Shrinking Projection Methods for Accelerating Relaxed Inertial Tseng-Type Algorithm with Applications

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Our main goal in this manuscript is to accelerate the relaxed inertial Tseng-type (RITT) algorithm by adding a shrinking projection (SP) term to the algorithm. Hence, strong convergence results were obtained in a real Hilbert space (RHS). A novel structure was used to solve an inclusion and a minimization problem under proper hypotheses. Finally, numerical experiments to elucidate the applications, performance, quickness, and effectiveness of our procedure are discussed.

1. Introduction

The standard form of the variational inclusion problem (VIP) on a RHS Σ is

\[ 0 \in (\varphi + \Upsilon)\vartheta^*, \tag{1} \]

where \( \vartheta^* \) is the unknown point that we need to find, for an operator \( \varphi: \Sigma \rightarrow \Sigma \) and a set-valued operator \( \Upsilon: \Sigma \rightarrow 2^\Sigma \). VIP is a frequent problem in the optimization field, which has a lot of applications in many areas, including equilibrium, machine learning, economics, engineering, image processing, and transportation problems [1–16].

The vintage technique to solve problem (1) which is denoted by \((\varphi + \Upsilon)^{-1} (0)\) is the forward-backward splitting method [17–22] which is defined as follows: \( \vartheta_1 \in \Sigma \) and

\[ \vartheta_{n+1} = (I + \ell \Upsilon)^{-1} (I - \ell \varphi)\vartheta_n, \quad n \geq 1, \tag{2} \]

where \( \ell > 0 \). In (2), each step of iterates includes only the forward step \( \varphi \) and the backward step \( \Upsilon \), but not \( \varphi + \Upsilon \). This technique involves the proximal point algorithm [23–25] and the gradient method [26–28] as special cases.

In a RHS, nice splitting iterative procedures presented by Lions and Mercier [29] are shown as follows:

\[ \vartheta_{n+1} = (2J_{\ell,1}^* - I)(2J_{\ell,1}^* - I)\vartheta_n, \quad n \geq 1, \tag{3} \]

and

\[ \vartheta_{n+1} = J_{\ell,1}^*(2J_{\ell,1}^* - I)\vartheta_n + (I - J_{\ell,1}^*)\vartheta_n, \quad n \geq 1, \tag{4} \]

where \( J_{\ell,1}^* = (I + \ell \varphi)^{-1} \). Permanently, two algorithms are weakly convergent [30], knowing that algorithm (3) is called Peaceman–Rachford algorithm [19] and scheme (4) is called Douglas–Rachford algorithm [31].

A lot of works are concerned with problem (1) for accretive operators and two monotone operators, for instance, a stationary solution to the initial-valued problem of the evolution equation

\[ 0 \in \frac{\partial \varphi}{\partial t} - \Xi \vartheta, \quad \vartheta (0) = \vartheta_0. \tag{5} \]

can be adjusted as (1) when the governing maximal monotone \( \Xi = \varphi + \Upsilon \) [29].

[1] is used to solve a minimization problem as follows:

\[ \min_{\vartheta \in \Sigma} \varpi (\vartheta) + \sigma (\vartheta), \tag{6} \]

where \( \varpi, \sigma: \Sigma \rightarrow (-\infty, \infty] \) are proper and lower semi-continuous convex functions such that \( \varpi \) is differentiable with \( L \)-Lipschitz gradient, and the proximal mapping of \( \sigma \) is
where $\Lambda$ and $\beta$ are extrapolation and relaxation parameters, respectively. Under this algorithm, they discussed the weak convergence to the solution point of VIP (1) and the problem of image recovery. Note that the extrapolation step works to accelerate but not for the desired acceleration.

The concept of the SP method was discussed by Takahashi et al. [46] as in the following algorithm:

\[
\begin{align*}
\theta_0 & \in \Upsilon \text{ be arbitrarily fixed,} \\
C_1 &= C, \theta_1 = P_C \theta_0, \\
\omega_n &= \Lambda_n \theta_n + (1 - \Lambda_n) \nu_n \theta_n, \\
C_n &= \{ \eta \in C : \| \omega_n - \eta \| \leq \| \theta_n - \eta \| \}, \\
\theta_{n+1} &= P_{C_{n+1}} \theta_n.
\end{align*}
\]

They just selected one closed convex (CC) set for a family of nonexpansive mappings $\{ \nu_n \}$ to modify Mann’s iteration method [47] and proved that the sequence $\{ \theta_n \}$ converges strongly to $P_{\text{fix}(\theta)}$, provided $\Lambda_n \leq c$ for all $n \geq 1$ and for some $0 < c < 1$.

In 2019, Yang and Liu [48] selected the stepsize sequence for the iterative algorithm for monotone variational inequalities, which are based on Tseng’s extragradient method and Moudafi viscosity scheme that does not require either the knowledge of the Lipschitz constant of the operator or additional projections.

With the incorporation of results of [45, 46, 48], we accelerate RITT algorithm by adding the SP method to algorithm (12). In a RHS, strong convergence results are given under a proposed algorithm. As applications, our algorithm was used to find the solution to a VIP and minimization problem under certain conditions. Eventually, numerical experiments to illustrate the applications, performance, acceleration, and effectiveness of the proposed algorithm are presented.

### 2. Preparatory Lemmas and Definitions

Suppose that $C$ is a nonempty closed convex subset (CCS) of a RHS $\Upsilon$; we shall refer to $\rightarrow^*$ as the strong convergence, and $P_C : \Upsilon \rightarrow C$ is the nearest point projection, that is, for all $\theta \in \Upsilon$ and $\omega \in C$, $\| \theta - P_C \theta \| \leq \| \theta - \omega \|$. $P_C$ is called the metric projection. It is obvious that $P_C$ verifies the following inequality:

\[
\| P_C \theta - P_C \omega \| \leq \langle P_C \theta - P_C \omega, \theta - \omega \rangle,
\]

for all $\theta, \omega \in \Upsilon$. In other words, the metric projection $P_C$ is firmly nonexpansive. Hence, $\langle \theta - P_C \theta, \omega - P_C \omega \rangle \leq 0$ holds for all $\theta \in \Upsilon$ and $\omega \in C$, see [49, 50].

The following inequality holds in a HS [51]:

\[
\| l \pm m \| = \| l \| + \| m \| \pm 2 \langle l, m \rangle,
\]

for all $l, m \in \Upsilon$.

**Lemma 1** (see [52]). Let $C$ be a nonempty CCS of a RHS $\Upsilon$. For each $\theta, \omega \in \Upsilon$ and $\nu \in R$, the following set is closed and convex:

\[
\{ \theta \in C : \theta - \nu \theta \in C \}
\]
\[ \{ \eta \in C : \| \omega - \eta \|^2 \leq \| \theta - \eta \|^2 + \langle \nu, \eta \rangle + \delta \}. \] (16)

**Lemma 2** (see [38]). Let \( C \) be a nonempty CCS of a RHS \( \gamma \) and \( P_C : \gamma \rightarrow \gamma \) be the metric projection. Then,
\[ \| \omega - P_C \theta \|^2 + \| \theta - P_C \theta \|^2 \leq \| \theta - \omega \|^2, \] (17)
for all \( \theta \in \gamma \) and \( \omega \in C \).

**Definition 1.** Suppose that \( D(\gamma) \subset \gamma \) and \( R(\gamma) \subset \gamma \) are the domain and the range of an operator \( \gamma \), respectively. For all \( \delta, \omega \in D(\gamma) \), an operator \( \gamma \) is called

1. **Monotone if**
   \[ \langle \theta - \omega, \gamma \theta - \gamma \omega \rangle \geq 0. \] (18)

2. **L–Lipschitz if**
   \[ \| \gamma \theta - \gamma \omega \| \leq L \| \theta - \omega \|. \] (19)

3. **\( \beta \)–Strongly monotone if there exists \( \beta > 0 \) such that**
   \[ \langle \theta - \omega, \gamma \theta - \gamma \omega \rangle \geq \beta \| \theta - \omega \|^2. \] (20)

4. **\( \Lambda \)–Inverse strongly monotone (\( \Lambda \)-ism) if there exists \( \Lambda > 0 \) such that**
   \[ \langle \theta - \omega, \gamma \theta - \gamma \omega \rangle \geq \Lambda \| \gamma \theta - \gamma \omega \|^2. \] (21)

**Lemma 3** (see [44]). Let \( \gamma \) be a RHS, \( \gamma : \gamma \rightarrow \gamma \) be an \( \Lambda \)-ism operator, and \( \gamma : \gamma \rightarrow \gamma \) be a maximal monotone operator. For each \( \ell > 0 \), we define
\[ \Sigma_{\ell} = f_{\ell}^{-1} \circ (I - \ell \gamma) = (I + \ell \gamma)^{-1}(I - \ell \gamma). \] (22)
Then, we get
(i) For \( \ell > 0 \), fix \( \Sigma_{\ell} = (\gamma + \gamma^{\ell})^{-1}(0) \)
(ii) For \( 0 < s \leq \ell \) and \( \delta \in \gamma \), \( \| \theta - \Sigma_{\ell} \delta \| \leq 2 \| \theta - \Sigma_{\ell} \delta \| \)

**Lemma 4.** Let \( \gamma \) be a RHS, \( \gamma : \gamma \rightarrow \gamma \) be an \( \Lambda \)-ism operator, and \( \gamma : \gamma \rightarrow \gamma \) be a maximal monotone operator. For each \( \ell > 0 \), we have
\[ \| \Sigma_{\ell} \theta - \Sigma_{\ell} \omega \|^2 \leq \| \theta - \omega \|^2 - \ell (2\Lambda - \ell) \| \gamma \theta - \gamma \omega \|^2, \] (23)
for all \( \theta, \omega \in \gamma \).

**Proof.** For all \( \theta, \omega \in \gamma \), we get
\[ \| \Sigma_{\ell} \theta - \Sigma_{\ell} \omega \|^2 \leq \| \theta - \omega \|^2 - \ell (2\Lambda - \ell) \| \gamma \theta - \gamma \omega \|^2, \] (24)

The proof is ended. \( \Box \)

### 3. Shrinking Projection Relaxed Inertial Tseng-Type Algorithm

We provide a method consisting of the forward-backward splitting method with an inertial factor and an explicit stepsize formula, which are being used to ameliorate the convergence average of the iterative scheme and to make the manner independent of the Lipschitz constants. The detailed method is provided in Algorithm 1.

Note that
(i) Since \( \gamma \) is an \( \Lambda \)-ism operator, it is a Lipschitz function with a constant \( L, \gamma \mathcal{N}_n \neq \mathcal{Y}_n \), and we get
\[ \rho \| \mathcal{N}_n - \mathcal{Y}_n \| \geq \rho \frac{\| \mathcal{N}_n - \mathcal{Y}_n \|}{L}. \] (25)

It is obvious for \( \mathcal{N}_n = \mathcal{Y}_n \) that inequality (25) is satisfied. Hence, it follows that \( \ell_n \approx \min \{ (\rho/L), \ell_0 \} \). This implies that the generated sequence \( \{ \ell_n \} \) is bounded below by \( \min \{ (\rho/L), \ell_0 \} \), i.e., \( \{ \ell_n \} \) is monotonically decreasing.

(ii) By (i) and (25), we have
\[ \ell_n \| \mathcal{N}_n - \mathcal{Y}_n \| \leq \rho \| \mathcal{N}_n - \mathcal{Y}_n \| \] (26)
i.e., the update (28) is well defined.

(iii) If we delete the shrinking projection term from our algorithm, we get the algorithms of the papers [22, 45, 53].

**Theorem 1.** Let \( \gamma \) be a RHS and the operators \( \gamma : \gamma \rightarrow \gamma \) be an \( \Lambda \)-ism on \( \gamma \), and \( \gamma : \gamma \rightarrow \gamma \) is maximally monotone. If feasible set \( \Omega = (\gamma + \gamma)^{-1}(0) \) is a nonempty CCS of a RHS \( \gamma \), then the sequence \( \{ \theta_n \} \) generated by Algorithm 1 converges strongly to a point \( \theta = P_\gamma (\theta_1) \), provided that
(i) \( 0 < \liminf_{n \to \infty} \ell_n \leq \limsup_{n \to \infty} \ell_n < 2\lambda \).
(ii) \( \lim_{n \to \infty} \| \psi_n - \mathcal{N}_n \| = 0. \)

**Proof.** The proof will be divided as follows: \( \Box \)
Initialization: select initial \( \delta_0, \delta_1 \in \mathbb{R}, \rho \in (0, 1), \lambda \geq 0, \zeta_0 > 0, \) and \( 0 < \beta < 1. \)

St. (i). Put \( \mathcal{S}_n \) as:
\[
\mathcal{S}_n = \delta_n + \Lambda (\delta_n - \delta_{n-1}).
\]

St. (ii). Calculate:
\[
\psi_n = (1 + \xi_n^2)^{-1} (1 - \xi_n^2) \mathcal{S}_n,
\]
If \( \mathcal{S}_n = \psi_n, \) discontinue. \( \mathcal{S}_n \) is a solution of (1), otherwise, continue to St. (iii)

St. (iii). Calculate:
\[
\phi_n = (1 - \beta) \mathcal{S}_n + \beta \psi_n + \beta \ell_n (\mathcal{Y} \mathcal{S}_n - \psi_n),
\]
where \( \ell_{n+1} \) is stepsize sequence revised as follows:
\[
\ell_{n+1} = \min \left\{ \ell_{n}, \min \left( \binom{\min(\ell_{n}, (\rho \mathcal{S}_n - \psi_n))/((\mathcal{Y} \mathcal{S}_n - \psi_n))}{\ell_{n+1}} \right) \right\}.
\]

St. (iv). Calculate:
\[
C_{n+1} = \left\{ \eta \in C_n : \| \psi_n - \eta \|^2 \leq \| \mathcal{S}_n - \eta \|^2 + \Lambda^2 \| \mathcal{S}_n - \delta_n \|^2 - 2 \Delta (\mathcal{S}_n - \eta, \mathcal{S}_n - \delta_n) - \beta \Lambda (\mathcal{S}_n - \psi_n) \right\}.
\]
where \( \Delta = (2 - \beta - 2 \rho (1 - \beta) \ell_n / \ell_{n+1}). \)

St. (v). Compute
\[
\delta_{n+1} = P_{C_{n+1}}(\delta_n),
\]
put \( n = n + 1, \) and return to St. (i).

Algorithm 1: Splitting method for the VIP.

Part 1. Demonstrate that \( P_{C_n} \delta_1 \) is well-defined, for each \( \delta_1 \in \mathbb{R}, n \geq 1, \) and \( \Omega \subset C_{n+1}. \) It follows from condition (i) and Lemma 4 that \( \mathcal{S}_n = (I + \xi_n^2)^{-1} (I - \xi_n^2) \) is a nonexpansive mapping. Lemma 3 implies that \( \Omega \) is a closed and convex set, and Lemma 1 clarifies that \( C_{n+1} \) is closed and convex, for all \( n \geq 1. \)

Let \( \eta \in \Omega; \) we have
\[
\|
\mathcal{S}_n - \eta^2 \| = \| (\delta_n - \eta - \Lambda (\delta_n - \delta_{n-1}) \|^2 = \| \delta_n - \eta \|^2 - 2 \Lambda (\delta_n - \eta, \delta_n - \delta_{n-1}) + \| \delta_n - \delta_n \|^2. \tag{27}
\]

Since the resolvent \( \mathcal{S}_n \) is firmly a nonexpansive mapping and by Lemma 3, we have
\[
\langle \psi_n - \eta, \mathcal{S}_n - \psi_n - \ell_n (\mathcal{Y} \mathcal{S}_n + \psi_n) \rangle \geq 0, \tag{29}
\]
Hence, by (28), we get
\[
\langle \psi_n - \eta, \mathcal{S}_n - \psi_n - \ell_n (\mathcal{Y} \mathcal{S}_n + \psi_n) \rangle \geq 0, \tag{29}
\]
which leads to
\[
2 \langle \mathcal{S}_n - \psi_n, \psi_n - \eta \rangle - 2 \ell_n \langle \mathcal{Y} \mathcal{S}_n + \psi_n, \psi_n - \eta \rangle \geq 0. \tag{30}
\]
It is obvious that
\[
\|
\phi_n - \eta \|^2 = \| (1 - \beta) \mathcal{S}_n + \beta \psi_n + \beta \ell_n (\mathcal{Y} \mathcal{S}_n - \psi_n) - \eta \|^2 = \| (1 - \beta) (\mathcal{S}_n - \eta) + \beta (\psi_n - \eta) + \beta \ell_n (\mathcal{Y} \mathcal{S}_n - \psi_n) \|^2
\]
\[
= (1 - \beta) \| \mathcal{S}_n - \eta \|^2 + \beta^2 \| \psi_n - \eta \|^2 + \beta^2 \| \mathcal{Y} \mathcal{S}_n - \psi_n \|^2 + 2 \beta (1 - \beta) \langle \mathcal{S}_n - \eta, \psi_n - \eta \rangle
\]
\[
+ 2 \beta \ell_n (1 - \beta) \langle \mathcal{S}_n - \eta, \mathcal{Y} \mathcal{S}_n - \psi_n \rangle + 2 \beta^2 \ell_n \langle \psi_n - \eta, \mathcal{Y} \mathcal{S}_n - \psi_n \rangle. \tag{33}
\]
Illustrate that Part 2. It follows from (32), (35), and (26) that

\[
\|\phi_n - \eta\|^2 \leq (1 - \beta)\|\phi_n - \eta\|^2 + \beta \left( \|\phi_n - \phi\|^2 - 2\epsilon_n (\phi \cdot \phi, \phi_n - \eta) \right) \leq (1 - \beta)\|\phi_n - \eta\|^2 + \beta \left( \|\phi_n - \phi\|^2 - 2\epsilon_n (\phi \cdot \phi, \phi_n - \eta) \right)
\]

(35)

Applying (34) in (33), we get

\[
\|\phi_n - \eta\|^2 \leq \|\phi_n - \eta\|^2 + \beta \left( \|\phi_n - \phi\|^2 - 2\epsilon_n (\phi \cdot \phi, \phi_n - \eta) \right) - \beta (1 - \beta)\|\phi_n - \phi\|^2
\]

(36)

Applying (27) in (36), we have

\[
\|\phi_n - \eta\|^2 \leq \|\phi_n - \eta\|^2 + \lambda^2 \|\phi_n - \phi\|^2 - 2\lambda (\phi_n - \phi, \phi_n - \eta) - \beta\|\phi_n - \eta\|^2.
\]

(37)

It is clear that \(\Omega \subset C_1 = \mathbb{T}\). Assume that \(\Omega \subset C_n\) for some \(n \geq 1\). Then, \(\eta \in C_n\) and by (37), we have for all \(n \geq 1\), \(\eta \in C_{n+1}\). Thus, \(\Omega \subset C_{n+1}\) for all \(n \geq 1\), i.e., \(P_{C_{n+1}} \phi\) is well-defined and bounded.

Part 2. Illustrate that \(\{\theta_n\}\) is bounded. Since \(\Omega \neq \emptyset\) and closed and convex subset of \(\mathbb{T}\), there is a unique \(u \in \Omega\) such that \(u = P_{\Omega} \phi\). This leads to \(\theta_0 = P_{C_1} \phi\), \(C_n \subset C_{n+1}\), and \(\theta_n \in C_n\) for all \(n \geq 1\), and we have

\[
\|\theta_n - \theta_0\| \leq \|\theta_{n+1} - \theta_n\|.
\]

(38)

Furthermore, as \(\Omega \subset C_n\), for all \(n \geq 1\), we obtain

\[
\|\theta_n - \theta_0\| \leq \|u - \theta_0\|.
\]

(39)

It follows by (38) and (39) that \(\lim_{n \to \infty} \|\theta_n - \theta_0\|\) exists. Hence, \(\{\theta_n\}\) is bounded.

Part 3. Fulfillment of \(\lim_{n \to \infty} \theta_n = \tau\). By the definition of \(C_n\), for \(m \geq n\), we observe that \(\delta_m = P_{C_n} \delta_1 \in C_m \subset C_n\). From Lemma 2, we have

\[
\|\theta_m - \theta_0\|^2 \leq \|\theta_m - \theta_0\|^2 - \|\theta_n - \theta_0\|^2.
\]

(40)

By Part 2, we conclude that \(\lim_{n \to \infty} \|\theta_n - \theta_0\|^2 = 0\). Thus, \(\{\theta_n\}\) is a Cauchy sequence. Hence, \(\lim_{n \to \infty} \theta_n = \tau\). Additionally, we get

\[
\lim_{n \to \infty} \|\theta_m - \theta_n\| = 0.
\]

(41)

Part 4. Prove that \(\tau \in \Omega\). It follows from (41) that

\[
\|\theta_n - \theta_0\| = \lambda \|\theta_0 - \delta_n\| \to 0 \text{ as } n \to \infty.
\]

(42)

Also, by (42) and condition (ii), we can write

\[
\|\psi_n - \theta_n\| \leq \|\psi_n - C_n\| + \|C_n - \theta_n\| \to 0 \text{ as } n \to \infty.
\]

(43)
From triangle inequality on the norm and (42) and (43), we obtain
\[
\|\mathcal{F}_n - \psi_n\| \leq \|\mathcal{F}_n - \beta_n\| + \|\psi_n - \beta_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\]
(44)

Replacing \(\eta\) with \(\beta_n\) in (36) and using (41) and (44), we have
\[
\|\phi_n - \beta_n\| \leq \Lambda^2 \|\theta_{n-1} - \beta_n\| + \beta \Lambda \|\mathcal{F}_n - \psi_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\]
(45)

Applying (41), (42), and (45), we can write
\[
\|\mathcal{G}_n - \mathcal{F}_n\| \leq \|\mathcal{G}_n - \mathcal{F}_n\| + \|\mathcal{F}_n - \mathcal{F}_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty,
\]
\[
\|\mathcal{G}_n - \mathcal{F}_n\| \leq \|\mathcal{G}_n - \mathcal{F}_n\| + \|\phi_n - \phi_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty,
\]
\[
\|\mathcal{G}_n - \mathcal{F}_n\| \leq \|\mathcal{G}_n - \mathcal{F}_n\| + \|\mathcal{F}_n - \mathcal{F}_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\]
(46)

It follows from (44) that
\[
\lim_{n \longrightarrow \infty} \|\mathcal{F}_n - \mathcal{F}_n\| = \lim_{n \longrightarrow \infty} \|\psi_n - \mathcal{F}_n\| = 0.
\]
(47)

Since \(\lim inf_{n \longrightarrow \infty} \ell_n > 0\), there is \(\varepsilon > 0\) such that \(\ell_n \geq \varepsilon\) and \(\varepsilon \in (0, 2\Lambda)\) for all \(n \geq 1\). Then, by Lemma 3 (ii) and (47), we get
\[
\|\mathcal{G}_n - \mathcal{F}_n\| \leq \|\mathcal{G}_n - \mathcal{F}_n\| \leq \|\mathcal{G}_n - \mathcal{F}_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\]
(48)

From (45) and (46), since \(\theta_n \longrightarrow \tau\) as \(n \longrightarrow \infty\), we have also \(\mathcal{F}_n \longrightarrow \tau\) as \(n \longrightarrow \infty\). Since \(\mathcal{G}_n\) is a nonexpansive and continuous mapping, from (47), we conclude that \(\tau \in \Omega\).

Part 5. Show that \(\tau = P_{\Omega} (\Theta_1)\). Since \(\Theta_n = P_{C_n} \Theta_1\) and \(\Omega \subset C_n\), we can get
\[
\langle \Theta_1 - \Theta_n, \Theta_1 - \Theta_n \rangle \geq 0, \quad \forall \Theta_n \in \Omega.
\]
(49)

Setting \(n \longrightarrow \infty\) in (49), we have
\[
\langle \Theta_1 - \Theta, \Theta_1 - \Theta_1 \rangle \geq 0, \quad \forall \Theta \in \Omega.
\]
(50)

This shows that \(\tau = P_{\Omega} (\Theta_1)\). This finishes the proof.

4. Solve a Minimization Problem

As an application of our theorem, we solve the following constrained convex minimization problem:
\[
\min_{\theta \in C} \Xi (\theta),
\]
(51)
where \(\Xi: \Omega \longrightarrow \mathbb{R}\) is a convex function. We suppose that the function \(\Xi\) is differentiable such that \(\nabla \Xi\) is an \(\Lambda\)-ism operator.

It is easy to see that problem (51) is equivalent to the following problem:
\[
\min_{\theta \in \Omega} \Xi (\theta) + \partial_{\mathcal{C}} (\theta),
\]
(52)
where \(\partial_{\mathcal{C}}\) is the indicator function of \(\mathcal{C}\). Thus, this problem becomes the problem of finding an element \(\theta^* \in \Omega\) such that
\[
\nabla \Xi (\theta^*) + \partial_{\mathcal{C}} (\theta^*) \ni 0,
\]
(53)
where \(\partial_{\mathcal{C}}\) is the subdifferential of \(\partial_{\mathcal{C}}\). We know that \(\partial_{\mathcal{C}}\) is a maximal monotone operator, and \((\mathcal{I} + m \partial_{\mathcal{C}})^{-1} = P_{\mathcal{C}}\) for all \(m > 0\).

For solving problem (51), we state the theorem in the following, which is similar to Theorem 1.

**Theorem 2.** Let the sequence \(\{\ell_n\}\) be bounded below by \(\min\{\rho/\ell_1\}, \rho > 0\), where \(\rho \in (0, 1)\) and \(\ell_0 > 0\). Given a parameter \(\Lambda \geq 0\) such that \(0 < \inf \{\ell_n\} \leq \sup \{\ell_n\} < 2\Lambda\). Let \(\{\eta_n\}\) be the sequence in \(\Omega\) which is defined by \(\theta_0, \theta_1 \in \Omega, \mathcal{C} \subset \Omega, 0 < \beta < 1, \) and
\[
\begin{aligned}
\mathcal{F}_n &= \theta_n + \Lambda (\theta_n - \theta_{n-1}), \\
\psi_n &= P_{\mathcal{C}} (\mathcal{F}_n - \ell_n \mathcal{Y} \mathcal{F}_n), \\
\phi_n &= (1 - \beta)\mathcal{F}_n + \beta \psi_n + \beta \ell_n (\psi_n - \psi_n),
\end{aligned}
\]
where \(\ell_{n+1} = \min\left\{\ell_n, \rho \frac{\|\psi_n - \psi_n\|}{\|\mathcal{F}_n - \psi_n\|}\right\}, \) if \(\mathcal{F}_n \neq \psi_n,
\]
\[
\ell_{n+1} = \ell_n, \quad \text{else,}
\]
\[
\begin{cases}
\eta_n \in \Omega: \|\phi_n - \phi_n\| \leq \|\theta_n - \eta_n\| + \Lambda^2 \|\theta_n - \beta_n\| \leq 2\Lambda^2 \|\theta_n - \beta_n\| \\
-2\Lambda (\theta_n - \eta_n, \theta_n - \beta_n) - \beta \Lambda \|\mathcal{F}_n - \psi_n\| > 0,
\end{cases}
\]
(54)

where \(\mathcal{Y}: \Omega \longrightarrow \mathcal{Y}\) is \(\Lambda\)-ism on a RHS \(\Omega, \mathcal{Y}: \Omega \longrightarrow 2^\Omega\) is a maximally monotone operator, and \(\Delta = (2 - \beta - 2\rho (1 - \beta)/\ell_{n+1} - \beta \rho \ell_{n+1} / \ell_n)\). If \(\Omega \neq \emptyset\), then the set sequence \(\{\theta_n\}\) converges strongly to \(\tau = P_{\Omega} (\theta_1),\) provided that \(\lim_{n \longrightarrow \infty} \|\psi_n - \mathcal{F}_n\| = 0\).

5. Solve a Split Feasibility Problem

In this section, we investigated the application of our proposed methods to the split convex feasibility problem (SCFP). Let \(T: \Omega \longrightarrow \Omega\) be a bounded linear operator and \(T^*\) its adjoint defined on the two RHSS \(\Omega_1\) and \(\Omega_2\). Assume that \(\mathcal{C} \subset \Omega_1\) and \(\mathcal{C} \subset \Omega_2\) are nonempty CCSSs. The SCFP [54] take the shape as follows:
\[
\text{create a point } \theta \in \mathcal{C} \text{ so that } T(\theta) \in \mathcal{D}.
\]
(55)

In a HS, SFP was initiated by Censor and Elfving [54], and they used a multistart approach to find an adaptive approach for resolving it. Many of the problems that emerge from state retrieval and restoration of medical image can be formulated as SVFP [55, 56]. SFP is also used in a variety of disciplines such as dynamic emission tomographic image.
reconstruction, image restoration, and radiation therapy treatment planning [57–59]. Let us consider
\[ \nabla (\theta) := \sqrt{\frac{1}{2} \| T \theta - P_Q(T \theta) \|^2} = T^* (I - P_Q) T \theta \]  
for the metric projection \( P_Q \) on to \( Q \), the gradient \( \nabla \), and \( \nabla = \partial_i \theta \). Due to the above construction, problem (55) has an inclusion format as described in (1). It can be seen that \( \nabla \) is Lipschitz continuous with constant \( L = \| T \|^2 \), and \( \nabla \) is maximal monotone, see, e.g., [60].

Let \( C \) be a nonempty CCS of a RHS \( \nabla \), and a normal cone of \( C \) at \( \theta \in C \) is defined by
\[ N_C (\theta) = \{ z \in \nabla : \langle z, y - \theta \rangle \leq 0, \forall y \in C \}. \]
(57)

Suppose \( \nabla : \nabla \longrightarrow (-\infty, +\infty) \) is a proper, lower semicontinuous, and convex function. For each \( \theta \in \nabla \), the subdifferential of \( \nabla \) is given by
\[ \partial g (\theta) = \{ z \in \nabla : g (y) - g (\theta) \geq \langle z, y - \theta \rangle, \forall y \in C \}. \]
(58)

Let \( i_C \) be a nonempty CCS of \( \nabla \), and the indicator function \( i_C \)
of \( C \) is defined by
\[ i_C (\theta) = \begin{cases} 0, & \text{if } \theta \in C \ln ; \\ \infty, & \text{otherwise.} \end{cases} \]
(59)

It is obvious that the indicator function \( i_C \) is proper, convex, and lower semicontinuous on \( C \). A subdifferential \( \partial i_C \) of \( i_C \) is a maximal monotone operator, and
\[ \partial i_C (\theta) = \{ z \in C : i_C (y) - i_C (\theta) \geq \langle z, y - \theta \rangle, \forall y \in C \} \]
\[ = \{ z \in \nabla : \langle z, y - \theta \rangle \leq 0, \forall y \in C \} = N_C (\theta). \]
(60)

For each \( \theta \in \nabla \), now we define the resolvent of an indicator function \( \partial i_C \) for each \( \lambda > 0 \) in the following manner:

\[ \ell_{n+1} = \min \left\{ \ell_n, \lambda \| i_C - i_C \| \right\}, \]
\[ \ell_n, \]
\[ = \min \left\{ \ell_n, \frac{\lambda \| i_C - i_C \|}{\| T^* (I - P_Q) T V_n \|} \right\}, \]
\[ = \min \left\{ \ell_n, \frac{\lambda \| i_C - i_C \|}{\| T^* (I - P_Q) T V_n \|} \right\}, \]
\[ \ell_n, \]
\[ \text{otherwise.} \]

St. (iv): calculate
\[ C_{n+1} = \left\{ \eta \in \nabla : \| \phi_n - \eta \|^2 \leq \| \theta_n - \eta \|^2 + \Lambda^2 \| \theta_{n-1} - \theta_n \|^2 \right\}, \]
\[ = \left\{ \eta \in \nabla : \| \phi_n - \eta \|^2 \leq \| \theta_n - \eta \|^2 + \Lambda^2 \| \theta_{n-1} - \theta_n \|^2 \right\}, \]
\[ \text{otherwise.} \]

where \( \Delta = (2 - \beta - 2 \beta (1 - \beta) \ell_n / \ell_{n+1} - \beta \rho \ell_n^2 / \ell_{n+1} \).

St. (v): compute
\[ \theta_{n+1} = P_{C_{n+1}} (\theta_n), \quad n \geq 1. \]
Put $n = n + 1$, and return to St. (i). If the solution set $\Gamma_{\text{SFP}}$ is nonempty, then the sequence $\{\vartheta_n\}$ converges weakly to an element of $\Gamma_{(\text{SFP})}$.

6. Numerical Discussion

This part is devoted to present a numerical solution to a SCFP in an infinite HS, which is a special inclusion problem as explained in Section 5. The problem setting is taken from [61]. We provide the comparison of Algorithm 1 (Alg1) in [45] and our proposed Algorithm 1 (Alg2).

Example 1. Let $\tau_1 = \tau_2 = L_2([0, 2\pi])$ be two HSs with an inner product

$$\langle \vartheta, y \rangle = \int_0^{2\pi} \vartheta(t) y(t) dt, \quad \forall \vartheta, y \in L_2([0, 2\pi]),$$

and the induced norm defined by

$$\|\vartheta\| = \sqrt{\int_0^{2\pi} |\vartheta(t)|^2 dt}, \quad \forall \vartheta \in L_2([0, 2\pi]).$$

Next, consider the feasible set $\mathcal{D} \subset \tau_1$ as
Consider the mapping $T: \Omega_1 \longrightarrow \Omega_2$ such that $T(\theta)(s) = \theta(s)$, $\theta \in \Omega_1$. Then, $(T^* \theta)(s) = \bar{\theta}(s)$, and $\|T\| = 1$. So, we shall solve the following problem:

$$\text{create } \bar{\theta}^* \in \mathcal{C} \text{ so that } T(\bar{\theta}^*) \in \mathcal{Q}. \quad (73)$$

We can also observe that since $(T\theta)(s) = \theta(s)$, $\theta \in \Omega_1$, the above problem is actually a CFP of the form

$$\mathcal{C} = \left\{ \theta \in \Omega_1: \int_0^{2\pi} \theta(t) \, dt \leq 1 \right\}, \quad (71)$$

and $\mathcal{Q} \subset \Omega_2$ is

$$\mathcal{Q} = \left\{ \theta \in \Omega_2: \int_0^{2\pi} |\theta(t) - \sin(t)|^2 \, dt \leq 16 \right\}. \quad (72)$$

\begin{align*}
\text{Figure 4: Numerical conduct of Alg2 by choosing different values of } \ell_0. \\
\text{Figure 5: Numerical comparison of Alg2 with Alg1 by assuming values of } \theta_\text{in} = \theta_0 = t. 
\end{align*}
create $\theta^* \in \mathcal{C} \cap \mathcal{Q}$.  

(74)

Figures 1–9 and Tables 1 and 2 show the numerical results by assuming $D_n = \|\theta_n - \theta_n\| \leq 10^{-6}$.

Remark 1. It is well known that the success of any iterative method depends on two main things: first, the number of iterations: when the number of iterations is small, the method is successful in saving effort. Second, time factor: the method that needs less time in implementation is excellent than its counterpart, which needs a lot of time and is considered successful in saving time. So, from figures and tables, we observe that our algorithm needs fewer iterations and less time than Algorithm 1 [45]. This illustrates that our method is successful in speeding up Algorithm 1 [45] and solving problem (55). Also, the performance of our algorithm is good because it saves time and effort in studying the convergence rate.
Figure 8: Numerical comparison of Alg2 with Alg1 by assuming values of $\theta_{-1} = \theta_0 = e^t \sin(t)$.

Figure 9: Numerical comparison of Alg2 with Alg1 by assuming values of $\theta_{-1} = \theta_0 = (t^2 - e^t)\cos(t)$.

Table 1: Numerical comparison of Alg2 with Alg1 by assuming different values of $\ell_0$.

<table>
<thead>
<tr>
<th>$\theta_{-1} = \theta_0$</th>
<th>$\rho$</th>
<th>$\Lambda$</th>
<th>$\ell_0$</th>
<th>Alg1</th>
<th>Alg2</th>
<th>Alg1</th>
<th>Alg2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5 exp $(t/2)^{3/4}$</td>
<td>0.27</td>
<td>0.50</td>
<td>1.00</td>
<td>56</td>
<td>50</td>
<td>0.0136</td>
<td>0.0190</td>
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<td>0.50</td>
<td>0.80</td>
<td>62</td>
<td>52</td>
<td>0.0219</td>
<td>0.0150</td>
</tr>
<tr>
<td>1/5 exp $(t/2)^{3/4}$</td>
<td>0.27</td>
<td>0.50</td>
<td>0.60</td>
<td>72</td>
<td>56</td>
<td>0.0186</td>
<td>0.0205</td>
</tr>
<tr>
<td>1/5 exp $(t/2)^{3/4}$</td>
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<td>0.50</td>
<td>0.40</td>
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<td>62</td>
<td>0.0160</td>
<td>0.0183</td>
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<tr>
<td>1/5 exp $(t/2)^{3/4}$</td>
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<td>0.50</td>
<td>0.20</td>
<td>104</td>
<td>72</td>
<td>0.0252</td>
<td>0.0225</td>
</tr>
</tbody>
</table>
Data Availability

Data sharing is not applicable to this article as no datasets are generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest concerning the publication of this article.

Authors’ Contributions

All authors contributed equally and significantly to writing this article.

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Authors’ Contributions

All authors contributed equally and significantly to writing this article.

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