 11.64T/h & N_{w0} = 0.05\% \\ H_{10} = 650\,mm\,H_2O & H_{20} = 190\,mm\,H_2O \\ N_{10} = 0.35\% & N_{20} = 0.34\% \end{array} \tag{109}$$

into equations (105)–(108), we finally get the bilinear model for the headbox system as follows:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{2} A_i x(t) u_i(t) + Bu(t), \tag{110}$$

where state vector $x(t)$, input vector $u(t)$, and system matrices are given as follows:

$$\begin{aligned}
x(t) &= \begin{bmatrix} H_1 \\ H_2 \\ N_1 \\ N_2 \end{bmatrix}, \\
A_0 &= \begin{bmatrix} -1.930 & 0 & 0 & 0 \\ 0.394 & -0.426 & 0 & 0 \\ 0 & 0 & -0.63 & 0 \\ 0.095 & -0.103 & 0.413 & -0.426 \end{bmatrix}, \\
u(t) &= \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} G_p \\ G_w \end{bmatrix}, \\
B &= \begin{bmatrix} 1.274 & 1.274 \\ 0 & 0 \\ 1.34 & -0.65 \\ 0 & 0 \end{bmatrix}, \\
A_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.755 & 0 & -0.718 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.366 & 0 & -0.718 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{111}$$

In our paper, we assume the following:

(i) All state variables of the considered bilinear system are measurable and accompanied by two time delays in states and nonconstant initial conditions. According to state model (110), it comes out that

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{l=1}^{2} D_{l0} x(t-\tau_l) + \sum_{i=1}^{2} A_i x(t) u_i(t) \\ + \sum_{l=1}^{2}\sum_{i=1}^{2} D_{li} x(t-\tau_l) u_i(t) + Bu(t), \\ x(t) = \zeta(t) = \left[\, 1-\cos(t) \;\; \sin(t) \;\; 1-\cos(t) \;\; \sin(t) \,\right]^T, \\ \qquad t \in [-3, 0], \end{cases} \tag{112}$$

with

$$\begin{aligned}
D_{10} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
D_{20} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
D_{11} &= \begin{bmatrix} 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
D_{12} &= \begin{bmatrix} 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
D_{21} &= \begin{bmatrix} -0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
D_{22} &= \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{113}$$

and $\tau_1 = 2s$ and $\tau_2 = 3s$.

(ii) Most matrices of the adopted time delays bilinear system (112) are affected by additional uncertainties satisfying assumptions (5) and (6), with

$$\begin{aligned}
\gamma_1 &= 0, \\
\gamma_2 &= 0, \\
\eta_1 &= 0.1, \\
\eta_2 &= 0.1, \\
\overline{\eta}_1 &= 0.1, \\
\overline{\eta}_2 &= 0.1.
\end{aligned} \tag{114}$$

(iii) The output vector is given by

$$y(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}=\begin{bmatrix} H_2 \\ 2N_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} x(t)=Cx(t). \tag{115}$$

(iv) The actuator vector $u_i(t)$ for each $i \in \{1,2\}$ should satisfy the constraint conditions (8), with

$$\begin{aligned} -1 &\le u_1(t) \le 1, \\ -1 &\le u_2(t) \le 1. \end{aligned} \tag{116}$$

5.2. Control Specifications and Considered Reference Model. The main objective is to synthesize a memory state feedback control with feedforward gain (7) in order to keep the outputs of the controlled system to the value of 1. For this purpose, let us consider the following matrices system of the reference model (13):

$$E=\begin{bmatrix} -0.5 & -0.4 & 0 & -0.2 \\ 0.4 & -0.45 & 0 & 0 \\ 0 & 0 & -1 & -0.2 \\ 0.1 & -0.1 & 0.42 & -0.42 \end{bmatrix}, \quad F=\begin{bmatrix} 0.95 & 0.1 \\ 0 & 0 \\ -0.12 & 0.7 \\ 0 & 0 \end{bmatrix}, \quad G=C. \tag{117}$$

5.3. Simulation Results and Discussion. With respect to a unit step input $y_{c1}(t)=y_{c2}(t)=1$, the analytical solution of the linear reference can be easily found. In order to provide the suitable choice of the number of block-pulse functions, we compare the exact solution $x_r(t)$ with approximate solution, given by

$$x_r(t)=X_{rN}S_N(t). \tag{118}$$

For $N=2^6$ (number of BPFs) and $T=16$s, it can be observed from Figure 2 that we have an excellent approximation of the exact solution $x_r(t)$.

The implementation of the proposed tracking control approach leads to the following control gains:

$$\begin{aligned} \overline{N} &= \begin{bmatrix} 0.250 & 0.347 \\ 0.485 & -0.320 \end{bmatrix}, \\ K &= \begin{bmatrix} -0.360 & 0.100 & 0.186 & 0.150 \\ -0.750 & 0.210 & -0.186 & 0 \end{bmatrix}, \\ \overline{K}_1 &= \begin{bmatrix} 0 & -0.5025 & 0.0385 & -0.0231 \\ 0 & 0.5025 & 0.0792 & -0.0476 \end{bmatrix}, \\ \overline{K}_2 &= \begin{bmatrix} -0.0385 & 0.0231 & 0 & 0.5025 \\ -0.0792 & 0.0476 & 0 & -0.5025 \end{bmatrix}. \end{aligned} \tag{119}$$

In Figure 3, the step responses of the reference model and the controlled nominal bilinear system with multiple time-delayed states under bounded input control are plotted. The figure shows that both level and consistency of the stock in headbox increase quickly to their final nonzero values at settling time of 15 s and without overshoot. During the transient state, the errors between the reference model outputs and the controlled system outputs are due to the large values of time delays, which are taken into account in modeling step of the paper-making machine.

It can be seen that the memory state feedback controllers with compensator gain, applied to the considered system, permit achieving the purpose. In fact, the closed-loop performance of the system is judged to be very satisfactory with respect to the effectiveness trackability of the reference input vector. This is confirmed by a zero steady-state error in level and consistency of the stock in headbox.

From what has been stated above, it can be deduced that the dynamic of the controlled system through the proposed control law is essentially characterized by a satisfactory speed convergence time, acceptable damping behavior, and excellent tracking performance.

Figure 4 illustrates the variation of the control signals $u_1(t)$ and $u_2(t)$. It can be observed from the simulation results that the proposed memory state feedback controller can cope well with the hard input constraints (116).

By taking into account the fact that the actuator limitations have been incorporated into control design, the potential ability of the designed closed-loop system to withstand these constraints can be seen from simulation results.

The step responses of the reference model and the closed-loop bilinear system with both time delays and uncertainty are depicted in Figure 5. The obtained results show the applicability of the robust tracking controller designed and its ability to cope with model uncertainties. Thus, it appears that the proposed robust tracking controller design approach can achieve an admissible tracking performance with an acceptable settling time and a little overshoot transient response. Hence, this can be evaluated as a second advantage of the investigated approach in this paper.

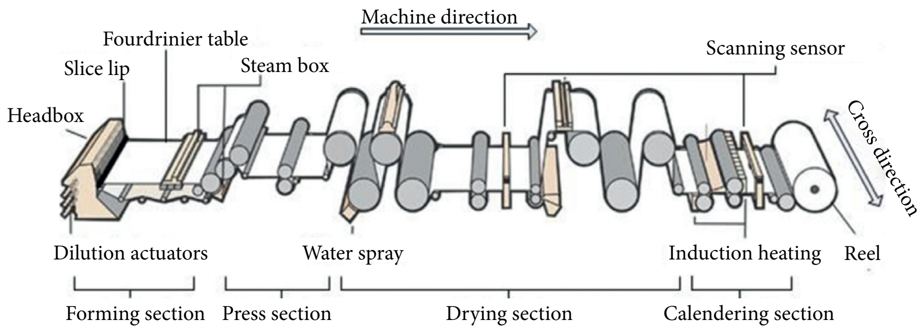

Figure 1: Sections of the paper-making process.

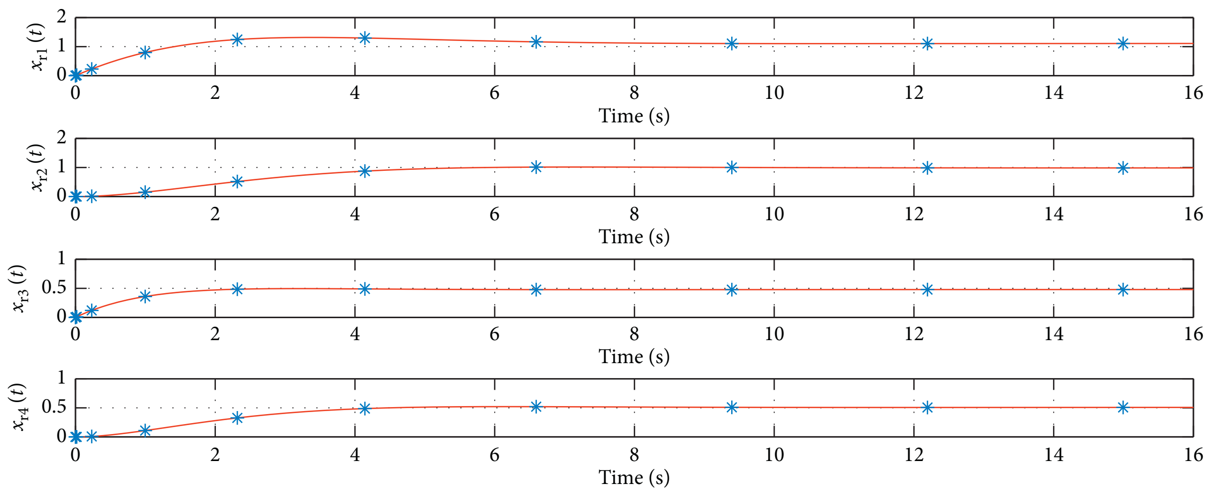

Figure 2: State variables of the reference model: –, exact solutions; ∗, BPFs approximations.

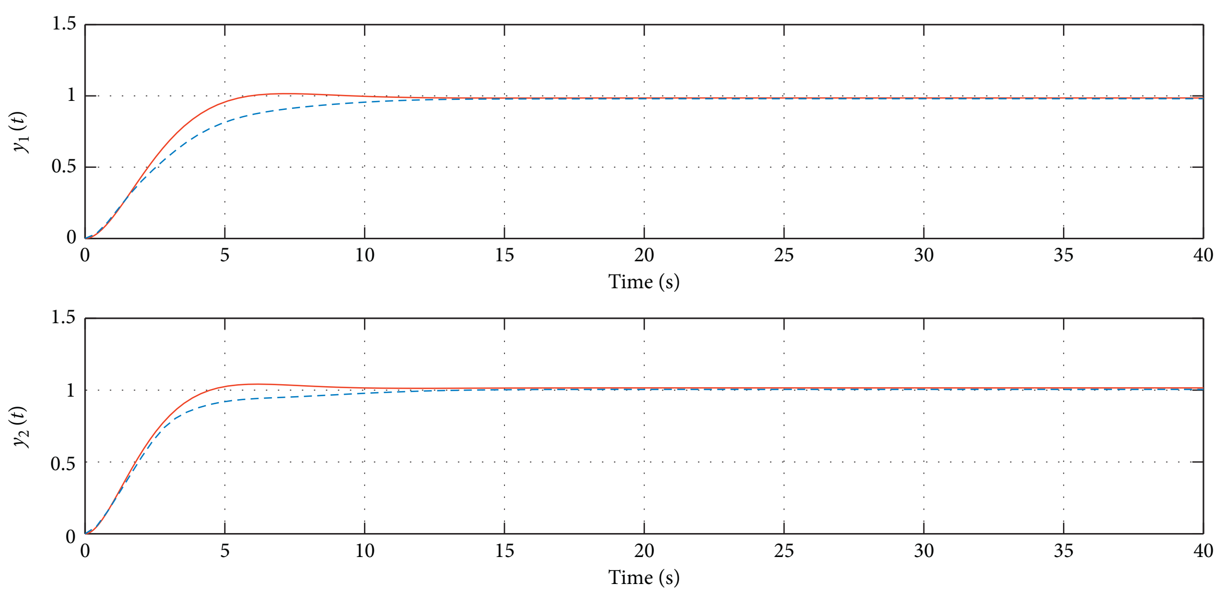

Figure 3: Step responses of the reference model, –; the controlled nominal bilinear system with multiple time-delayed states under bounded input control,– –.

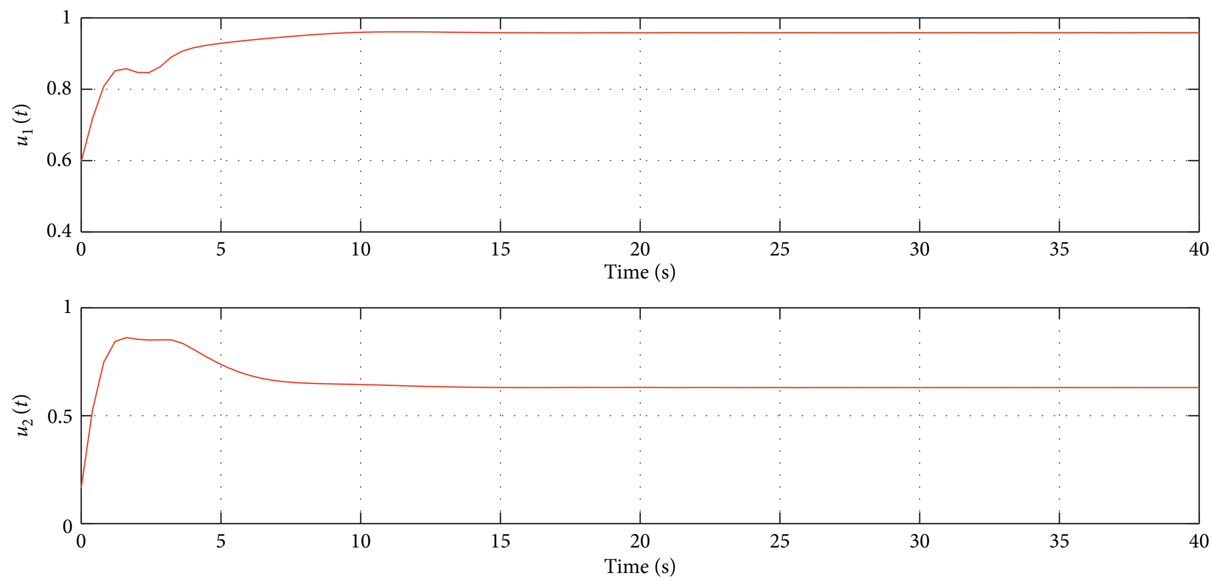

Figure 4: Variation of the control signals $u_1(t)$ and $u_2(t)$.

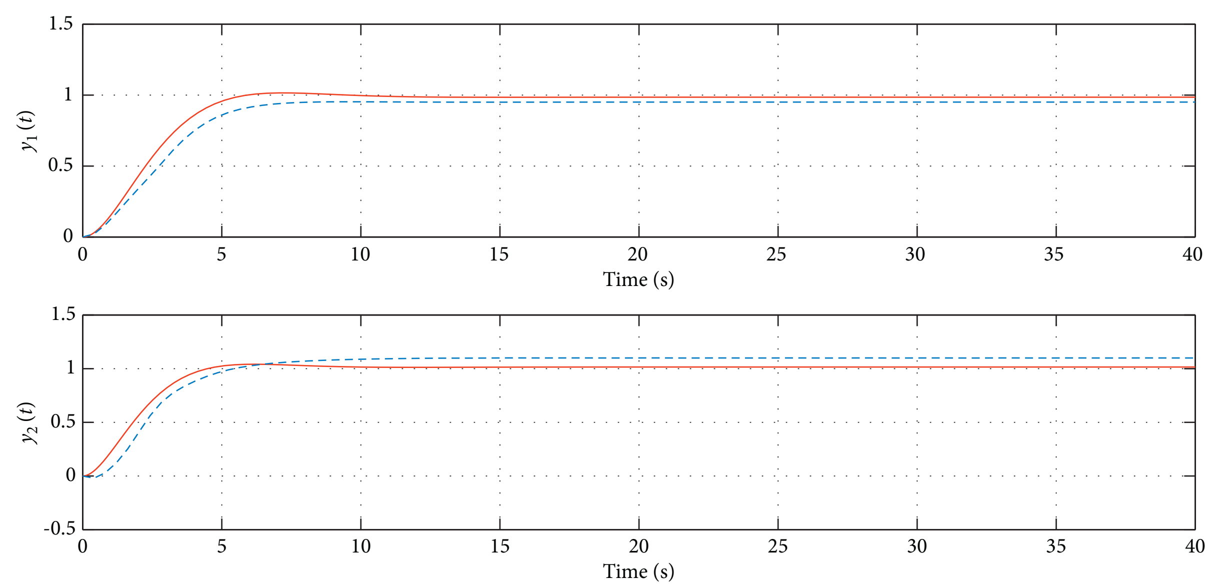

Figure 5: Step responses of the reference model, –; the controlled uncertain time delays bilinear system,– –.

5.4. Practical Stability Test. Now, we can verify that all eigenvalues of matrix M_0 have a strictly negative real part, and

$$\left\| e^{M_0 t} \right\| \le 1.05 e^{-0.4t}, \quad \forall t \ge 0, \tag{120}$$

which corresponds to inequality (70) with $\lambda = 1.05$ and $\omega = -0.4$ and satisfying condition (68) with $Q_1 = 0.3797$; then, from Theorem 1, we can conclude that the closed-loop system is practically stable for each initial state satisfying $\|x(0)\| < 0.0002$.

6. Conclusion

In this paper, we have developed a new algebraic approach to design memory tracking controllers with compensator gain for bilinear time delays systems with unstructured norm bounded uncertainties and under bounded input control. Due to the elegant operational properties of block-pulse functions as a basis, the proposed approach of control design has been formulated as a constrained and unstructured linear system of algebraic equations, depending on the parameters of the feedback regulator, which has been treated as an optimization problem. Sufficient conditions for the practical stability of the controlled system have been derived. Lastly, a paper-making machine as an example has been used to illustrate our results.

Note that this algebraic technique based on the projection of the system model on an orthogonal functions basis has the ability to be investigated as a promising method for the saturated tracking control of more complex dynamic systems.

In the future work, we intend to extend the actual study to the synthesis of the saturated tracking controller for continuous-time nonlinear polynomial systems affected with both unstructured uncertainties and time delays in states.

Appendix

A. Block-Pulse Functions and Their Properties

N-set of block-pulse functions (BPF) over the interval $[0, T]$ is defined in [30] as follows:

$$\varphi_i(t) = \begin{cases} 1, & \frac{iT}{N} \le t \le \frac{(i+1)T}{N}, \text{ for } i = 0, \ldots, N-1, \\ 0, & \text{elsewhere}, \end{cases} \tag{A.1}$$

with a positive integer value for N. Also, $\varphi_i(t)$ denotes the i-th block-pulse functions. There are some properties for BPFs, and the most important properties are disjointness, orthogonality, and completeness.

So, a vector function $x(t)$ of n-dimensional components which are square-integrable in $[0, T]$ can be represented approximately by a finite block-pulse series:

$$x(t) \cong \sum_{i=0}^{N-1} x_i \varphi_i(t) = X_N S_N(t), \tag{A.2}$$

with

$$\begin{aligned} X_N &= [\, x_0 \ \ldots \ x_{N-1} \,], \\ S_N(t) &= [\, \varphi_0(t) \ \ldots \ \varphi_{N-1}(t) \,]^T, \end{aligned} \tag{A.3}$$

and x_i are the block-pulse coefficients of $x(t)$, as obtained from the orthogonality of the block-pulse functions:

$$x_i = \frac{N}{T} \int_{((iT/N))}^{(((i+1)T/N))} x(t) \mathrm{d}t. \tag{A.4}$$

Assuming that the vector $x(t)$ has its initial as

$$x(t) = \zeta(t), \quad -\tau \le t \le 0, \tag{A.5}$$

with $0 < \tau < T$, the time delay vector $x(t-\tau)$ is expressed for $t \in [0, T]$ as follows:

$$x(t-\tau) = \begin{cases} \zeta(t-\tau), & \text{for } 0 \le t \le \tau, \\ x(t-\tau), & \text{for } \tau < t \le T. \end{cases} \tag{A.6}$$

The block-pulse series approximation of time delay vector $x(t-\tau)$ is given in [31] by

$$x(t-\tau) \cong \sum_{i=0}^{N-1} x_i^*(\tau) \varphi_i(t) = X_N^*(\tau) S_N(t), \tag{A.7}$$

with

$$X_N^*(\tau) = [\, x_0^*(\tau) \ \ldots \ x_{N-1}^*(\tau) \,], \tag{A.8}$$

where

$$x_i^*(\tau) = \frac{N}{T} \int_{(iT/N)}^{((i+1)T/N)} x(t-\tau) \mathrm{d}t = \begin{cases} \zeta_i^*(\tau), & \text{for } i \le \mu, \\ x_{i-\mu}, & \text{for } i > \mu, \end{cases} \tag{A.9}$$

with μ being the number of BPFs considered over $0 \le t \le \tau$, and

$$\zeta_i^*(\tau) = \frac{N}{T} \int_{(iT/N)}^{((i+1)T/N)} \zeta(t-\tau) \mathrm{d}t. \tag{A.10}$$

Operational Matrix of Integration

The integration matrix of the BPFs is given in [30] by

$$\int_0^t S_N(t) \mathrm{d}t = P_N S_N(t), \tag{A.11}$$

$$P_N = \frac{T}{2N} \begin{bmatrix} 1 & 2 & \ldots & 2 \\ 0 & 1 & \ldots & 2 \\ \vdots & \ddots & \ddots & 2 \\ 0 & \ldots & 0 & 1 \end{bmatrix}. \tag{A.12}$$

Operational Matrix of Product

We denote by e_i^p p-dimensional unit vector which has 1 in the i^{th} element and zero elsewhere. The elementary matrix is defined by

$$E_{i,j}^{p \times q} = e_i^p \otimes e_j^{q^T}, \tag{A.13}$$

where $\otimes$ is the symbol of the Kronecker product.

Based on the disjointness property of BPFs, we have [33]

$$S_N(t) \otimes S_N(t) = \begin{bmatrix} E_{1,1}^{N \times N} \\ \vdots \\ E_{N,N}^{N \times N} \end{bmatrix} S_N(t) = M_N S_N(t). \tag{A.14}$$

B. vec Operator and Kronecker Product Properties

An important vector valued function of matrix denoted as vec( ) was defined in [35] as follows:

$$H = [\, h_1 \ h_2 \ \ldots \ h_q \,], \tag{B.1}$$

where, for all $i \in \{1, \ldots, q\}$, $h_i \in \mathbb{R}^p$ are the columns of H:

$$\mathrm{vec}(H) = [\, h_1^T \ h_2^T \ \ldots \ h_q^T \,] \in \mathbb{R}^{pq}. \tag{B.2}$$

Property 1. For any matrices X, Y, and Z having appropriate dimensions, the following property of the Kronecker product is given in [35] by

$$\mathrm{vec}(XYZ) = \left(Z^T \otimes X\right)\mathrm{vec}(Y). \tag{B.3}$$

Property 2. Let the matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$; we have [34]

$$\begin{aligned}\mathrm{vec}(A\otimes B) &= \mathrm{vec}\left(\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{i,j}^{m\times n}\right)\otimes B\right)\\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\mathrm{vec}\left(E_{i,j}^{m\times n}\otimes B\right) = \Pi_{m,n}(B)\mathrm{vec}(A),\end{aligned} \tag{B.4}$$

where

$$\Pi_{m,n}(B) = \begin{bmatrix} \mathrm{vec}\left(E_{11}^{m\times n}\otimes B\right) \vdots \dots \vdots \mathrm{vec}\left(E_{m1}^{m\times n}\otimes B\right) \vdots \\ \mathrm{vec}\left(E_{22}^{m\times n}\otimes B\right) \vdots \dots \vdots \mathrm{vec}\left(E_{m2}^{m\times n}\otimes B\right) \vdots \dots \vdots \\ \mathrm{vec}\left(E_{1n}^{m\times n}\otimes B\right) \vdots \dots \vdots \mathrm{vec}\left(E_{mn}^{m\times n}\otimes B\right) \end{bmatrix}. \tag{B.5}$$

C. Two Useful Lemmas

Lemma 1. *Given $A \in \mathbb{R}^{m\times n}$ with $m > n$, $B \in \mathbb{R}^m$, and positive real numbers (δ_A, δ_B), the following optimization problem,*

$$\min_{\theta\in\mathbb{R}^n}\max_{\|\Delta A\|\le\delta_A,\|\Delta B\|\le\delta_B}\|(A+\Delta A)\theta - (B+\Delta B)\|, \tag{C.1}$$

is equivalent to the following minimization problem [36]:

$$\min_{\theta\in\mathbb{R}^n}\|A\theta - B\| + \delta_A\|\theta\| + \delta_B, \tag{C.2}$$

Lemma 2. *Let $\sigma(t), \xi(t)$ and $v(t)$ be real-valued continuous functions defined on interval $[0,\infty)$. If $v(t)$ is differentiable in the interval $[0,\infty)$ and satisfies the differential inequality:*

$$\frac{dv(t)}{\mathrm{d}t} \prec \sigma(t)v(t) + \xi(t), \tag{C.3}$$

and then $v(t)$ is bounded by

$$v(t) \prec e^{\int_0^t \sigma(\tau)\mathrm{d}\tau}\left[v(0) + \int_0^t \xi(\tau)e^{-\int_0^\tau \sigma(\mu)\mathrm{d}\mu}\mathrm{d}\tau\right]. \tag{C.4}$$

Data Availability

The system parameters used for simulation in this paper are given in the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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