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Research Article

Shrinking Projection Methods for Accelerating Relaxed Inertial Tseng-Type Algorithm with Applications

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Our main goal in this manuscript is to accelerate the relaxed inertial Tseng-type (RITT) algorithm by adding a shrinking projection (SP) term to the algorithm. Hence, strong convergence results were obtained in a real Hilbert space (RHS). A novel structure was used to solve an inclusion and a minimization problem under proper hypotheses. Finally, numerical experiments to elucidate the applications, performance, quickness, and effectiveness of our procedure are discussed.

1. Introduction

The standard form of the variational inclusion problem (VIP) on a RHS $\daleth$ is

$$0 \in (¥ + \Upsilon)\vartheta^*, \tag{1}$$

where ϑ^* is the unknown point that we need to find, for an operator $¥: \daleth \longrightarrow \daleth$ and a set-valued operator $\Upsilon: \daleth \longrightarrow 2^{\daleth}$. VIP is a frequent problem in the optimization field, which has a lot of applications in many areas, including equilibrium, machine learning, economics, engineering, image processing, and transportation problems [1–16].

The vintage technique to solve problem (1) which is denoted by $(¥ + \Upsilon)^{-1}(0)$ is the forward-backward splitting method [17–22] which is defined as follows: $\vartheta_1 \in \daleth$ and

$$\vartheta_{n+1} = (I + \ell\Upsilon)^{-1}(I - \ell¥)\vartheta_n, \quad n \geq 1, \tag{2}$$

where $\ell > 0$. In (2), each step of iterates includes only the forward step ¥ and the backward step Υ, but not $¥ + \Upsilon$. This technique involves the proximal point algorithm [23–25] and the gradient method [26–28] as special cases.

In a RHS, nice splitting iterative procedures presented by Lions and Mercier [29] are shown as follows:

$$\vartheta_{n+1} = \left(2J_\ell^{¥} - I\right)\left(2J_\ell^{\Upsilon} - I\right)\vartheta_n, \quad n \geq 1, \tag{3}$$

and

$$\vartheta_{n+1} = J_\tau^{¥}\left(2J_\ell^{\Upsilon} - I\right)\vartheta_n + \left(I - J_\ell^{\Upsilon}\right)\vartheta_n, \quad n \geq 1, \tag{4}$$

where $J_\ell^{\mathfrak{R}} = (I + \ell\mathfrak{R})^{-1}$. Permanently, two algorithms are weakly convergent [30], knowing that algorithm (3) is called Peaceman–Rachford algorithm [19] and scheme (4) is called Douglas–Rachford algorithm [31].

A lot of works are concerned with problem (1) for accretive operators and two monotone operators, for instance, a stationary solution to the initial-valued problem of the evolution equation

$$0 \in \frac{\partial \varpi}{\partial t} - \Xi\varpi, \varpi(0) = \varpi_\circ \tag{5}$$

can be adjusted as (1) when the governing maximal monotone $\Xi = ¥ + \Upsilon$ [29].

[1] is used to solve a minimization problem as follows:

$$\min_{\vartheta \in \daleth} \sqsupset(\vartheta) + \sigma(\vartheta), \tag{6}$$

where $\sqsupset, \sigma: \daleth \longrightarrow (-\infty, \infty]$ are proper and lower semicontinuous convex functions such that $\sqsupset$ is differentiable with L-Lipschitz gradient, and the proximal mapping of σ is

$$\vartheta \longmapsto \arg\min_{\omega\in\daleth} \sigma(\omega) + \frac{\|\vartheta-\omega\|^2}{2\ell}. \quad (7)$$

In particular, if $¥ = \nabla\sqsupset$ and $\Upsilon = \partial\sigma$, where $\nabla\sqsupset$ is the gradient of $\sqsupset$ and $\partial\sigma$ is the subdifferential of σ which takes the form $\partial\sigma(\vartheta) = \{\lambda \in \daleth: \sigma(\omega) \geq \sigma(\vartheta) + \langle\lambda, \omega-\vartheta\rangle \forall\omega \in \daleth\}$, problem (1) becomes (6), and (3) becomes

$$\vartheta_{n+1} = \mathrm{prox}_{\ell\sigma}(\vartheta_n - \ell\nabla\sqsupset(\vartheta_n)), \quad n \geq 1, \quad (8)$$

where $\ell > 0$ is the stepsize and $\mathrm{prox}_{\ell\sigma} = (I + \ell\,\partial\sigma)^{-1}$ is the proximity operator of σ.

The concept of merging the inertial term with the backward step was initiated by Alvarez and Attouch [32] and studied extensively in [33, 34]. For maximal monotone operators, it was called the inertial proximal point (IPP) algorithm, and they defined it by

$$\begin{cases} \mathfrak{I}_n = \vartheta_n + \Lambda_n(\vartheta_n - \vartheta_{n-1}), \\ \vartheta_{n+1} = (I + \ell_n\Upsilon)^{-1}\mathfrak{I}_n, \; n \geq 1. \end{cases} \quad (9)$$

It was proved that if $\{\ell_n\}$ is nondecreasing and $\{\Lambda_n\} \subset [0,1)$ with

$$\sum_{n=1}^{\infty} \Lambda_n \|\vartheta_n - \vartheta_{n-1}\|^2 < \infty, \quad (10)$$

then algorithm (9) converges weakly to zero of Υ. In particular, condition (10) is true for $\Lambda_n < 1/3$. Here, Λ_n is an extrapolation factor, and the inertia is represented by the term $\Lambda_n(\vartheta_n - \vartheta_{n-1})$. Note that the inertial term improves the performance of the procedure and has good convergence results [35–37].

Inertial term was merged with forward-backward algorithm by authors [38]. They added Lipschitz-continuous, a single-valued, cocoercive operator ¥ into the IPP algorithm:

$$\begin{cases} \mathfrak{I}_n = \vartheta_n + \Lambda_n(\vartheta_n - \vartheta_{n-1}), \\ \vartheta_{n+1} = (I + \ell_n\Upsilon)_n^{-1}(\mathfrak{I}_n - \ell_n ¥ \mathfrak{I}_n), \quad n \geq 1. \end{cases} \quad (11)$$

Via assumption (10), provided $\ell_n < 2/L$ with L, the Lipschitz constant of ¥, they obtained a weak convergence result. Note that, for $\Lambda_n > 0$, algorithm (11) does not take the form of (2), in spite of ¥ is still evaluated at the points $¥_n$.

Relaxation techniques and inertial effects have many advantages in solving monotone inclusion and convex optimization problems; this effect appeared in several names such as relaxed inertial proximal method, relaxed inertial forward-backward method, and relaxed inertial Douglas–Rachford algorithm; for more details, refer to [22, 24, 39–44].

Abubakar et al. [45] introduced the RITT method as follows:

$$\begin{cases} \mathfrak{I}_n = \vartheta_n + \Lambda(\vartheta_n - \vartheta_{n-1}), \\ \psi_n = (1 + \ell_n\Upsilon)^{-1}(1 - \ell_n ¥)\mathfrak{I}_n, \\ \phi_{n+1} = (1-\beta)\mathfrak{I}_n + \beta\psi_n + \beta\ell_n(¥\mathfrak{I}_n - ¥\psi_n), \; n \geq 1, \end{cases} \quad (12)$$

where Λ and β are extrapolation and relaxation parameters, respectively. Under this algorithm, they discussed the weak convergence to the solution point of VIP (1) and the problem of image recovery. Note that the extrapolation step works to accelerate but not for the desired acceleration.

The concept of the SP method was discussed by Takahashi et al. [46] as in the following algorithm:

$$\begin{cases} \vartheta_0 \in \daleth \text{ be arbitrarily fixed}, \\ C_1 = C, \vartheta_1 = P_{C_1}\vartheta_0, \\ \omega_n = \Lambda_n\vartheta_n + (1-\Lambda_n)\hbar_n\vartheta_n, \\ C_n = \{\eta \in C: \|\omega_n - \eta\| \leq \|\vartheta_n - \eta\|\}, \\ \vartheta_{n+1} = P_{C_{n+1}}\vartheta_0. \end{cases} \quad (13)$$

They just selected one closed convex (CC) set for a family of nonexpansive mappings $\{\hbar_n\}$ to modify Mann's iteration method [47] and proved that the sequence $\{\vartheta_n\}$ converges strongly to $P_{\mathrm{Fix}(\hbar)}\vartheta_0$, provided $\Lambda_n \leq e$ for all $n \geq 1$ and for some $0 < e < 1$.

In 2019, Yang and Liu [48] selected the stepsize sequence for the iterative algorithm for monotone variational inequalities, which are based on Tseng's extragradient method and Moudafi viscosity scheme that does not require either the knowledge of the Lipchitz constant of the operator or additional projections.

With the incorporation of results of [45, 46, 48], we accelerate RITT algorithm by adding the SP method to algorithm (12). In a RHS, strong convergence results are given under a proposed algorithm. As applications, our algorithm was used to find the solution to a VIP and minimization problem under certain conditions. Eventually, numerical experiments to illustrate the applications, performance, acceleration, and effectiveness of the proposed algorithm are presented.

2. Preparatory Lemmas and Definitions

Suppose that C is a nonempty closed convex subset (CCS) of a RHS $\daleth$; we shall refer to " $\longrightarrow$ " as the strong convergence, and P_C: $\daleth \longrightarrow C$ is the nearest point projection, that is, for all $\vartheta \in \daleth$ and $\omega \in C$, $\|\vartheta - P_C\vartheta\| \leq \|\vartheta - \omega\|$. P_C is called the metric projection. It is obvious that P_C verifies the following inequality:

$$\|P_C\vartheta - P_C\omega\|^2 \leq \langle P_C\vartheta - P_C\omega, \vartheta - \omega\rangle, \quad (14)$$

for all $\vartheta, \omega \in \daleth$. In other words, the metric projection P_C is firmly nonexpansive. Hence, $\langle\vartheta - P_C\vartheta, \omega - P_C\omega\rangle \leq 0$ holds for all $\vartheta \in \daleth$ and $\omega \in C$, see [49, 50].

The following inequality holds in a HS [51]:

$$\|l \pm m\|^2 = \|l\|^2 + \|m\|^2 \pm 2\langle l, m\rangle, \quad (15)$$

for all $l, m \in \daleth$.

Lemma 1 (see [52]). *Let C be a nonempty CCS of a RHS $\daleth$. For each $\vartheta, \omega, v \in \daleth$ and $\in \mathbb{R}$, the following set is closed and convex:*

$$\left\{\eta \in C\colon \|\omega-\eta\|^2 \le \|\vartheta-\eta\|^2 + \langle \upsilon, \eta\rangle + \delta\right\}. \quad (16)$$

Lemma 2 (see [38]). *Let C be a nonempty CCS of a RHS ℸ and P_C: ℸ ⟶ C be the metric projection. Then,*

$$\left\|\omega - P_C\vartheta\right\|^2 + \left\|\vartheta - P_C\vartheta\right\|^2 \le \|\vartheta-\omega\|^2, \quad (17)$$

for all $\vartheta \in$ ℸ *and* $\omega \in C$.

Definition 1. Suppose that $D(¥) \subset$ ℸ and $R(¥) \subset$ ℸ are the domain and the range of an operator ¥, respectively. For all $\vartheta, \omega \in D(¥)$, an operator ¥ is called

(1) Monotone if

$$\langle \vartheta-\omega, ¥\vartheta - ¥\omega\rangle \ge 0. \quad (18)$$

(2) L–Lipschitz if

$$\|¥\vartheta - ¥\omega\| \le L\|\vartheta-\omega\|. \quad (19)$$

(3) β–Strongly monotone if there exists $\beta > 0$ such that

$$\langle \vartheta-\omega, ¥\vartheta - ¥\omega\rangle \ge \beta\|\vartheta-\omega\|^2. \quad (20)$$

(4) Λ–Inverse strongly monotone (Λ–ism) if there exists $\Lambda > 0$ such that

$$\langle \vartheta-\omega, ¥\vartheta - ¥\omega\rangle \ge \Lambda\|¥\vartheta - ¥\omega\|^2. \quad (21)$$

Lemma 3 (see [44]). *Let ℸ be a RHS, ¥: ℷ ⟶ ℷ be an Λ–ism operator, and Υ: ℸ ⟶ $2^{ℸ}$ be a maximal monotone operator. For each $\ell > 0$, we define*

$$\mho_\ell = J_\ell^\Upsilon (I - \ell ¥) = (I + \ell\Upsilon)^{-1}(I - \ell ¥). \quad (22)$$

Then, we get

(i) For $\ell > 0$, $fix(\mho_\ell) = (¥ + \Upsilon)^{-1}(0)$

(ii) For $0 < s \le \ell$ and $\vartheta \in$ ℸ, $\|\vartheta - \mho_s\vartheta\| \le 2\|\vartheta - \mho_\ell\vartheta\|$

Lemma 4. *Let ℸ be a RHS, ¥: ℸ ⟶ ℸ be an Λ–ism operator, and Υ: ℸ ⟶ $2^{ℸ}$ be a maximal monotone operator. For each $\ell > 0$, we have*

$$\left\|\mho_\ell\vartheta - \mho_\ell\omega\right\|^2 \le \|\vartheta-\omega\|^2 - \ell(2\Lambda - \ell)\|¥\vartheta - ¥\omega\|^2, \quad (23)$$

for all $\vartheta, \omega \in$ ℸ.

Proof. For all $\vartheta, \omega \in$ ℸ, we get

$$\begin{aligned}\left\|\mho_\ell\vartheta - \mho_\ell\omega\right\|^2 &= \left\|J_r^\Upsilon (I - \ell ¥)\vartheta - J_r^\Upsilon (I - \ell ¥)\omega\right\|^2\\ &\le \|(I - \ell ¥)\vartheta - (I - \ell ¥)\omega\|^2\\ &= \|(\vartheta-\omega) - \ell(¥\vartheta - ¥\omega)\|^2\\ &= \|\vartheta-\omega\|^2 - 2\ell\langle \vartheta-\omega, ¥\vartheta - ¥\omega\rangle + \ell^2\|¥\vartheta - ¥\omega\|^2\\ &\le \|\vartheta-\omega\|^2 - 2\ell\Lambda\|¥\vartheta - ¥\omega\|^2 + \ell^2\|¥\vartheta - ¥\omega\|^2\\ &= \|\vartheta-\omega\|^2 - \ell(2\Lambda - \ell)\|¥\vartheta - ¥\omega\|^2.\end{aligned} \quad (24)$$

The proof is ended. □

3. Shrinking Projection Relaxed Inertial Tseng-Type Algorithm

We provide a method consisting of the forward-backward splitting method with an inertial factor and an explicit stepsize formula, which are being used to ameliorate the convergence average of the iterative scheme and to make the manner independent of the Lipschitz constants. The detailed method is provided in Algorithm 1.

Note that

(i) Since ¥ is an Λ–ism operator, it is a Lipschitz function with a constant L, $¥\mathfrak{I}_n \ne ¥\psi_n$, and we get

$$\frac{\rho\left\|\mathfrak{I}_n - \psi_n\right\|}{\left\|¥\mathfrak{I}_n - ¥\psi_n\right\|} \ge \frac{\rho}{L}. \quad (25)$$

It is obvious for $¥\mathfrak{I}_n = ¥\psi_n$ that inequality (25) is satisfied. Hence, it follows that $\ell_n \ge \min\{(\rho/L), \ell_0\}$. This implies that the generated sequence $\{\ell_n\}$ is bounded below by $\min\{(\rho/L), \ell_0\}$, i.e., $\{\ell_n\}$ is monotonically decreasing.

(ii) By (i) and (25), we have

$$\ell_{n+1}\left\|¥\mathfrak{I}_n - ¥\psi_n\right\| \le \rho\left\|\mathfrak{I}_n - \psi_n\right\|, \quad (26)$$

i.e., the update (28) is well defined.

(iii) If we delete the shrinking projection term from our algorithm, we get the algorithms of the papers [22, 45, 53].

Theorem 1. *Let ℸ be a RHS and the operators ¥: ℸ ⟶ ℸ be Λ–ism on ℸ, and Υ: ℸ ⟶ $2^{ℸ}$ is maximally monotone. If feasible set $\Omega = (¥ + \Upsilon)^{-1}(0)$ of (1) is a nonempty CCS of a RHS ℸ, then the sequence $\{\vartheta_n\}$ generated by Algorithm 1 converges strongly to a point $\tau = P_\Omega(\vartheta_1)$, provided that*

(i) $0 < \liminf_{n\longrightarrow\infty} \ell\ell_n \le \limsup_{n\longrightarrow\infty} \ell_n < 2\Lambda$.

(ii) $lim_{n\longrightarrow\infty}\|\psi_n - \mathfrak{I}_n\| = 0$.

Proof. The proof will be divided as follows: □

Initialization: select initial $\vartheta_0, \vartheta_1 \in \urcorner$, $\rho \in (0,1)$, $\Lambda \geq 0$, $\ell_0 > 0$, and $0 < \beta < 1$.
St. (i). Put $\mathfrak{I}_n$ as:
$\mathfrak{I}_n = \vartheta_n + \Lambda(\vartheta_n - \vartheta_{n-1})$,
St. (ii). Calculate:
$\psi_n = (1 + \ell_n \Upsilon)^{-1}(1 - \ell_n ¥)\mathfrak{I}_n$.,
If $\mathfrak{I}_n = \psi_n$, discontinue. $\mathfrak{I}_n$ is a solution of (1), otherwise, continue to **St. (iii)**
St. (iii). Calculate:
$\phi_n = (1-\beta)\mathfrak{I}_n + \beta\psi_n + \beta\ell_n(¥\mathfrak{I}_n - ¥\psi_n)$,
where ℓ_{n+1} is stepsize sequence revised as follows:

$$\ell_{n+1} = \begin{cases} \min\{\ell_n, (\rho\|\mathfrak{I}_n - \psi_n\|)/(\|¥\mathfrak{I}_n - ¥\psi_n\|)\}, & \text{if } ¥\mathfrak{I}_n \neq ¥\psi_n, \\ \ell_n, & \text{else,} \end{cases}$$

St. (iv). Calculate:
$C_{n+1} = \{\eta \in C_n: \|\phi_n - \eta\|^2 \leq \|\vartheta_n - \eta\|^2 + \Lambda^2\|\vartheta_{n-1} - \vartheta_n\|^2 - 2\Lambda\langle\vartheta_n - \eta, \vartheta_{n-1} - \vartheta_n\rangle - \beta\Delta\|\mathfrak{I}_n - \psi_n\|^2\}$,
where $\Delta = (2 - \beta - 2\rho(1-\beta)\ell_n/\ell_{n+1} - \beta\rho^2\ell_n^2/\ell_{n+1}^2)$.
St. (v). Compute
$\vartheta_{n+1} = P_{C_{n+1}}(\vartheta_1)$, $n \geq 1$,
put $n = n+1$, and return to **St. (i).**

Algorithm 1: Splitting method for the VIP.

Part 1. Demonstrate that $P_{C_{n+1}}\vartheta_1$ is well-defined, for each $\vartheta_1 \in \urcorner, n \geq 1$, and $\Omega \subset C_{n+1}$. It follows from condition (i) and Lemma 4 that $\mho_{\ell_n} = (I + \ell_n\Upsilon)^{-1}(I - \ell_n ¥)$ is a nonexpansive mapping. Lemma 3 implies that Ω is a closed and convex set, and Lemma 1 clarifies that C_{n+1} is closed and convex, for all $n \geq 1$.

Let $\eta \in \Omega$; we have

$$\|\mathfrak{I}_n - \eta^2\| = \|(\vartheta_n - \eta) - \Lambda(\vartheta_{n-1} - \vartheta_n)\|^2 = \|\vartheta_n - \eta\|^2 - 2\Lambda\langle\vartheta_n - \eta, \vartheta_{n-1} - \vartheta_n\rangle + \Lambda^2\|\vartheta_{n-1} - \vartheta_n\|^2. \tag{27}$$

Since the resolvent $\mho_{\ell_n}$ is firmly a nonexpansive mapping and by Lemma 3, we have

$$\begin{aligned} \langle\psi_n - \eta, \mathfrak{I}_n - \psi_n - \ell_n ¥\mathfrak{I}_n\rangle &= \langle J_\ell^\Upsilon(I - \ell_n ¥)\mathfrak{I}_n - J_\ell^\Upsilon(I - \ell_n ¥)\eta, (I - \ell_n ¥)\mathfrak{I}_n - (I - \ell_n ¥)\eta + (I - \ell_n ¥)\eta - \psi_n\rangle \\ &\geq \|\psi_n - \eta\|^2 + \langle\psi_n - \eta, \eta - \psi_n\rangle - \langle\psi_n - \eta, \ell_n ¥\psi_n\rangle = -\langle\psi_n - \eta, \ell_n ¥\psi_n\rangle. \end{aligned} \tag{28}$$

Hence, by (28), we get

$$\langle\psi_n - \eta, \mathfrak{I}_n - \psi_n - \ell_n(¥\mathfrak{I}_n + ¥\psi_n)\rangle \geq 0, \tag{29}$$

which leads to

$$2\langle\mathfrak{I}_n - \psi_n, \psi_n - \eta\rangle - 2\ell_n\langle¥\mathfrak{I}_n + ¥\psi_n, \psi_n - \eta\rangle \geq 0. \tag{30}$$

It is obvious that

$$2\langle\mathfrak{I}_n - \psi_n, \psi_n - \eta\rangle = \|\mathfrak{I}_n - \eta\|^2 - \|\mathfrak{I}_n - \psi_n\|^2 - \|\psi_n - \eta\|^2. \tag{31}$$

Applying (31) in (30), we can write

$$\|\psi_n - \eta\|^2 \leq \langle\mathfrak{I}_n - \eta\rangle^2 - \|\mathfrak{I}_n - \psi_n\|^2 - 2\ell_n\langle¥\mathfrak{I}_n - ¥\psi_n, \psi_n - \eta\rangle. \tag{32}$$

Now, from definition ϕ_n, we have

$$\begin{aligned} \|\phi_n - \eta\|^2 &= \|(1-\beta)\mathfrak{I}_n + \beta\psi_n + \beta\ell_n(¥\mathfrak{I}_n - ¥\psi_n) - \eta\|^2 = \|(1-\beta)(\mathfrak{I}_n - \eta) + \beta(\psi_n - \eta) + \beta\ell_n(¥\mathfrak{I}_n - ¥\psi_n)\|^2 \\ &= (1-\beta)^2\|\mathfrak{I}_n - \eta\|^2 + \beta^2\|\psi_n - \eta\| + \beta^2\ell_n^2\|¥\mathfrak{I}_n - ¥\psi_n\|^2 + 2\beta(1-\beta)\langle\mathfrak{I}_n - \eta, \psi_n - \eta\rangle \\ &\quad + 2\beta\ell_n(1-\beta)\langle\mathfrak{I}_n - \eta, ¥\mathfrak{I}_n - ¥\psi_n\rangle + 2\beta^2\ell_n\langle\psi_n - \eta, ¥\mathfrak{I}_n - ¥\psi_n\rangle. \end{aligned} \tag{33}$$

From equation (15), one can write

$$2\langle \mathfrak{I}_n-\eta,\psi_n-\eta\rangle = \|\mathfrak{I}_n-\eta\|^2-\|\mathfrak{I}_n-\psi_n\|^2+\|\psi_n-\eta\|^2. \tag{34}$$

Applying (34) in (33), we get

$$\begin{aligned}\|\phi_n-\eta\|^2 ={}& (1-\beta)\|\mathfrak{I}_n-\eta\|^2+\beta\|\psi_n-\eta\|^2-\beta(1-\beta)\|\psi_n-\mathfrak{I}_n\|^2+\beta^2\ell_n^2\|¥\mathfrak{I}_n-¥\psi_n\|^2+2\beta\ell_n(1-\beta)\langle\mathfrak{I}_n-\eta,¥\mathfrak{I}_n-¥\psi_n\rangle\\ &+2\beta^2\ell_n\langle\psi_n-\eta,¥\mathfrak{I}_n-¥\psi_n\rangle.\end{aligned} \tag{35}$$

It follows from (32), (35), and (26) that

$$\begin{aligned}\|\phi_n-\eta\|^2 \le{}& (1-\beta)\|\mathfrak{I}_n-\eta\|^2+\beta\left[\|\mathfrak{I}_n-\eta\|^2-\|\mathfrak{I}_n-\psi_n\|^2-2\ell_n\langle ¥\mathfrak{I}_n-¥\psi_n,\psi_n-\eta\rangle\right]-\beta(1-\beta)\|\psi_n-\mathfrak{I}_n\|^2\\ &+\beta^2\ell_n^2\|¥\mathfrak{I}_n-¥\psi_n\|^2+2\beta\ell_n(1-\beta)\langle\mathfrak{I}_n-\eta,¥\mathfrak{I}_n-¥\psi_n\rangle+2\beta^2\ell_n\langle\psi_n-\eta,¥\mathfrak{I}_n-¥\psi_n\rangle\\ \le{}& \|\mathfrak{I}_n-\eta\|^2-\beta(2-\beta)\|\mathfrak{I}_n-\psi_n\|^2-2\beta\ell_n\langle ¥\mathfrak{I}_n-¥\psi_n,\psi_n-\eta\rangle+\beta^2\ell_n^2\|¥\mathfrak{I}_n-¥\psi_n\|^2\\ &+2\beta\ell_n(1-\beta)\langle\mathfrak{I}_n-\eta,¥\mathfrak{I}_n-¥\psi_n\rangle+2\beta\ell_n\langle\psi_n-\eta,¥\mathfrak{I}_n-¥\psi_n\rangle\\ \le{}& \|\mathfrak{I}_n-\eta\|^2-\beta(2-\beta)\|\mathfrak{I}_n-\psi_n\|^2+\beta^2\ell_n^2\|¥\mathfrak{I}_n-¥\psi_n\|^2+2\beta\ell_n(1-\beta)\langle\mathfrak{I}_n-\psi_n,¥\mathfrak{I}_n-¥\psi_n\rangle\\ \le{}& \|\mathfrak{I}_n-\eta\|^2-\beta(2-\beta)\|\mathfrak{I}_n-\psi_n\|^2+\beta^2\ell_n^2\frac{\rho^2}{\ell_{n+1}^2}\|\mathfrak{I}_n-\psi_n\|^2+2\beta\ell_n(1-\beta)\frac{\rho}{\ell_{n+1}}\|\mathfrak{I}_n-\psi_n\|^2\\ ={}& \|\mathfrak{I}_n-\eta\|^2-\beta\left[2-\beta-2\rho(1-\beta)\frac{\ell_n}{\ell_{n+1}}-\beta\rho^2\frac{\ell_n^2}{\ell_{n+1}^2}\right]\|\mathfrak{I}_n-\psi_n\|^2=\|\mathfrak{I}_n-\eta\|^2-\beta\Delta_n\|\mathfrak{I}_n-\psi_n\|^2.\end{aligned} \tag{36}$$

Applying (27) in (36), we have

$$\begin{aligned}\|\phi_n-\eta\|^2 \le{}& \|\vartheta_n-\eta\|^2+\Lambda^2\|\vartheta_{n-1}-\vartheta_n\|^2\\ &-2\Lambda\langle\vartheta_n-\eta,\vartheta_{n-1}-\vartheta_n\rangle-\beta\Delta_n\|\mathfrak{I}_n-\psi_n\|^2.\end{aligned} \tag{37}$$

It is clear that $\Omega\subset C_1=\daleth$. Assume that $\Omega\subset C_n$ for some $n\ge 1$. Then, $\eta\in C_n$ and by (37), we have for all $n\ge 1$, $\eta\in C_{n+1}$. Thus, $\Omega\subset C_{n+1}$ for all $n\ge 1$, i.e., $P_{C_{n+1}}\vartheta_1$ is well-defined and bounded.

Part 2. Illustrate that $\{\vartheta_n\}$ is bounded. Since $\Omega\ne\emptyset$ and closed and convex subset of $\daleth$, there is a unique $u\in\Omega$ such that $u=P_\Omega\vartheta_1$. This leads to $\vartheta_n=P_{C_n}\vartheta_1$, $C_n\subset C_{n+1}$, and $\vartheta_{n+1}\in C_n$ for all $n\ge 1$, and we have

$$\|\vartheta_n-\vartheta_1\|\le\|\vartheta_{n+1}-\vartheta_1\|. \tag{38}$$

Furthermore, as $\Omega\subset C_n$, for all $n\ge 1$, we obtain

$$\|\vartheta_n-\vartheta_1\|\le\|u-\vartheta_1\|. \tag{39}$$

It follows by (38) and (39) that $\lim_{n\longrightarrow\infty}\|\vartheta_n-\vartheta_1\|$ exists. Hence, $\{\vartheta_n\}$ is bounded.

Part 3. Fulfillment of $\lim_{n\longrightarrow\infty}\vartheta_n=\tau$. By the definition of C_n, for $m>n$, we observe that $\vartheta_m=P_{C_m}\vartheta_1\in C_m\subset C_n$. From Lemma 2, we have

$$\|\vartheta_m-\vartheta_n\|^2\le\|\vartheta_m-\vartheta_1\|^2-\|\vartheta_n-\vartheta_1\|^2. \tag{40}$$

By Part 2, we conclude that $\lim_{n,m\longrightarrow\infty}\|\vartheta_m-\vartheta_n\|^2=0$. Thus, $\{\vartheta_n\}$ is a Cauchy sequence. Hence, $\lim_{n\longrightarrow\infty}\vartheta_n=\tau$. Additionally, we get

$$\lim_{n\longrightarrow\infty}\|\vartheta_{n+1}-\vartheta_n\|=0. \tag{41}$$

Part 4. Prove that $\tau\in\Omega$. It follows from (41) that

$$\|\mathfrak{I}_n-\vartheta_n\|=\Lambda\|\vartheta_n-\vartheta_{n-1}\|\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{42}$$

Also, by (42) and condition (ii), we can write

$$\|\psi_n-\vartheta_n\|\le\|\psi_n-\mathfrak{I}_n\|+\|\mathfrak{I}_n-\vartheta_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{43}$$

From triangle inequality on the norm and (42) and (43), we obtain

$$\|\mathfrak{I}_n - \psi_n\| \leq \|\mathfrak{I}_n - \vartheta_n\| + \|\psi_n - \vartheta_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{44}$$

Replacing η with ϑ_n in (36) and using (41) and (44), we have

$$\|\phi_n - \vartheta_n\|^2 \leq \Lambda^2\|\vartheta_{n-1} - \vartheta_n\|^2 - \beta\Delta_n\|\mathfrak{I}_n - \psi_n\|^2 \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{45}$$

Applying (41), (42), and (45), we can write

$$\begin{aligned} \|\vartheta_{n+1} - \mathfrak{I}_n\| &\leq \|\vartheta_{n+1} - \vartheta_n\| + \|\mathfrak{I}_n - \vartheta_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty, \\ \|\vartheta_{n+1} - \phi_n\| &\leq \|\vartheta_{n+1} - \vartheta_n\| + \|\phi_n - \vartheta_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty, \\ \|\phi_n - \mathfrak{I}_n\| &\leq \|\phi_n - \vartheta_n\| + \|\mathfrak{I}_n - \vartheta_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{46}$$

It follows from (44) that

$$\lim_{n\longrightarrow\infty}\|\mho_{\ell_n}\mathfrak{I}_n - \mathfrak{I}_n\| = \lim_{n\longrightarrow\infty}\|\psi_n - \mathfrak{I}_n\| = 0. \tag{47}$$

Since $\liminf_{n\longrightarrow\infty}\ell_n > 0$, there is $\varepsilon > 0$ such that $\ell_n \geq \varepsilon$ and $\varepsilon \in (0, 2\Lambda)$ for all $n \geq 1$. Then, by Lemma 3 (ii) and (47), we get

$$\|\mho_\varepsilon\mathfrak{I}_n - \mathfrak{I}_n\| \leq 2\|\mho_{\ell_n}\mathfrak{I}_n - \mathfrak{I}_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{48}$$

From (45) and (46), since $\vartheta_n \longrightarrow \tau$ as $n \longrightarrow \infty$, we have also $\mathfrak{I}_n \longrightarrow \tau$ as $n \longrightarrow \infty$. Since $\mho_\varepsilon$ is a nonexpansive and continuous mapping, from (47), we conclude that $\tau \in \Omega$.

Part 5. Show that $\tau = P_\Omega(\vartheta_1)$. Since $\vartheta_n = P_{C_n}\vartheta_1$ and $\Omega \subset C_n$, we can get

$$\langle\vartheta_1 - \vartheta_n, \vartheta_n - \eta\rangle \geq 0, \quad \forall\eta \in \Omega. \tag{49}$$

Setting $n \longrightarrow \infty$ in (49), we have

$$\langle\vartheta_1 - \tau, \tau - \eta\rangle \geq 0, \quad \forall\eta \in \Omega. \tag{50}$$

This shows that $\tau = P_\Omega(\vartheta_1)$. This finishes the proof.

4. Solve a Minimization Problem

As an application of our theorem, we solve the following constrained convex minimization problem:

$$min_{\vartheta\in C}\sqsupset(\vartheta), \tag{51}$$

where $\sqsupset: \urcorner \longrightarrow \mathbb{R}$ is a convex function. We suppose that the function $\sqsupset$ is differentiable such that $\nabla\sqsupset$ is an Λ–ism operator.

It is easy to see that problem (51) is equivalent to the following problem:

$$\min_{\vartheta\in\urcorner}[\sqsupset(\vartheta) + \wp_C(\vartheta)], \tag{52}$$

where $\wp_C$ is the indicator function of C. Thus, this problem becomes the problem of finding an element $\vartheta^* \in \urcorner$ such that

$$\nabla\sqsupset(\vartheta^*) + \partial\wp_C(\vartheta^*) \ni 0, \tag{53}$$

where $\partial\wp_C$ is the subdifferential of $\wp_C$. We know that $\partial\wp_C$ is a maximal monotone operator, and $(I + m\,\partial\wp_C)^{-1} = P_C$ for all $m > 0$.

For solving problem (51), we state the theorem in the following, which is similar to Theorem 1.

Theorem 2. *Let the sequence $\{\ell_n\}$ be bounded below by $\min\{(\rho/L), \ell_0\}$, where $\rho \in (0,1)$ and $\ell_0 > 0$. Given a parameter $\Lambda \geq 0$ such that $0 < \inf\{\ell_n\} \leq sup_n\{\ell_n\} < 2\Lambda$. Let $\{\vartheta_n\}$ be the sequence in $\urcorner$ which is defined by $\vartheta_0, \vartheta_1 \in \urcorner$, $C_1 = \urcorner$, $0 < \beta < 1$, and*

$$\begin{cases} \mathfrak{I}_n = \vartheta_n + \Lambda(\vartheta_n - \vartheta_{n-1}), \\ \psi_n = P_C(\mathfrak{I}_n - \ell_n\nabla\sqsupset\mathfrak{I}_n), \\ \phi_n = (1-\beta)\mathfrak{I}_n + \beta\psi_n + \beta\ell_n(¥\mathfrak{I}_n - ¥\psi_n), \\ \text{where,} \quad \ell_{n+1} = \begin{cases} \min\left\{\ell_n, \dfrac{\rho\|\mathfrak{I}_n - \psi_n\|}{\|¥\mathfrak{I}_n - ¥\psi_n\|}\right\}, & \text{if } ¥\mathfrak{I}_n \neq ¥\psi_n, \\ \ell_n, & \text{else,} \end{cases} \\ C_{n+1} = \left\{\begin{aligned} &\eta \in \urcorner: \|\phi_n - \eta\|^2 \leq \|\vartheta_n - \eta\|^2 + \Lambda^2\|\vartheta_{n-1} - \vartheta_n\|^2 \\ &-2\Lambda\langle\vartheta_n - \eta, \vartheta_{n-1} - \vartheta_n\rangle - \beta\Delta\|\mathfrak{I}_n - \psi_n\|^2 \end{aligned}\right\}, \\ \vartheta_{n+1} = P_{C_{n+1}}(\vartheta_1), \; n \geq 1, \end{cases} \tag{54}$$

where $¥: \urcorner \longrightarrow \urcorner$ is Λ–ism on a RHS $\urcorner$, $\Upsilon: \urcorner \longrightarrow 2^\urcorner$ is a maximally monotone operator, and $\Delta = (2-\beta - 2\rho(1-\beta)\ell_n/\ell_{n+1} - \beta\rho^2\ell_n^2/\ell_{n+1}^2)$. If $\Omega \neq \varnothing$, then the sequence $\{\vartheta_n\}$ converges strongly to $\tau = P_\Omega(\vartheta_1)$, provided that $lim_{n\longrightarrow\infty}\|\psi_n - \mathfrak{I}_n\| = 0$.

5. Solve a Split Feasibility Problem

In this section, we investigated the application of our proposed methods to the split convex feasibility problem (SCFP). Let $T: \urcorner_1 \longrightarrow \urcorner_2$ be a bounded linear operator and T^* its adjoint defined on the two RHSs $\urcorner_1$ and $\urcorner_2$. Assume that $\mathscr{C} \subset \urcorner_1$ and $\mathscr{Q} \subset \urcorner_2$ are nonempty CCSs. The SCFP [54] take the shape as follows:

$$\text{create a point } \vartheta \in \mathscr{C} \text{ so that } T(\vartheta) \in \mathscr{Q}. \tag{55}$$

In a HS, SFP was initiated by Censor and Elfving [54], and they used a multidistance approach to find an adaptive approach for resolving it. Many of the problems that emerge from state retrieval and restoration of medical image can be formulated as SVFP [55, 56]. SFP is also used in a variety of disciplines such as dynamic emission tomographic image reconstruction, image restoration, and radiation therapy treatment planning [57–59]. Let us consider

$$¥(\vartheta) := \nabla\left(\frac{1}{2}\|T\vartheta - P_Q(T\vartheta)\|^2\right) = T^*\left(I - P_Q\right)T\vartheta \tag{56}$$

for the metric projection P_Q on to Q, the gradient ∇, and $\Upsilon = \partial i_{\mathscr{C}}$. Due to the above construction, problem (55) has an inclusion format as described in (1). It can be seen that ¥ is Lipschitz continuous with constant $L = \|T\|^2$, and Υ is maximal monotone, see, e.g., [60].

Let $\mathscr{C}$ be a nonempty CCS of a RHS $\daleth$, and a normal cone of $\mathscr{C}$ at $\vartheta \in \mathscr{C}$ is defined by

$$N_{\mathscr{C}}(\vartheta) = \{z \in \daleth: \langle z, y - \vartheta\rangle \leq 0, \forall y \in \mathscr{C}\}. \tag{57}$$

Suppose $g: \daleth \longrightarrow (-\infty, +\infty)$ is a proper, lower semicontinuous, and convex function. For each $\vartheta \in \daleth$, the subdifferential ∂g of g is given by

$$\partial g(\vartheta) = \{z \in \daleth: g(y) - g(\vartheta) \geq \langle z, y - \vartheta\rangle, \quad \forall y \in \mathscr{C}\}. \tag{58}$$

For any nonempty CCS $\mathscr{C}$ of $\daleth$, the indicator function $i_{\mathscr{C}}$ of $\mathscr{C}$ is defined by

$$i_{\mathscr{C}}(\vartheta) = \begin{cases} 0, & \text{if } \vartheta \in \mathscr{C} \\ \infty, & \text{otherwise.} \end{cases} \tag{59}$$

It is obvious that the indicator function $i_{\mathscr{C}}$ is proper, convex, and lower semicontinuous on $\daleth$. A subdifferential $\partial i_{\mathscr{C}}$ of $i_{\mathscr{C}}$ is a maximal monotone operator, and

$$\begin{aligned}\partial i_{\mathscr{C}}(\vartheta) &= \{z \in \daleth: i_{\mathscr{C}}(y) - i_{\mathscr{C}}(\vartheta) \geq \langle z, y - \vartheta\rangle, \forall y \in \mathscr{C}\} \\ &= \{z \in \daleth: \langle z, y - \vartheta\rangle \leq 0, \forall y \in \mathscr{C}\} = N_{\mathscr{C}}(\vartheta).\end{aligned} \tag{60}$$

For each $\vartheta \in \daleth$, now we define the resolvent of an indicator function $\partial i_{\mathscr{C}}$ for each $\lambda > 0$ in the following manner:

$$J_{\lambda}^{\partial i_{\mathscr{C}}} = \left(\mathrm{Id} + \lambda\,\partial i_{\mathscr{C}}\right)^{-1}. \tag{61}$$

Hence, we can observe that

$$\begin{aligned}&y = J_{\lambda}^{\partial i_{\mathscr{C}}}(\vartheta) \Longleftrightarrow \vartheta \in \left(y + \lambda\,\partial i_{\mathscr{C}}(y)\right)^{-1} \Longleftrightarrow \vartheta - y \in \lambda\,\partial i_{\mathscr{C}}(y) \\ &\Longleftrightarrow y = P_{\mathscr{C}}(\vartheta).\end{aligned} \tag{62}$$

Now, on the basis of the above, Algorithm 1 may be reduced to the following scheme.

Theorem 3. *Let $\{\vartheta_n\}$ be a sequence generated by the following scheme: choose $\vartheta_{-1}, \vartheta_0 \in \mathscr{C}$, $\rho \in (0,1)$, $\Lambda \geq 0$, $\ell_0 > 0$, and $0 < \beta < 1$.*

St. (i): compute $\mathfrak{I}_n$ in the following way:

$$\mathfrak{I}_n = \vartheta_n + \Lambda\left(\vartheta_n - \vartheta_{n-1}\right). \tag{63}$$

St. (ii): calculate

$$\psi_n = P_{\mathscr{C}}\left[\mathfrak{I}_n - \ell_n T^*\left(I - P_Q\right)T\mathfrak{I}_n\right]. \tag{64}$$

If $\mathfrak{I}_n = \psi_n$, stop, and $\mathfrak{I}_n$ is a solution of problem (55); otherwise, continue to St. (iii).

St. (iii): calculate

$$\phi_n = (1-\beta)\mathfrak{I}_n + \beta\psi_n + \beta\ell_n\left[T^*\left(I - P_Q\right)T\mathfrak{I}_n - T^*\left(I - P_Q\right)T\psi_n\right], \tag{65}$$

where ℓ_{n+1} is the stepsize sequence revised in the following way:

$$\ell_{n+1} = \begin{cases} \min\left\{\ell_n, \dfrac{\rho\|\mathfrak{I}_n - \psi_n\|}{\left[T^*\left(I - P_Q\right)T\mathfrak{I}_n - T^*\left(I - P_Q\right)T\psi_n\right]}\right\}, & \text{if } T^*\left(I - P_Q\right)T\mathfrak{I}_n \neq T^*\left(I - P_Q\right)T\psi_n, \\ \ell_n, & \text{otherwise.} \end{cases} \tag{66}$$

St. (iv): calculate

$$C_{n+1} = \left\{\eta \in \daleth: \|\phi_n - \eta\|^2 \leq \|\vartheta_n - \eta\|^2 + \Lambda^2\|\vartheta_{n-1} - \vartheta_n\|^2 - 2\Lambda\langle\vartheta_n - \eta, \vartheta_{n-1} - \vartheta_n\rangle - \beta\Delta\|\mathfrak{I}_n - \psi_n\|^2\right\}, \tag{67}$$

where $\Delta = (2 - \beta - 2\rho(1-\beta)\ell_n/\ell_{n+1} - \beta\rho^2\ell_n^2/\ell_{n+1}^2)$.

St. (v): compute

$$\vartheta_{n+1} = P_{C_{n+1}}(\vartheta_1), \quad n \geq 1. \tag{68}$$

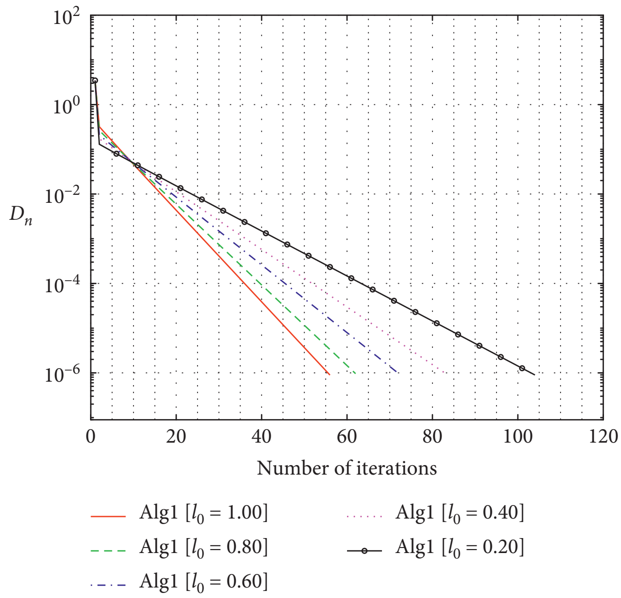


Figure 1: Numerical conduct of Alg1 by choosing different values of ℓ_0.

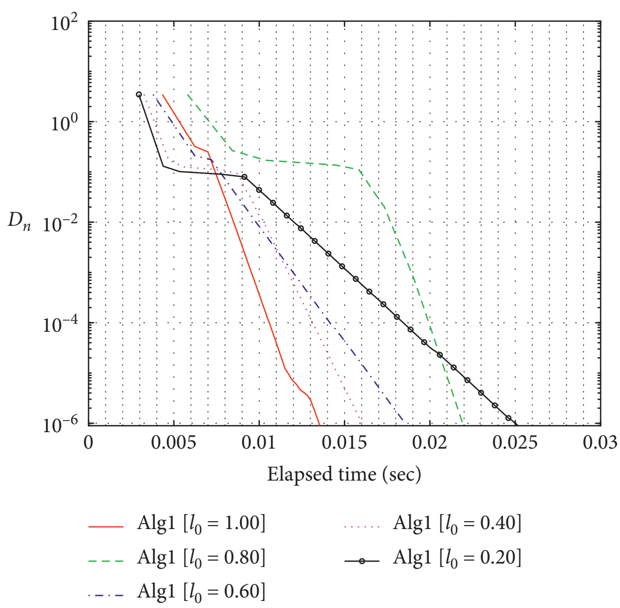


Figure 2: Numerical conduct of Alg1 by choosing different values of ℓ_0.

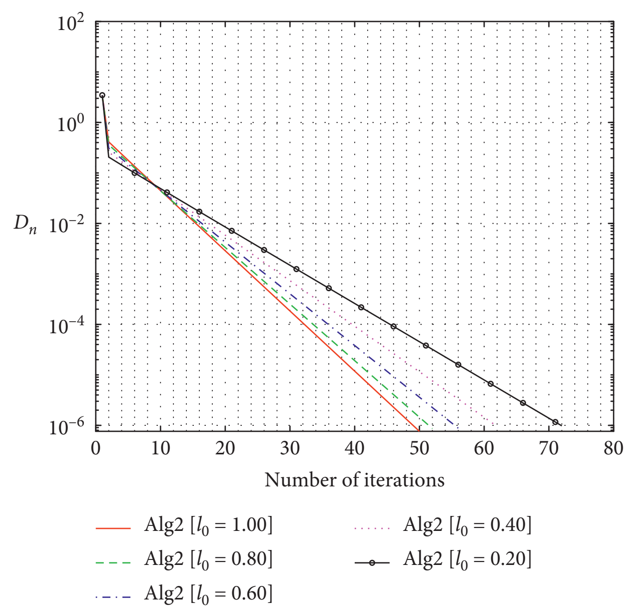


Figure 3: Numerical conduct of Alg2 by choosing different values of ℓ_0.

Put $n = n + 1$, and return to St. (i). If the solution set Γ_{SFP} is nonempty, then the sequence $\{\vartheta_n\}$ converges weakly to an element of $\Gamma_{(SFP)}$.

6. Numerical Discussion

This part is devoted to present a numerical solution to a SCFP in an infinite HS, which is a special inclusion problem as explained in Section 5. The problem setting is taken from [61]. We provide the comparison of Algorithm 1 (Alg1) in [45] and our proposed Algorithm 1 (Alg2).

Example 1. Let $\daleth_1 = \daleth_2 = L_2([0, 2\pi])$ be two HSs with an inner product

$$\langle \vartheta, y \rangle := \int_0^{2\pi} \vartheta(t) y(t) \mathrm{d}t, \quad \forall \vartheta, y \in L_2([0, 2\pi]), \tag{69}$$

and the induced norm defined by

$$\|\vartheta\| := \sqrt{\int_0^{2\pi} |\vartheta(t)|^2 \mathrm{d}t}, \quad \forall \vartheta \in L_2([0, 2\pi]). \tag{70}$$

Next, consider the feasible set $\mathscr{C} \subset \daleth_1$ as

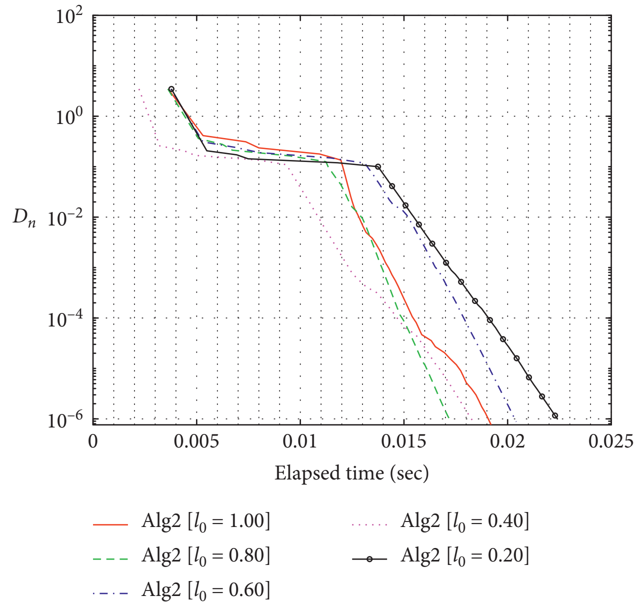


FIGURE 4: Numerical conduct of Alg2 by choosing different values of ℓ_0.

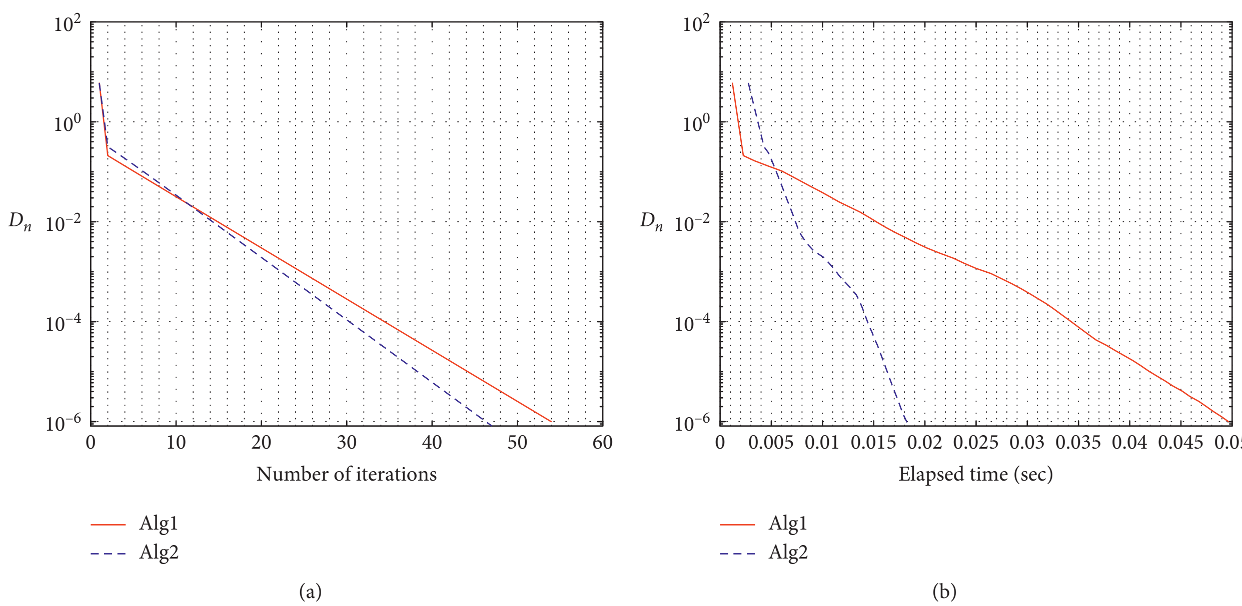


FIGURE 5: Numerical comparison of Alg2 with Alg.1 by assuming values of $\vartheta_{-1} = \vartheta_0 = t$.

$$\mathscr{C} = \left\{ \vartheta \in \daleth_1 : \int_0^{2\pi} \vartheta(t) dt \le 1 \right\}, \tag{71}$$

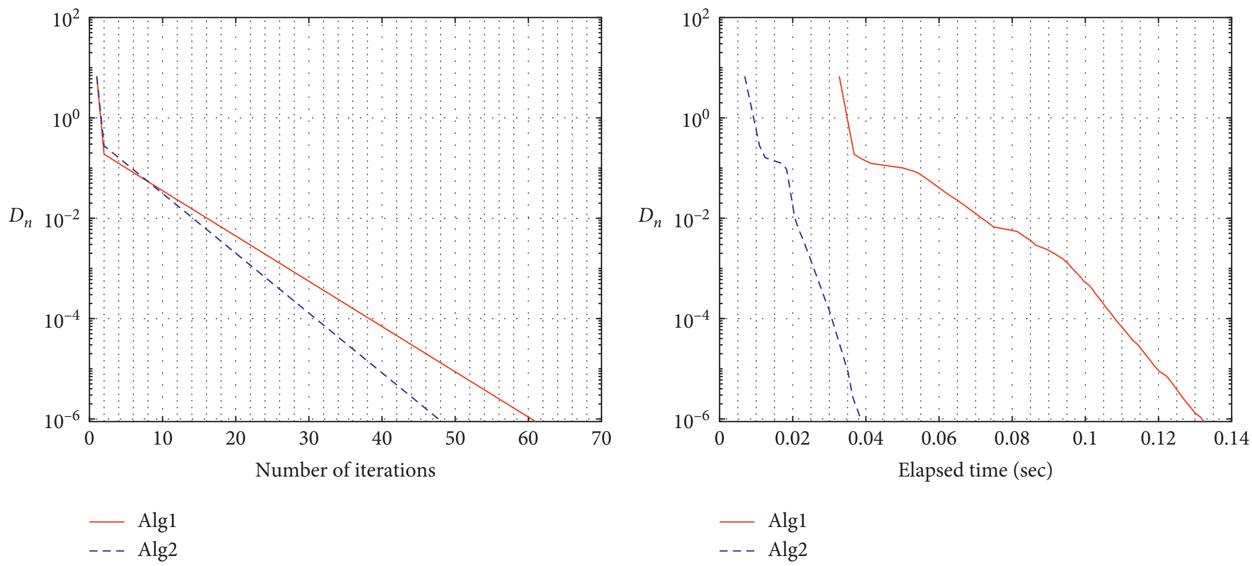
and $\mathscr{Q} \subset \daleth_2$ is

$$\mathscr{Q} = \left\{ \vartheta \in \daleth_2 : \int_0^{2\pi} |\vartheta(t) - \sin(t)|^2 dt \le 16 \right\}. \tag{72}$$

Consider the mapping $T: \daleth_1 \longrightarrow \daleth_2$ such that $(T\vartheta)(s) = \vartheta(s)$, $\vartheta \in \daleth_1$. Then, $(T^*\vartheta)(s) = \vartheta(s)$, and $\|T\| = 1$. So, we shall solve the following problem:

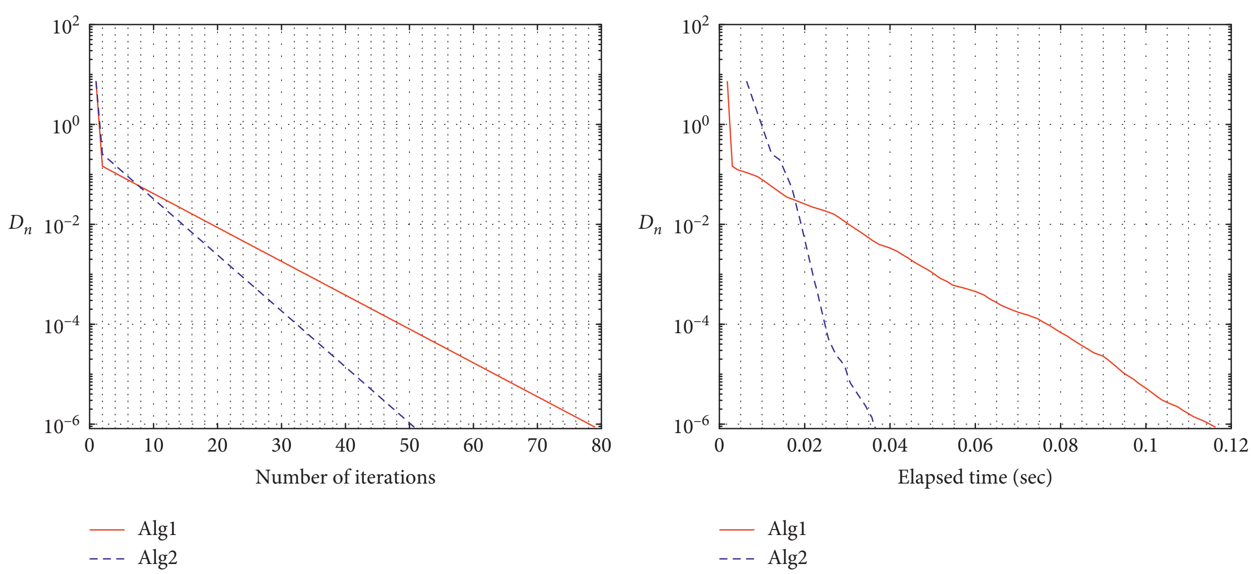
$$\text{create } \vartheta^* \in \mathscr{C} \text{ so that } T(\vartheta^*) \in \mathscr{Q}. \tag{73}$$

We can also observe that since $(T\vartheta)(s) = \vartheta(s)$, $\vartheta \in \daleth_1$, the above problem is actually a CFP of the form



(a) (b)

Figure 6: Numerical comparison of Alg2 with Alg.1 by assuming values of $\vartheta_{-1} = \vartheta_0 = t^2/5$.



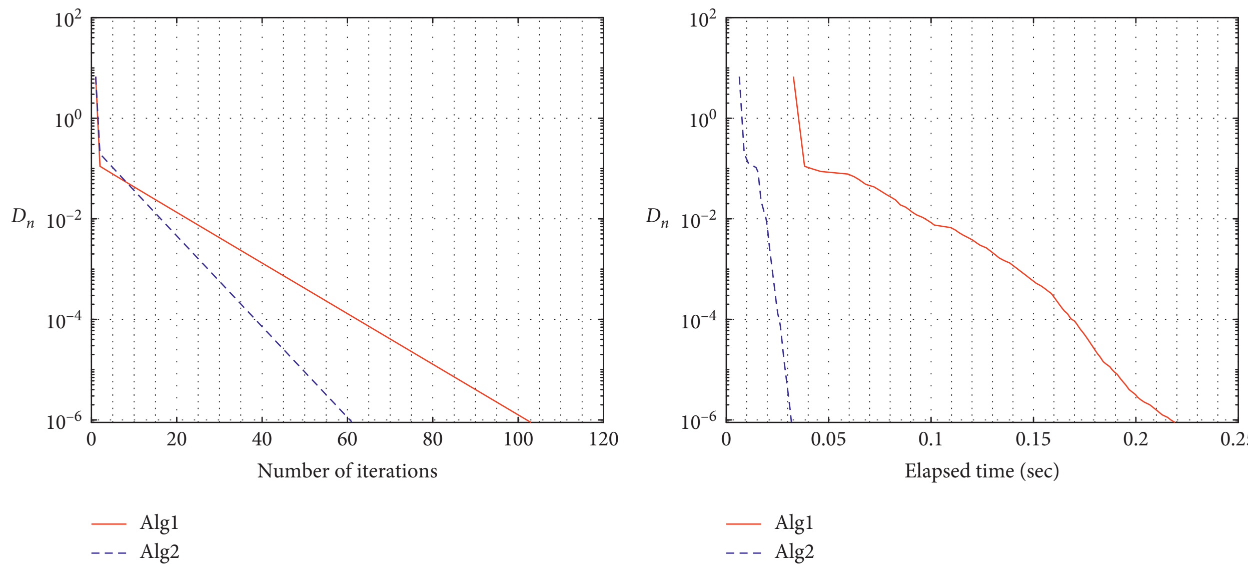
(a) (b)

Figure 7: Numerical comparison of Alg2 with Alg1 by assuming values of $\vartheta_{-1} = \vartheta_0 = 2e^t t^5$.

$$\text{create } \vartheta^* \in \mathscr{C} \cap \mathscr{Q}. \tag{74}$$

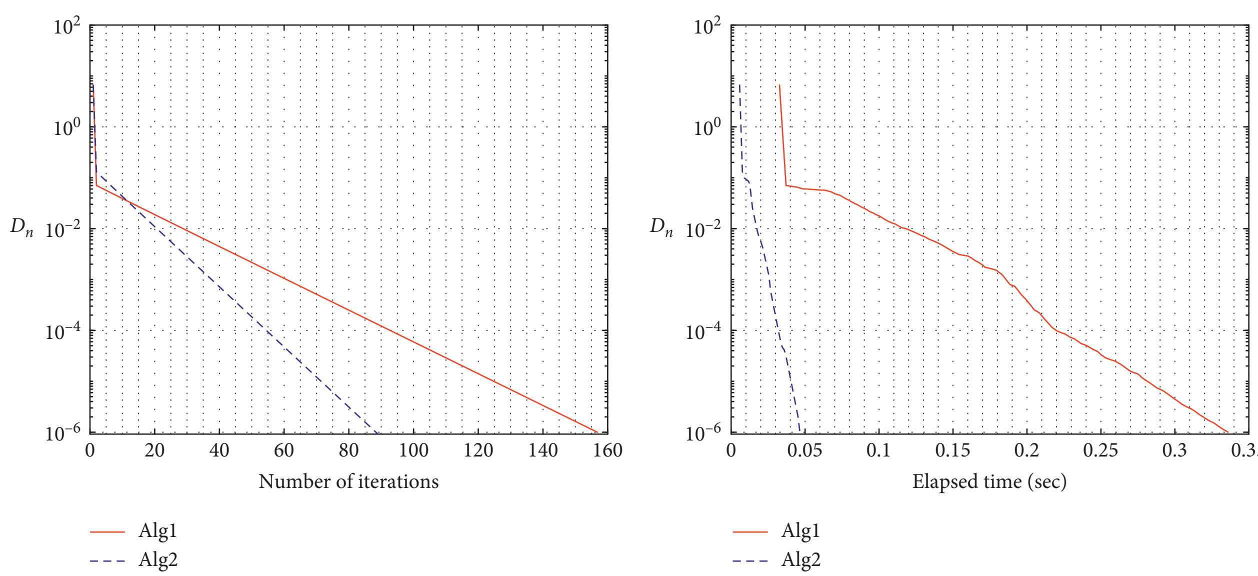
Figures 1–9 and Tables 1 and 2 show the numerical results by assuming $D_n = \|\vartheta_n - \vartheta_{n_1}\| \leq 10^{-6}$.

Remark 1. It is well known that the success of any iterative method depends on two main things: first, the number of iterations: when the number of iterations is small, the method is successful in saving effort. Second, time factor: the method that needs less time in implementation is excellent than its counterpart, which needs a lot of time and is considered successful in saving time. So, from figures and tables, we observe that our algorithm needs fewer iterations and less time than Algorithm 1 [45]. This illustrates that our method is successful in speeding up Algorithm 1 [45] and solving problem (55). Also, the performance of our algorithm is good because it saves time and effort in studding the convergence rate.



(a) (b)

Figure 8: Numerical comparison of Alg2 with Alg1 by assuming values of $\vartheta_{-1} = \vartheta_0 = e^t \sin(t)$.



(a) (b)

Figure 9: Numerical comparison of Alg2 with Alg1 by assuming values of $\vartheta_{-1} = \vartheta_0 = (t^2 - e^t)\cos(t)$.

Table 1: Numerical comparison of Alg2 with Alg1 by assuming different values of ℓ_0.

				Number of iterations		Execution time in seconds	
$\vartheta_{-1} = \vartheta_0$	ρ	Λ	ℓ_0	Alg1	Alg2	Alg1	Alg2
$1/5 \exp(t/2)^{5/4}$	0.27	0.50	1.00	56	50	0.0136	0.0190
$1/5 \exp(t/2)^{5/4}$	0.27	0.50	0.80	62	52	0.0219	0.0150
$1/5 \exp(t/2)^{5/4}$	0.27	0.50	0.60	72	56	0.0186	0.0205
$1/5 \exp(t/2)^{5/4}$	0.27	0.50	0.40	83	62	0.0160	0.0183
$1/5 \exp(t/2)^{5/4}$	0.27	0.50	0.20	104	72	0.0252	0.0225

Table 2: Numerical comparison of Alg2 with Alg1.

				Number of iterations		Execution time in seconds	
$\vartheta_{-1} = \vartheta_0$	ρ	Λ	ℓ_0	Alg1	Alg2	Alg1	Alg2
t	0.33	0.35	0.50	54	47	0.0497	0.0184
$t^2/5$	0.33	0.35	0.50	61	48	0.1325	0.0390
$2e^t t^5$	0.33	0.35	0.50	71	51	0.1166	0.0366
$e^t \sin(t)$	0.3	0.35	0.50	103	61	0.2193	0.0318
$(t^2 - e^t)\cos(t)$	0.33	0.35	0.50	157	89	0.3363	0.0467

Data Availability

Data sharing is not applicable to this article as no datasets are generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest concerning the publication of this article.

Authors' Contributions

All authors contributed equally and significantly to writing this article.

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