Research Article

Design and Numerical Solutions of a Novel Third-Order Nonlinear Emden–Fowler Delay Differential Model

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In this study, the design of a novel model based on nonlinear third-order Emden–Fowler delay differential (EF-DD) equations is presented along with two types using the sense of delay differential and standard form of the second-order EF equation. The singularity at $\xi = 0$ at single or multiple points of each type of the designed EF-DD model are discussed. The detail of shape factors and delayed points is provided for both types of the designed third-order EF-DD model. For the verification and validation of the model, two numerical examples are presented of each case and numerical results have been performed using the artificial neural network along with the hybrid of global and local capabilities. The comparison of the obtained numerical results with the exact solutions shows the perfection and correctness of the designed third-order EF-DD model.

1. Introduction

The delay differential (DD) equation is known as one of the historical and important equations. Recently, DD equation has attained much attention of the researcher's community due to its vast applications in many biological models, as well as scientific phenomena such as communication system model, dynamical population model, economical systems, engineering system, and transport and propagation model [1–5]. It is always interested to find the solution of DD equations and many researchers have applied different numerical/analytical techniques. Brunner et al. [6] solved DD equation by applying a discontinuous Galerkin numerical scheme. Hsiao and Wu [7] applied Haar wavelet to solve DD equations, while Wang [8] presented the solution of DD equations using Legendre wavelet. Adomian and Rach [9] solved DD equation using the Adomian decomposition scheme. Shakeri and Dehghan [10] found the solutions of DD initial value problems using the homotopy

perturbation scheme. Erdogan et al. [11] implemented finite difference approach on layer-adapted mesh using the singularly perturbed DD equations. The general form of the DD model is written as [12, 13]

$$\begin{cases} \frac{d^3 u}{d\xi^3} = h\left(\xi, u(\xi - \tau), \frac{du(\xi - \tau)}{d\xi}, \frac{d^2 u(\xi - \tau)}{d\xi^2}\right), \\ u(0) = A, \frac{du(0)}{d\xi} = B, \frac{d^2 u(0)}{d\xi^2} = C, \end{cases} \quad (1)$$

where h shows the linear/nonlinear function and τ is the delayed term, whereas A , B , and C are the constants.

The singular study has become very significant in the modern era due to the variety of applications in technology, engineering, and biological and physical sciences. The singular nature models are always difficult, grim, and challengeable to solve for the research community. One of the important, famous, historical, and singular models is

Emden–Fowler (EF) model that shows the singularity at the origin. Since its invention, this model has been solved by various analytical and numerical schemes, and it has a number of applications in the study of relativistic mechanics, fluid dynamics, population growth model, pattern creation, and the study of chemical reactor models. The literature form of the EF model is written as [14–16]

$$\begin{cases} \frac{d^2 u}{d\xi^2} + \frac{\kappa}{\xi} \frac{du}{d\xi} + g(\xi)h(u) = 0, \\ u(0) = A_1, \frac{du(0)}{d\xi} = A_2, \end{cases} \quad (2)$$

where $\kappa \geq 1$ is the shape vector. The EF model (1) becomes the Lane–Emden model by taking $h(u) = 1$ and is written as follows:

$$\begin{cases} \frac{d^2 u}{d\xi^2} + \frac{\kappa}{\xi} \frac{du}{d\xi} + h(u) = 0, \\ u(0) = A_1, \frac{du(0)}{d\xi} = A_2. \end{cases} \quad (3)$$

The above singular models have been achieved from the work of Homer Lane and Robert Emden. These models designate inner construction of polytropic stars, gas cloud model, cluster galaxies, and radiative cooling. Due to the worth of these models, no one can deny the value and importance of such models, which has vast applications in the physical science field [17], isotropic continuous media [18], density of gaseous star [19], morphogenesis [20], dusty fluid models [21], stellar structure models [22], reactions based on catalytic diffusion [23], oscillating magnetic systems [24], isothermal gas sphere models [25], mathematical physics [26], catalytic diffusion reactions [23], classical/quantum mechanics [27], and electromagnetic theory [28].

Due to the fame of these models, the researcher's community is interested to solve these models and only a few methods are available in the literature that has been investigated. One of the well-known methods used to solve these models is the Adomian decomposition method, which is proposed by Shawagfeh and Wazwaz [29, 30]. Parand and Razzaghi [31] implemented a famous numerical scheme to solve singular equations. Liao [32] applied an analytic technique to avoid the difficulty of singular points. Bender et al. [33] proposed a perturbative scheme to solve the singular models. Nouh [34] presented two techniques' power series and Pade approximation to solve the singular models.

The aim of this study is to design a novel third-order Emden–Fowler delay differential (EF-DD) model along with two types. Two examples of the designed third-order EF-DD model have been presented for both of the types. For the correctness of the model, the numerical investigations have been performed by using an artificial neural network along with its global/local competences. The singular ordinary differential equations are much important and have many applications in engineering as well as scientific applications,

e.g., optimization and control theory, reactant application in the area of chemical reactor, theory of boundary layer, and biological sciences.

The structure of remaining paper is summarized as follows. Section 2 defines the construction of the third-order EF-DD model along with two types. Methodology and the detail of the results for solving the third-order EF-DD equations are provided in the Section 3. The conclusions along with future research directions are drawn in the Section 4.

2. Construction of Third-Order EF-DD Model

In this section, two different types are presented based on the third-order EF-DD model. The construction of the third-order EF-DD model along with the singular points, delayed points, and shape factors for both of the types is discussed. The initial conditions of the designed third-order EF-DD model are achieved using the standard form of the Lane–Emden. To derive the third-order EF-DD model system of Emden–Fowler equations, the mathematical form is used as follows:

$$\xi^{-k} \frac{d^p}{d\xi^p} \left(\xi^k \frac{d^q}{d\xi^q} \right) u(\xi - \tau) + g(\xi)h(u) = 0, \quad (4)$$

where k is real positive number. To determine the third-order DD-EF model, the values of p and q should be designated as follows:

$$p + q = 3, \quad p, q \geq 1. \quad (5)$$

The following two possibilities satisfy equation (5) as follows:

$$p = 2, \quad (6)$$

$$q = 1,$$

$$p = 1, \quad (7)$$

$$q = 2.$$

2.1. Type 1. Using equations (6), the updated form of equation (4) is

$$\xi^{-k} \frac{d^2}{d\xi^2} \left(\xi^k \frac{d}{d\xi} \right) u(\xi - \tau) + g(\xi)h(u) = 0. \quad (8)$$

The derivative part of the above equation is obtained as follows:

$$\frac{d^2}{d\xi^2} \left(\xi^k \frac{d}{d\xi} \right) u(\xi - \tau) = \xi^k \frac{d^3}{d\xi^3} u(\xi - \tau) + 2k\xi^{k-1} \frac{d}{d\xi} u(\xi - \tau) + k(k-1)u(\xi - \tau)\xi^{k-2} \frac{d}{d\xi} u(\xi - \tau). \quad (9)$$

Using the above expression in equation (8), the third-order EF-DD equation becomes

$$\begin{cases} \frac{d^3}{d\xi^3}u(\xi - \tau) + \frac{2k}{\xi} \frac{d^2}{d\xi^2}u(\xi - \tau) + \frac{k(k-1)}{\xi^2} \frac{d}{d\xi}u(\xi - \tau) + g(\xi)h(u) = 0, \\ u(0) = \alpha, \frac{du(0)}{d\xi} = 0, \frac{d^2u(0)}{d\xi^2} = 0, \end{cases} \quad (10)$$

where the singular point at $\xi = 0$ appears two times as $\xi = 0$ and $\xi^2 = 0$. The shape factors expressed in equation (10) are $2k$ and $k(k-1)$, respectively. The multiple delays have been noticed in the first, second, and third term of equation (10). Moreover, the third expression vanishes for $k=1$ and the shape factor reduces to 2.

2.2. *Type 2.* Equation (4) by putting $p = 1$ and $q = 2$ takes the form as follows:

$$\xi^{-k} \frac{d}{d\xi} \left(\xi^k \frac{d^2}{d\xi^2} \right) u(\xi - \tau) + g(\xi)h(u) = 0. \quad (11)$$

The derivative part of the above equation is obtained as follows:

$$\frac{d}{d\xi} \left(\xi^k \frac{d^2}{d\xi^2} \right) u(\xi - \tau) = \xi^k \frac{d^3}{d\xi^3} u(\xi - \tau) + k\xi^{k-1} \frac{d^2}{d\xi^2} u(\xi - \tau). \quad (12)$$

Using the above value in equation (11), the third-order EF-DD model becomes as follows:

$$\begin{cases} \frac{d^3}{d\xi^3}u(\xi - \tau) + \frac{k}{\xi} \frac{d^2}{d\xi^2}u(\xi - \tau) + g(\xi)h(u) = 0, \\ u(0) = \alpha, \frac{du(0)}{d\xi} = \beta, \frac{d^2u(0)}{d\xi^2} = 0. \end{cases} \quad (13)$$

The single singularity at $\xi=0$ has been noticed in the above equation (13). The shape factor is k and delayed expression appears twice in the above equation.

Some prime features of the designed model are presented as follows:

The design of third-order Emden–Fowler delay differential model is presented by using the sense of standard Emden–Fowler equation and delay-differential equation

Two types of the designed model are presented and two numerical nonlinear examples of each type are designed based on the designed model

The shape factors, delay expressions, and singularities are discussed in both of the types

The artificial neural network is used to check the perfection and correctness of the designed third-order Emden–Fowler model

3. Methodology and Numerical Examples

Two numerical examples based on the EF-DD novel model are presented in this section. The numerical investigations of the examples are performed using the artificial neural network. The error function is provided by using the sense of the differential equations and initial conditions. The optimization of the error function is performed using the hybrid of global and local search captainties, which are genetic algorithm (GA) and active-set method (ASM). The artificial neural network is famous and widely applied in many well-known recent applications, see [35–41]. To approximate the results, feedforward ANN system along with its respective derivatives is used as follows:

$$\hat{u} = \sum_{i=1}^m l_i P(\alpha_i \xi + b_i), \quad (14)$$

$$\hat{u}^{(n)} = \sum_{i=1}^m l_i P^{(n)}(\alpha_i \xi + b_i), \quad (15)$$

where l_i , m_i , and n_i are the i th components of l , α , and b vectors, while n is the order of derivative. An activation log-sigmoid function, i.e., $P(\xi) = (1 + \text{Exponential}E;^{-\xi})^{-1}$ along with its third derivative is used as follows:

$$\hat{u} = \sum_{i=1}^m l_i \left(1 + e^{-(\alpha_i \xi + b_i)} \right)^{-1}, \quad (16)$$

$$\hat{u}^{(n)} = \sum_{i=1}^m l_i \frac{d^n}{d\xi^n} \left(\left(1 + e^{-(\alpha_i \xi + b_i)} \right)^{-1} \right). \quad (17)$$

The third-order derivative is provided as follows:

$$\begin{aligned} \hat{u}'''(\xi) = \sum_{i=1}^m l_i \xi_i^3 & \left(\frac{6e^{-3(\alpha_i \xi + b_i)}}{\left(1 + e^{-(\alpha_i \xi + b_i)} \right)^4} - \frac{6e^{-2(\alpha_i \xi + b_i)}}{\left(1 + e^{-(\alpha_i \xi + b_i)} \right)^3} \right. \\ & \left. + \frac{e^{-(\alpha_i \xi + b_i)}}{\left(1 + e^{-(\alpha_i \xi + b_i)} \right)^2} \right). \end{aligned} \quad (18)$$

The fitness function is given as follows:

$$E = E_1 + E_2, \quad (19)$$

where E_1 and E_2 are the respective error functions related to differential equation and initial conditions.

FIGURE 1: Optimization variables, learning curves, and comparison of results of the GA-AS scheme for nonlinear EF-DD equations (1) and (2) of type 1. (a) Set of best weights and current function values for 10 neurons based on third-order nonlinear EF-DD equation (1). (b) Set of best weights and current function values for 10 neurons based on third-order nonlinear EF-DD equation (2). (c) Comparison of the numerical and exact solutions of third-order nonlinear EF-DD equation (1). (d) Comparison of the numerical and exact solutions of third-order nonlinear EF-DD equation (2).

3.1. EF-DD Equation of Type 1. In this type, two different third-order EF-DD-based equations will be discussed. The updated form of equation (10) using $k = 2$ is given as follows.

Example 1. Consider the nonlinear third-order EF-DD equation having multiple singularities is shown as follows:

$$\begin{cases} \frac{d^3}{d\xi^3} u(\xi - 1) + \frac{4}{\xi} \frac{d^2}{d\xi^2} u(\xi - 1) + \frac{2}{\xi^2} \frac{d}{d\xi} u(\xi - 1) + \xi u^2 = \xi^7 + 2\xi^4 + \xi + 30 - \frac{36}{\xi} + \frac{6}{\xi^2}, \\ u(0) = 1, \quad \frac{du(0)}{d\xi} = 0, \quad \frac{d^2u(0)}{d\xi^2} = 0. \end{cases} \quad (20)$$

FIGURE 2: Optimization variables, learning curves, and comparison of results of the GA-AS scheme for nonlinear EF-DD equations (1) and (2) of type 2. (a) Set of best weights and current function values for 10 neurons based on third-order nonlinear EF-DD equation (1) of type 2. (b) Set of best weights and current function values for 10 neurons based on third-order nonlinear EF-DD equation (2) of type 2. (c) Comparison of the numerical and exact solutions of third-order nonlinear EF-DD equation (1) of type 2. (d) Comparison of the numerical and exact solutions of third-order nonlinear EF-DD equation (2) of type 2.

The exact solution of equation (20) is $1 + \xi^3$.

The exact solution of equation (21) is $\cos \xi + (1/2)\xi^2$.

Example 2. Consider the nonlinear third-order EF-DD equation having multiple singularities and trigonometric functions is written as follows:

$$\begin{cases} \frac{d^3}{d\xi^3}u(\xi-1) + \frac{4}{\xi} \frac{d^2}{d\xi^2}u(\xi-1) + \frac{2}{\xi^2} \frac{d}{d\xi}u(\xi-1) + \xi u^2 = \frac{\xi^5}{4} - \frac{2}{\xi^2} + \frac{6}{\xi} + \\ \frac{\xi^2-2}{\xi^2} \sin(\xi-1) - \frac{4}{\xi} \cos(\xi-1) + \xi^3 \cos \xi + \xi \cos^2 \xi, \\ u(0) = 1, \frac{du(0)}{d\xi} = 0, \frac{d^2u(0)}{d\xi^2} = 0. \end{cases} \quad (21)$$

3.2. EF-DD Equation of Type 2. In this type, two different third-order EF-DD-based equations will be discussed. The updated form of equation (13) using $k=1$ is given in the form of two examples.

Example 3. Consider the nonlinear third-order EF-DD equation having exponential function is given as follows:

$$\begin{cases} \frac{d^3}{d\xi^3}u(\xi-1) + \frac{1}{\xi} \frac{d^2}{d\xi^2}u(\xi-1) + \xi e^u = 12 - \frac{6}{\xi} + \xi e^{1+\xi+\xi^3}, \\ u(0) = 1, \frac{du(0)}{d\xi} = 1, \frac{d^2u(0)}{d\xi^2} = 0. \end{cases} \quad (22)$$

FIGURE 3: Absolute error based on the nonlinear EF-DD equations (1) and (2) of types 1 and 2. (a) AE for examples 1 and 2 of type 1. (b) AE for examples 1 and 2 of type 2.

The exact solution of equation (22) is $1 + \xi + \xi^3$.

Example 4. Consider the nonlinear third-order EF-DD equation having multi trigonometric function is given as follows:

$$\begin{cases} \frac{d^3}{d\xi^3} u(\xi - 1) + \frac{1}{\xi} \frac{d^2}{d\xi^2} u(\xi - 1) + \xi u^2 = \xi \sin^2 \xi + 2\xi \sin \xi + \xi - \cos(\xi - 1) - \frac{1}{\xi} \sin(\xi - 1), \\ u(0) = 1, \frac{du(0)}{d\xi} = 1, \frac{d^2u(0)}{d\xi^2} = 0. \end{cases} \quad (23)$$

The exact solution of equation (23) is $1 + \sin \xi$.

Figures 1 and 2 represent the current point and function values using 10 neurons based on the hybrid combination of GA-AS scheme for both of the examples of types 1 and 2. The current function values (CFVs) are 10^{-09} and 10^{-08} for both of the examples of type 1 and 10^{-07} and 10^{-09} for both of the examples of 2 using 10 numbers of neurons. The comparison of results is presented in the rest of the figures for both examples of types 1 and 2. The overlapping of the exact and obtained results shows the correctness and the perfection of the novel third-order nonlinear EF-DD model.

The plots of the absolute error (AE) for both types of examples 1 and 2 based on the third-order nonlinear EF-DD model are provided in Figure 3. It is clear that most of the values lie around 10^{-04} to 10^{-05} for both types of examples 1 and 2, which indicates the exactness of the designed model. These witnesses prove the correctness of the designed third-order nonlinear EF-DD model. Comparison of the obtained results from GA-ASM for solving the nonlinear EF-DD model based on both problems of both types is tabulated in

Tables 1 and 2. The exact solution, proposed results from GA-ASM, and the AE are provided in these tables. One can conclude on the behalf of AE the exactness and accurateness of the proposed model, as well as designed scheme.

4. Conclusion

In the present study, a novel design of third-order Emden–Fowler delay differential model is presented. The designed model is obtained by using the sense of fundamental Emden–Fowler model. The details of singular points, delay expressions, and the shape factors are also provided of the modeled equations of each type. The singularity at $\xi = 0$ appears twice in the first type, while single singularity is noticed in the second type. Similarly, the shape factor is unique in the standard form of the Emden–Fowler model, while the occurrence of shape factor is noticed twice in the type 1; however, single shape factor is noticed in type 2. For the perfection of the designed model, two nonlinear examples are presented of each type and numerical

TABLE 1: Comparison of the obtained results from GA-ASM for solving the nonlinear EF-DD model based on both problems of type 1.

ξ	Problem 1			Problem 2		
	Exact	GA-ASM	AE	Exact	GA-ASM	AE
0	1.00000000	0.99998788	1.212101E-05	1.00000000	0.99999570	4.2979068E-06
0.05	1.00012500	1.00010387	2.112830E-05	1.00000026	0.99999586	4.4025337E-06
0.1	1.00100000	1.00097307	2.693406E-05	1.00000417	0.99999959	4.5773499E-06
0.15	1.00337500	1.00335045	2.454778E-05	1.00002108	1.00001622	4.8547562E-06
0.2	1.00800000	1.00799333	6.670848E-06	1.00006658	1.00006131	5.2678390E-06
0.25	1.01562500	1.01566134	3.633505E-05	1.00016242	1.00015658	5.8417298E-06
0.3	1.02700000	1.02711600	1.160012E-04	1.00033649	1.00032991	6.5812201E-06
0.35	1.04287500	1.04311966	2.446627E-04	1.00062271	1.00061526	7.4548770E-06
0.4	1.06400000	1.06443360	4.336027E-04	1.00106099	1.00105262	8.3764421E-06
0.45	1.09112500	1.09181538	6.903829E-04	1.00169710	1.00168792	9.1849453E-06
0.5	1.12500000	1.12601542	1.015419E-03	1.00258256	1.00257294	9.6257224E-06
0.55	1.16637500	1.16777299	1.397990E-03	1.00377452	1.00376519	9.3353752E-06
0.6	1.21600000	1.21781202	1.812016E-03	1.00533561	1.00532778	7.8346627E-06
0.65	1.27462500	1.27683704	2.212044E-03	1.00733380	1.00732926	4.5343334E-06
0.7	1.34300000	1.34552994	2.529940E-03	1.00984219	1.00984343	1.2400050E-06
0.75	1.42187500	1.42454775	2.672747E-03	1.01293887	1.01294907	1.0196762E-05
0.8	1.51200000	1.51452198	2.521980E-03	1.01670671	1.01672970	2.2992646E-05
0.85	1.61412500	1.61605929	1.934293E-03	1.02123315	1.02127325	4.0102911E-05
0.9	1.72900000	1.72974297	7.429666E-04	1.02660997	1.02667161	6.1637149E-05
0.95	1.85737500	1.85613407	1.240925E-03	1.03293309	1.03302018	8.7089331E-05
1	2.00000000	1.99577063	4.229373E-03	1.04030231	1.04041732	1.1501021E-04

TABLE 2: Comparison of the obtained results from GA-ASM for solving the nonlinear EF-DD model based on both problems of type 2.

ξ	Problem 1			Problem 2		
	Exact	GA-ASM	AE	Exact	GA-ASM	AE
0	1.00000000	1.00022916	2.291613E-04	1.00000000	0.99963919	3.60806E-04
0.05	1.05012500	1.05035972	2.347211E-04	1.04997917	1.04960463	3.74535E-04
0.1	1.10100000	1.10123006	2.300602E-04	1.09983342	1.09944214	3.91281E-04
0.15	1.15337500	1.15357541	2.004064E-04	1.14943813	1.14902494	4.13190E-04
0.2	1.20800000	1.20812735	1.273485E-4	1.19866933	1.19822683	4.42503E-04
0.25	1.26562500	1.26561619	8.813729E-06	1.24740396	1.24692273	4.81224E-04
0.3	1.32700000	1.32677477	2.252260E-04	1.29552021	1.29498952	5.30689E-04
0.35	1.39287500	1.39234339	5.316061E-04	1.34289781	1.34230679	5.91015E-04
0.4	1.46400000	1.46307475	9.252507E-04	1.38941834	1.38875787	6.60468E-04
0.45	1.54112500	1.53973795	1.387048E-03	1.43496553	1.43423077	7.34760E-04
0.5	1.62500000	1.62312048	1.879521E-03	1.47942554	1.47861923	8.06304E-04
0.55	1.71637500	1.71402744	2.347561E-03	1.52268723	1.52182376	8.63464E-04
0.6	1.81600000	1.81327792	2.722078E-03	1.56464247	1.56375263	8.89845E-04
0.65	1.92462500	1.92169852	2.926483E-03	1.60518641	1.60432276	8.63645E-04
0.7	2.04300000	2.04011427	2.885735E-03	1.64421769	1.64346056	7.57124E-04
0.75	2.17187500	2.16933723	2.537774E-03	1.68163876	1.68110255	5.36205E-04
0.8	2.31200000	2.31015280	1.847204E-03	1.71735609	1.71719584	1.60246E-04
0.85	2.46412500	2.46330377	8.212322E-04	1.75128041	1.75169842	4.18013E-04
0.9	2.62900000	2.62947213	4.721265E-04	1.78332691	1.78457923	1.25232E-03
0.95	2.80737500	2.80925864	1.883638E-03	1.81341550	1.81581807	2.40257E-03
1	3.00000000	3.00316042	3.160417E-03	1.84147098	1.84540533	3.93434E-03

investigations have been performed using the powerful artificial neural networks. The comparison of the results is also plotted and overlapping of the proposed and exact solution enhanced more satisfaction of the model. The graphs of absolute error show that most of the values are found in good ranges for all examples of both types, which shows the exactness, worth, and the precision of the

designed third-order Emden–Fowler delay differential model.

In the future, the proposed scheme ANN-GA-ASM can be applied as an accurate and efficient stochastic numerical solver for nonlinear singular models [42–44], computational models of fluid dynamics [45–48], fractional models [49–52], and biological models [53–57].

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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