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Research Article

A Priori and a Posteriori Error Estimates of a WOPSIP DG Method for the Heat Equation

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We introduce and analyze a weakly overpenalized symmetric interior penalty method for solving the heat equation. We first provide optimal a priori error estimates in the energy norm for the fully discrete scheme with backward Euler time-stepping. In addition, we apply elliptic reconstruction techniques to derive a posteriori error estimators, which can be used to design adaptive algorithms. Finally, we present two numerical experiments to validate our theoretical analysis.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with Lipschitz boundary $\partial\Omega$, then we consider the following heat equation:

$$\begin{aligned} \partial_t u - \Delta u &= f, \quad \text{in}\, \Omega \times (0,T], \\ u &= 0, \quad \text{on}\, \partial\Omega \times (0,T], \\ u(\cdot,0) &= u_0, \end{aligned} \tag{1}$$

where $T > 0$ is finite time, $f \in L^2(0,T;L^2(\Omega))$ is the source term, and $u_0 \in L^2(\Omega)$ stands for the initial data. There has been much research on the a priori and a posteriori error estimates of finite element methods (FEM) for parabolic equations (see [1–12] and the references therein). However, most of the literature on this subject considers only conforming (or nonconforming) FEM in space. The error estimate of discontinuous Galerkin (DG) methods for such problems is still very rare. In the context of DG methods for space variable, see [13–15] for a priori error analysis, and see [16–21] for a posteriori error analysis. Recently, Ern et al. [22, 23] have developed a posteriori error estimates for the parabolic problem with DG discretization in time (see [24] for a posteriori error estimates of nonconforming Crouzeix–Raviart FEM for the heat equation). The object of the present work is to investigate a weakly overpenalized symmetric interior penalty (WOPSIP) method in space for problem (1), combined with an implicit Euler scheme in time.

The WOPSIP method is a kind of nonconsistent discontinuous Galerkin (DG) scheme, which was initially proposed in [25] for solving the second order elliptic equation, therein a priori error estimates were obtained. In recent years, DG methods have received significant attention since they are suitable for hp-adaptive computations. Besides this, they can also deal with nonhomogeneous boundary conditions and curved boundaries easily and allow for meshes with hanging nodes. Compared with the well-known DG methods in [26], the WOPSIP method has some advantages. For example, it has less computational complexity and thus is easy to implement [25]. Additionally, a high intrinsic parallelism property of the WOPSIP method was investigated in [27]. Therefore, the WOPSIP methods have been further developed to solve biharmonic problems [28] and Stokes equations [29], Reissner–Mindlin plate equations [30, 31], non-self-adjoint and indefinite problems [32], and variational inequalities [33]. In the present work, we shall extend the results of elliptic equations in [25] to the parabolic case. More precisely, the space variable is approximated by the WOPSIP method, and time variable is discretized by the backward Euler scheme. We shall give a detailed a priori and a posteriori error estimates. In this case, one may come across a difficulty that stems from the nonconsistency of the numerical method (for more details, see (34) in Theorem 2). On the contrary, a posteriori error analysis for parabolic problems is more involved than that for elliptic equations, since they involve both spatial error and temporal error. According to a framework stated in [18], we derive a posteriori error estimates which rely on the available estimates for elliptic problems [32, 34].

The rest of our paper is organized as follows. In Section 2, we state some notations and the numerical scheme. We establish a priori error estimates in the energy norm in Section 3. Section 4 is devoted to a posteriori error analysis which is based on the work [18]. We make some conclusions in Section 5. Finally, we provide some numerical results to validate theoretical analysis of a priori error estimates.

2. Preliminaries and WOPSIP Method

Let us first give some notations. For a bounded subdomain $\mathscr{D} \subset \Omega$, we denote by $H^m(\mathscr{D})\,(m \geq 0)$ the standard Sobolev space, associated with norm $\|\cdot\|_{m,\mathscr{D}}$ and seminorm $|\cdot|_{m,\mathscr{D}}$. When $m = 0$, $H^0(\mathscr{D})$ is the standard Lebesgue space $L^2(\mathscr{D})$, with the inner product defined by $(\cdot,\cdot)_D$. When $D = \Omega$, we will omit the index Ω.

To deal with functions of time and space, we also introduce the standard Bochner space $L^p(0,T;H^m(\mathscr{D}))$, which consists of all measurable functions u: $[0,T] \longrightarrow H^m(\mathscr{D})$ with norm

$$\|u\|_{L^p(0,T;H^m(\mathscr{D}))} = \left(\int_0^T \|u(s)\|_{m,\mathscr{D}}^p \mathrm{d}s\right)^{1/p}, \tag{2}$$

for $1 \leq p < \infty$.

The weak formulation of (1) reads: find $u \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ such that

$$\begin{gathered}(\partial_t u, v) + a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), t \in (0,T], \\ u(\cdot,0) = u_0,\end{gathered} \tag{3}$$

with $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$.

Let $\mathscr{T}_h$ be a family of conforming shape-regular meshes which decompose Ω into triangle elements $\{K\}$. Set $h_K = \operatorname{diam}(K)$ and $h = \max_{K\in\mathscr{T}_h} h_K$. Denote by $\mathscr{E}_h$ the set of all edges. Furthermore, $\mathscr{E}_h = \mathscr{E}_h^I \cup \mathscr{E}_h^\partial$, where $\mathscr{E}_h^I$ is the set of interior edges and $\mathscr{E}_h^\partial$ is the set of edges on $\partial\Omega$. In what follows, h_e stands for the length of the edge e. Given $\mathscr{D} \in \mathbb{R}^2$, we let $\mathbb{P}_r(\mathscr{D})$ be the space of polynomials of degree at most r on $\mathscr{D}$. Moreover, we associate a fixed unit normal $\mathbf{n}$ with each edge $e \in \mathscr{E}_h$ such that for edges on the boundary $\partial\Omega$, $\mathbf{n}$ is the exterior unit normal vector.

Let e be an interior edge in $\mathscr{E}_h^I$ shared by elements K^+ and K^-. For v: $\Omega \longrightarrow \mathbb{R}$, set $v^\pm = v|_{e\cap\partial K^\pm}$, we define the following quantities:

$$\begin{aligned}[\![v]\!] &= v^+ - v^-, \\ \{v\} &= \frac{1}{2}(v^+ + v^-).\end{aligned} \tag{4}$$

If $e \in \mathscr{E}_h^\partial$, set $[\![v]\!] = v$ and $\{v\} = v$. Furthermore, for $m \geq 1$, we also define

$$H^m(\mathscr{T}_h) = \left\{v \in L^2(\Omega)\colon v|_K \in H^m(K), \forall K \in \mathscr{T}_h\right\}. \tag{5}$$

Consider the discontinuous $\mathbb{P}_1$ finite element space:

$$V_h = \left\{v \in L^2(\Omega)\colon v|_K \in \mathbb{P}_1(K), \forall K \in \mathscr{T}_h\right\}. \tag{6}$$

The bilinear form of the WOPSIP DG method is defined by (see [25])

$$a_h(w,v) = \sum_{K\in\mathscr{T}_h}\int_K \nabla w\cdot\nabla v\,\mathrm{d}x + \sum_{e\in\mathscr{E}_h} h_e^{-2}\left(\Pi_e^0[\![w]\!]\right)\left(\Pi_e^0[\![v]\!]\right), \tag{7}$$

where $\Pi_e^0 v$ stands for the mean of v over e, that is,

$$\Pi_e^0 v = \frac{1}{h_e}\int_e v d_A. \tag{8}$$

For the time discretization, we introduce the uniform partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0,T]$, with time step $\tau = T/N$ and $t_n = n\tau$ $(n = 1,\ldots,N)$. The backward Euler WOPSIP DG method for solving heat equation (1) is to find $u_h^n \in V_h$ for $n = 1,\ldots,N$ such that

$$\begin{gathered}\left(\frac{u_h^n - u_h^{n-1}}{\tau}, v_h\right) + a_h(u_h^n, v_h) = (f^n, v_h), \quad \forall v_h \in V_h, \\ u_h^0 = u_{h0},\end{gathered} \tag{9}$$

where u_{h0} is a projection of u_0 onto V_h which will be specified later in Sections 3 and 4.

3. Stability and A Priori Error Analysis

We begin by defining the mesh-dependent norm $\|\cdot\|_h$ on $H^2(\mathscr{T}_h)$ as

$$\|v\|_h = \left(\sum_{K\in\mathscr{T}_h}\|\nabla v\|_{0,T}^2 + \sum_{e\in\mathscr{E}_h} h_e^{-2}\left(\Pi_e^0[\![v]\!]\right)^2\right)^{1/2}. \tag{10}$$

For the subsequent analysis, we need the following Poincaré–Friedrichs inequality (see [13, 35]):

$$\|v\|_0^2 \leq C\left(\sum_{K\in\mathscr{T}_h}\|\nabla v\|_{0,T}^2 + \sum_{e\in\mathscr{E}_h} h_e^{-1}\|[\![v]\!]\|_{0,e}^2\right), \quad \forall v \in H^2(\mathscr{T}_h). \tag{11}$$

Here and hereafter, we use C to denote a positive constant which is independent of h and τ, but may have different values at different places.

Moreover, from Lemma 3.1 in [25] we have

$$\sum_{e\in\mathscr{E}_h} h_e^{-1}\|[\![v]\!]\|_{0,e}^2 \leq C\|v\|_h^2, \quad \forall v \in H^2(\mathscr{T}_h). \tag{12}$$

Combining (11) and (12) gives

$$\|v\|_0 \leq C\|v\|_h, \quad \forall v \in H^2(\mathscr{T}_h). \tag{13}$$

We first prove that the numerical solutions satisfy the following stability result.

Theorem 1. *Let $\{u_h^n\}_{n\geq 0}$ be the solution of (9). It holds that, for all $m>0$,*

$$\|u_h^m\|_0^2+\tau\sum_{n=1}^{m}\|u_h^n\|_h^2\leq C\left(\|u_h^0\|_0^2+\tau\sum_{n=1}^{m}\|f^n\|_0^2\right). \tag{14}$$

Proof. Testing $v_h=u_h^n$ in (9) and using the definition of $\|\cdot\|_h$, we have

$$\frac{1}{\tau}\left(u_h^n-u_h^{n-1},u_h^n\right)+\|u_h^n\|_h^2=\left(f^n,u_h^n\right). \tag{15}$$

Then, using the equation $a(a-b)=(1/2)(a^2-b^2)+(1/2)(a-b)^2$ and Cauchy–Schwarz inequality gives

$$\frac{1}{2\tau}\left(\|u_h^n\|_0^2-\|u_h^{n-1}\|_0^2\right)+\frac{1}{2\tau}\|u_h^n-u_h^{n-1}\|_0^2+\|u_h^n\|_h^2\leq\|f^n\|_0\|u_h^n\|_0, \tag{16}$$

which together with the inequality (13) and Young's inequality implies that

$$\begin{aligned}\frac{1}{2\tau}\left(\|u_h^n\|_0^2-\|u_h^{n-1}\|_0^2\right)&+\frac{1}{2\tau}\|u_h^n-u_h^{n-1}\|_0^2+\|u_h^n\|_h^2\\&\leq C\|f^n\|_0\|u_h^n\|_h\\&\leq\frac{1}{2}\|u_h^n\|_h^2+C\|f^n\|_0^2.\end{aligned} \tag{17}$$

Thus,

$$\begin{aligned}\frac{1}{2\tau}\left(\|u_h^n\|_0^2-\|u_h^{n-1}\|_0^2\right)&+\frac{1}{2\tau}\|u_h^n-u_h^{n-1}\|_0^2+\frac{1}{2}\|u_h^n\|_h^2\\&\leq C\|f^n\|_0^2.\end{aligned} \tag{18}$$

Multiplying (18) by 2τ and summing from $n=1$ to $n=m$, we arrive at

$$\|u_h^m\|_0^2-\|u_h^0\|_0^2+\sum_{n=1}^{m}\left(\|u_h^n-u_h^{n-1}\|_0^2\right)+\tau\sum_{n=1}^{m}\|u_h^n\|_h^2\leq C\tau\sum_{n=1}^{m}\|f^n\|_0^2. \tag{19}$$

Noting that $\sum_{n=1}^{m}(\|u_h^n-u_h^{n-1}\|_0^2)\geq 0$, we get

$$\|u_h^m\|_0^2+\tau\sum_{n=1}^{m}\|u_h^n\|_h^2\leq\|u_h^0\|_0^2+C\tau\sum_{n=1}^{m}\|f^n\|_0^2. \tag{20}$$

The conclusion (14) follows immediately. □

To carry out a detailed error analysis, we introduce $\mathcal{I}_h u$ which denotes the continuous linear interpolation of u. We then have the following standard estimates (see [36]):

$$\|u-\mathcal{I}_h u\|_{m,K}\leq Ch^{2-m}|u|_{2,K},\quad m=0,1. \tag{21}$$

In addition, we introduce the following trace inequality:

$$h_e^{-1}\|w\|_{0,e}^2\leq C\left(h_K^{-2}\|w\|_{0,K}^2+\|\nabla w\|_{0,K}^2\right),\quad\forall w\in H^1(K), \tag{22}$$

with e being an edge of K.

Also, we need the elliptic projection operator $P_h\colon H^2(\Omega)\longrightarrow V_h$ defined by

$$a_h\left(v-P_hv,w_h\right)=0,\quad\forall w_h\in V_h. \tag{23}$$

It is well known that the projection operator P_h satisfies the following estimates (see [11]).

Lemma 1. *For any $v\in H^2(\Omega)$, it holds that*

$$\|v-P_hv\|_{0,\Omega}+h\|v-P_hv\|_h\leq Ch^2\|v\|_2. \tag{24}$$

For convenience, we use the following notation for any function $g(t,x)$: at each time step $t_n=n\tau, n=1,\ldots,N$, $g^n=g(t^n,x),\forall x\in\Omega$. In addition, Taylor expansion yields

$$\frac{g^n-g^{n-1}}{\Delta t}=\left(\partial_t g\right)^n+\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\left(t^{n-1}-s\right)\partial_{tt}g(s)\mathrm{d}s. \tag{25}$$

Now, we are in a position to state a priori error estimates, which is the main result of this section.

Theorem 2. *Let u and $\{u_h^n\}_{n\geq 0}$ be the solutions of (1) and (9), respectively. Assume that $\partial_t u\in L^2(0,T;H^2(\Omega))$, $\partial_{tt}u\in L^2(0,T;L^2(\Omega))$, and u_h^0 satisfying*

$$\|u_0-u_h^0\|_0\leq Ch^2\|u_0\|_2. \tag{26}$$

Then, for all $m>0$ it holds that

$$\begin{aligned}\tau\sum_{n=1}^{m}\|u^n-u_h^n\|_h^2\leq{}&Ch^2\left(\tau\sum_{n=1}^{m}\|u^n\|_2^2+h^2\|u_0\|_2^2+h^2\int_0^{t_m}\|\partial_t u(s)\|_2^2\mathrm{d}s\right)\\&+C\tau^2\int_0^{t_m}\|\partial_{tt}u(s)\|_0^2\mathrm{d}s.\end{aligned} \tag{27}$$

Proof. Integrating by parts and using the Taylor formulation (25), we have

$$\left(\frac{u^n - u^{n-1}}{\tau}, v_h\right) + a_h(u^n, v_h)$$
$$= (f^n, v_h) + (R_n, v_h) + \sum_{e\in\mathcal{E}_h}\int_e \{\nabla u^n \cdot \mathbf{n}\}[\![v_h]\!] d_A, \quad \forall v_h \in V_h, \tag{28}$$

where

$$R_n = -\frac{1}{\tau}\int_{t_{n-1}}^{t_n} (s - t_{n-1})\partial_{tt} u(s)\mathrm{d}s. \tag{29}$$

We then split the error $u^n - u_h^n$ into

$$u^n - u_h^n = (u^n - P_h u^n) + (P_h u^n - u_h^n) = \theta^n + \rho^n. \tag{30}$$

Thus, subtracting (9) from (28) and using the definition of θ^n and ρ^n gives

$$\left(\frac{\rho^n - \rho^{n-1}}{\tau}, v_h\right) + a_h(\rho^n, v_h)$$
$$= \sum_{e\in\mathcal{E}_h}\int_e \{\nabla u^n \cdot \mathbf{n}\}[\![v_h]\!] d_A - \left(\frac{\theta^n - \theta^{n-1}}{\tau}, v_h\right) + (R_n, v_h). \tag{31}$$

Testing $v_h = \rho^n$ in (31), using the definition of $\|\cdot\|_h$ and the formula $a(a-b) = (1/2)(a^2 - b^2) + (1/2)(a-b)^2$, we obtain

$$\frac{1}{2\tau}\left(\|\rho^n\|_0^2 - \|\rho^{n-1}\|_0^2\right) + \frac{1}{2\tau}\|\rho^n - \rho^{n-1}\|_0^2 + \|\rho^n\|_h^2$$
$$= \sum_{e\in\mathcal{E}_h}\int_e \{\nabla u^n \cdot \mathbf{n}\}[\![\rho^n]\!] d_A + (R_n, \rho^n) - \left(\frac{\theta^n - \theta^n}{\tau}, \rho^n\right)$$
$$\equiv S_1 + S_2 + S_3. \tag{32}$$

We now bound the first term S_1 of the above equation. We then employ the definition of Π_e^0 to find that

$$\begin{aligned} S_1 &= \sum_{e\in\mathcal{E}_h}\int_e \{\nabla u^n \cdot \mathbf{n}\}[\![\rho^n]\!] d_A \\ &= \sum_{e\in\mathcal{E}_h}\int_e \{\nabla(u^n - \mathcal{I}_h u^n) \cdot \mathbf{n}\}[\![\rho^n]\!] d_A \\ &\quad + \sum_{e\in\mathcal{E}_h}\int_e \{\nabla(\mathcal{I}_h u^n) \cdot \mathbf{n}\}\Pi_e^0[\![\rho^n]\!] d_A \\ &\equiv S_{11} + S_{12}. \end{aligned} \tag{33}$$

Furthermore, applying Cauchy–Schwarz inequality and Young's inequality and using (11), (21), and (22), we can estimate the terms S_{11} and S_{12} as follows:

$$\begin{aligned} S_{11} &= \sum_{e\in\mathcal{E}_h}\int_e \{\nabla(u^n - \mathcal{I}_h u^n) \cdot \mathbf{n}\}[\![\rho^n]\!] d_A \\ &\le \left(\sum_{e\in\mathcal{E}_h} h_e \|\{\nabla(u^n - \mathcal{I}_h u^n)\}\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\rho^n]\!]\|_{0,e}^2\right)^{1/2} \\ &\le Ch|u^n|_2\|\rho^n\|_h \\ &\le Ch^2\|u^n\|_2^2 + \frac{1}{8}\|\rho^n\|_h^2, \\ S_{12} &= \sum_{e\in\mathcal{E}_h}\int_e \{\nabla(\mathcal{I}_h u^n) \cdot \mathbf{n}\}\left(\Pi_e^0[\![\rho^n]\!]\right) d_A \\ &\le \left(\sum_{e\in\mathcal{E}_h} h_e^3 \|\{\nabla(\mathcal{I}_h u^n)\}\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h} h_e^{-3}\|\Pi_e^0[\![\rho^n]\!]\|_{0,e}^2\right)^{1/2} \\ &= \left(\sum_{e\in\mathcal{E}_h} h_e^3 \|\{\nabla(\mathcal{I}_h u^n)\}\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h} h_e^{-2}\left(\Pi_e^0[\![\rho^n]\!]\right)^2\right)^{1/2} \\ &\le \left(\sum_{e\in\mathcal{E}_h} h_e^3 \|\{\nabla(u^n - \mathcal{I}_h u^n)\}\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h} h_e^{-2}\left(\Pi_e^0[\![\rho^n]\!]\right)^2\right)^{1/2} \\ &\quad + \left(\sum_{e\in\mathcal{E}_h} h_e^3 \|\{\nabla u^n\}\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h} h_e^{-2}\left(\Pi_e^0[\![\rho^n]\!]\right)^2\right)^{1/2} \\ &\le Ch^2|u^n|_2\|\rho^n\|_h + C\left(h|u^n|_{1,\Omega} + h^2|u^n|_{2,\Omega}\right)\|\rho^n\|_h \\ &\le Ch\|u^n\|_2\|\rho^n\|_h \\ &\le Ch^2\|u^n\|_2^2 + \frac{1}{8}\|\rho^n\|_h^2. \end{aligned} \tag{34}$$

Next, we give bounds for S_2. Applying Cauchy–Schwarz inequality, the inequality (13) and Young's inequality yields

$$\begin{aligned} S_2 &= (R_n, \rho^n) \\ &\le \|R_n\|_0\|\rho^n\|_0 \\ &\le C\|R_n\|_0\|\rho^n\|_h \\ &\le C\|R_n\|_0^2 + \frac{1}{8}\|\rho^n\|_h^2. \end{aligned} \tag{35}$$

Similarly, it holds that

$$\begin{aligned} S_3 &= -\left(\frac{\theta^n - \theta^{n-1}}{\tau}, \rho^n\right) \\ &\le \left\|\frac{\theta^n - \theta^{n-1}}{\tau}\right\|_0 \left\|\rho^n\right\|_0 \\ &\le C\left\|\frac{\theta^n - \theta^{n-1}}{\tau}\right\|_0 \left\|\rho^n\right\|_h \\ &\le C\left\|\frac{\theta^n - \theta^{n-1}}{\tau}\right\|_0^2 + \frac{1}{8}\left\|\rho^n\right\|_h^2. \end{aligned} \tag{36}$$

Combining (32)–(36), we obtain

$$\begin{aligned} &\frac{1}{2\tau}\left(\left\|\rho^n\right\|_0^2 - \left\|\rho^{n-1}\right\|_0^2\right) + \frac{1}{2\tau}\left\|\rho^n - \rho^{n-1}\right\|_0^2 + \frac{1}{2}\left\|\rho^n\right\|_h^2 \\ &\quad \le C\left(h^2\left\|u^n\right\|_2^2 + \left\|R_n\right\|_0^2 + \left\|\frac{\theta^n - \theta^{n-1}}{\tau}\right\|_0^2\right). \end{aligned} \tag{37}$$

Moreover, applying Cauchy–Schwarz inequality for R_n gives

$$\begin{aligned} \left\|R_n\right\|_0^2 &\le \frac{1}{\tau^2}\left(\int_{t_{n-1}}^{t_n} \left(s - t_{n-1}\right)^2 \mathrm{d}s\right)\left(\int_{t_{n-1}}^{t_n} \left\|u_{tt}(s)\right\|_0^2 \mathrm{d}s\right) \\ &= \frac{\tau}{3}\left(\int_{t_{n-1}}^{t_n} \left\|\partial_{tt}u(s)\right\|_0^2 \mathrm{d}s\right). \end{aligned} \tag{38}$$

On the contrary,

$$\theta^n - \theta^{n-1} = \int_{t_{n-1}}^{t_n} \partial_t\theta(s)\mathrm{d}s. \tag{39}$$

Thus,

$$\begin{aligned} \left\|\frac{\theta^n - \theta^{n-1}}{\tau}\right\|_0^2 &= \frac{1}{\tau^2}\left\|\int_{t_{n-1}}^{t_n} \partial_t\theta(s)\mathrm{d}s\right\|_0^2 \\ &\le \frac{1}{\tau^2}\left(\int_{t_{n-1}}^{t_n} 1^2 \mathrm{d}t\right)\left(\int_{t_{n-1}}^{t_n} \left\|\partial_t\theta(s)\right\|_0^2 \mathrm{d}s\right) \\ &= \frac{1}{\tau}\left(\int_{t_{n-1}}^{t_n} \left\|\partial_t\theta(s)\right\|_0^2 \mathrm{d}s\right). \end{aligned} \tag{40}$$

Plugging (38) and (40) into (37) and then multiplying the result inequality by 2τ and summing from $n = 1$ to $n = m$, we obtain

$$\begin{aligned} &\left\|\rho^m\right\|_0^2 - \left\|\rho^0\right\|_0^2 + \sum_{n=1}^{m} \left\|\rho^n - \rho^{n-1}\right\|_0^2 + \tau\sum_{n=1}^{m} \left\|\rho^n\right\|_h^2 \\ &\quad \le C\left(h^2\tau\sum_{n=1}^{m} \left\|u^n\right\|_2^2 + \tau^2\int_0^{t_m} \left\|\partial_{tt}u(s)\right\|_0^2 \mathrm{d}s + \int_0^{t_m} \left\|\partial_t\theta(s)\right\|_0^2 \mathrm{d}s\right). \end{aligned} \tag{41}$$

Noting that $\sum_{n=1}^{m}\left\|\rho^n - \rho^{n-1}\right\|_0^2 \ge 0$ and $\tau\sum_{n=1}^{m}\left\|\rho^n\right\|_h^2 \ge 0$, and using the estimate (24), we infer that

$$\begin{aligned} \tau\sum_{n=1}^{m} \left\|\rho^n\right\|_h^2 &\le \left\|\rho^0\right\|_0^2 + C\left(h^2\tau\sum_{n=1}^{m} \left\|u^n\right\|_2^2 + \tau^2\int_0^{t_m} \left\|\partial_{tt}u(s)\right\|_0^2 \mathrm{d}s\right. \\ &\quad \left. + h^4\int_0^{t_m} \left\|\partial_t u(s)\right\|_2^2 \mathrm{d}s.\right. \end{aligned} \tag{42}$$

Moreover, it follows from (24) and (26) that

$$\begin{aligned} \left\|\rho^0\right\|_0 &= \left\|P_h u_0 - u_h^0\right\|_0 \\ &\le \left\|P_h u_0 - u_0\right\|_0 + \left\|u_0 - u_h^0\right\|_0 \\ &\le Ch^2\left\|u_0\right\|_2. \end{aligned} \tag{43}$$

Plugging (43) into (42) and then combining (30), the triangle inequality yields the desired estimate in (27). □

Remark 1. For simplicity, in the present work, we only consider the lowest linear finite element, and the extension of the results to high order methods [37] can be derived straightforwardly.

Remark 2. In this work, we only consider the constant coefficient parabolic equation, and the extension to more practical problems such as equations with variable coefficient can be derived by using the techniques developed in Section 6 in [25]. Additionally, the present work only addresses the conforming mesh, and the extension to meshes with hanging nodes can be also obtained by utilizing approaches stated in Section 6 in [25].

4. A Posteriori Error Analysis

We begin by recalling an a posteriori error estimator for the stationary problem (see [32]), which is a key step in the subsequent error analysis.

Theorem 3. *Let $z \in H_0^1(\Omega)$ be the solution of*

$$a(z, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{44}$$

and let $Z \in V_h$ be the solution satisfying

$$a_h(Z, v_h) = (f, v_h), \quad \forall v_h \in V_h. \tag{45}$$

Then, we have the following a posteriori error estimates:

$$\|z - Z\|_h \le C\{\mathscr{E}(Z, f, \mathscr{T}_h)\}, \tag{46}$$

with

$$\begin{aligned} \mathscr{E}(Z, f, \mathscr{T}_h) &= \left(\sum_{K\in\mathscr{T}_h} h_K\|f\|_{0,K} + \sum_{e\in\mathscr{E}_h^I} h_e\|[\![\nabla Z\cdot\mathbf{n}]\!]\|_{0,e}^2\right. \\ &\quad \left. + \sum_{e\in\mathscr{E}_h}\left(h_e^{-2}\left|\Pi_e^0[\![Z]\!]\right|^2 + h_e^{-1}\|[\![Z]\!]\|_{0,e}^2\right)\right)^{1/2}. \end{aligned} \tag{47}$$

Remark 3. It is worth mentioning that, the authors in [34] have proposed a different error estimator, which can also be applied directly in the forthcoming analysis.

The following useful fact is a standard result in studying a posteriori error estimates of DG methods. It shows that, given any discontinuous function $v_h \in V_h$, there exists a continuous polynomial function to approximate it (see [34, 38]).

Lemma 2. *For any $v_h \in V_h$, there exits an decomposition $v_h = v_h^c + v_h^d$ such that*

$$\sum_{K\in\mathcal{T}_h}\left(\left\|\nabla\left(v_h - v_h^c\right)\right\|_{0,K}^2 + h_K^{-2}\left\|v_h - v_h^c\right\|_{0,K}^2\right) \le C\sum_{e\in\mathcal{E}_h} h_e^{-1}\left\|[\![v_h]\!]\right\|_{0,e}^2, \tag{48}$$

where $v_h^c \in H_0^1(\Omega)\cap V_h$ and $v_h^d \in V_h$.

To proceed, we reformulate the DG method (9) to make it be suitable for adaptive computations. At each time step $n > 0$, we assume that a mesh $\mathcal{T}_n$ which can be obtained from $\mathcal{T}_{n-1}$ by locally refining and coarsening $\mathcal{T}_{n-1}$. Analogous to Section 3, we denote by $\mathcal{E}_n = \mathcal{E}_n^I \cup \mathcal{E}_n^\partial$ the set of edge of $\mathcal{T}_n$, and use V_h^n to stand for the finite element space with respect to $\mathcal{T}_n$. Let $I^n\colon V_h^{n-1} \longrightarrow V_h^n$ be a general data transfer operator, set $\tau_n = t_n - t_{n-1}$, the backward Euler WOPSIP DG method for approximating (1) can be reformulated as: find $u_h^n \in V_h^n$ $(n = 1, \ldots, N)$ such that

$$\begin{gathered}\left(\frac{u_h^n - I^n u_h^{n-1}}{\tau_n}, v_h^n\right) + B^n\left(u_h^n, v_h^n\right) = \left(f^n, v_h^n\right), \quad \forall v_h^n \in V_h^n,\\ u_h^0 = P_h^0 u_0,\end{gathered} \tag{49}$$

where

$$B^n(w, v) = \sum_{K\in\mathcal{T}_n}\int_K \nabla w\cdot\nabla v\,\mathrm{d}x + \sum_{e\in\mathcal{E}_n} h_e^{-2}\left(\Pi_e^0[\![w]\!]\right)\left(\Pi_e^0[\![v]\!]\right), \tag{50}$$

and $P_h^0\colon L^2(\Omega) \longrightarrow V_h^0$ is the L^2-projection operator onto V_h^0. The corresponding mesh-dependent norm is defined by $\|v\|_{\mathcal{T}_n}^2 = B^n(v, v)$. In view of $\{u_h^n\}$, we define a function which is piecewise linear continuous in time:

$$\begin{aligned} u_h(0) &= u_h^0,\\ u_h(t) &= l_{n-1}(t)u_h^{n-1} + l_n(t)u_h^n,\end{aligned} \tag{51}$$

for $t \in (t_{n-1}, t_n]$ $(n = 1, \ldots, N)$, where

$$l_n(t) = \frac{t - t_{n-1}}{\tau_n}\mathbb{1}_{[t_{n-1}, t_n]} + \frac{t_{n+1} - t}{\tau_{n+1}}\mathbb{1}_{[t_n, t_{n+1}]}, \tag{52}$$

are the Lagrange basic functions.

As in Lemma 2, at each time step t_n $(n = 1, \ldots, N)$, we can decompose u_h^n into

$$u_h^n = u_c^n + u_d^n, \quad \text{with respect to the mesh } \breve{\mathcal{T}}_n, \tag{53}$$

$$u_h^n = u_{c+}^n + u_{d+}^n, \quad \text{with respect to the mesh } \breve{\mathcal{T}}_{n+1}, \tag{54}$$

where $\breve{\mathcal{T}}_n$ refers to the coarsest common refinement. More precisely, it is the coarsest triangulation which satisfies $\mathcal{T}_n \le \breve{\mathcal{T}}_n$ and $\mathcal{T}_{n-1} \le \breve{\mathcal{T}}_n$. Here, we write $\mathcal{T}_n \le \breve{\mathcal{T}}_n$ to mean that $\breve{\mathcal{T}}_n$ is a refinement of $\mathcal{T}_n$. In addition, we define $u_h^c(t) = l_{n-1}(t)u_{c+}^{n-1} + l_n(t)u_c^n$ and $u_h^d(t) = l_{n-1}(t)u_{d+}^{n-1} + l_n(t)u_d^n$.

Moreover, we introduce the discrete elliptic operator $A^n\colon V_h^n \longrightarrow V_h^n$ by

$$\left(A^n\eta, \theta\right) = B^n(\eta, \theta), \quad \forall\theta \in V_h^n, \tag{55}$$

with $A^n\eta \in V_h^n$. Additionally, we define some elliptic reconstructions which are used commonly in the a posterior error analysis for parabolic equations [6].

Definition 1. The elliptic reconstruction $w^n \in H_0^1(\Omega)$ of u_h^n is defined as the solution of the elliptic problem:

$$B^n\left(w^n, v\right) = \left(A^n u_h^n, v\right), \quad \forall v \in H_0^1(\Omega). \tag{56}$$

Similarly, $w^{n-1} \in H_0^1(\Omega)$ satisfies

$$B^n\left(w^{n-1}, v\right) = \left(A^n I^n u_h^{n-1}, v\right), \quad \forall v \in H_0^1(\Omega). \tag{57}$$

Remark 4. We should point that the DG solution of w^n, denoted by w_h^n, is also the DG solution u_h^n. Indeed, we have $B^n(w_h^n, v_h) = (A^n u_h^n, v_h) = B^n(u_h^n, v_h)$, for all $v_h \in V_h^n$; thus, $w_h^n = u_h^n$.

Similar to $u_h(t)$ stated in (51), we define $w_h(t)$ by

$$w(t) = l_{n-1}(t)w^{n-1} + l_n(t)w^n. \tag{58}$$

Now, the error e can be decomposed into

$$\begin{aligned} e &= u_h - u = \varphi - \chi,\\ \varphi &= w - u,\\ \chi &= w - u_h,\end{aligned} \tag{59}$$

where φ is called the elliptic error and χ is the parabolic error. Furthermore, we define $e_c = u_h^c - u$ and $\chi_c = w - u_h^c$. Thus, $e = e_c + u_h^d$ and $\chi_c = \chi + u_h^d$.

Then, we can obtain the following result, which is a key step to prove the main result in Theorem 4. We remark that similar result can be found in Lemma 5.3 in [39].

Lemma 3. *For each $t \in (t_{n-1}, t_n]$ $(n = 1, 2, \ldots, N)$, we have*

$$\begin{aligned}&\left(\partial_t e, v\right) + B^n(\varphi, v)\\ &\quad = \left(\frac{\left(I^n u_h^{n-1} - u_h^{n-1}\right)}{\tau_n}, v\right) + l_{n-1}(t)\left(A^n I^n u_h^{n-1} - A^n u_h^n, v\right)\\ &\qquad + \left(P_h^n f^n - f, v\right),\end{aligned} \tag{60}$$

for all $v \in H_0^1(\Omega)$.

Proof. Since $\partial_t u_h = ((u_h^n - u_h^{n-1})/\tau^n)$, this combined with (49), (56), and (57) yields

$$\begin{aligned}(\partial_t u_h, v) &= ((I-P_h^n)\partial_t u_h, v) + (P_h^n \partial_t u_h, v)\\ &= ((I-P_h^n)\partial_t u_h, v) + (P_h^n \partial_t u_h, P_h^n v)\\ &= ((I-P_h^n)\partial_t u_h, v) + (\partial_t u_h, P_h^n v)\\ &= ((I-P_h^n)\partial_t u_h, v) + (f^n, P_h^n v)\\ &\quad + \left(\frac{(I^n u_h^{n-1} - u_h^{n-1})}{\tau_n}, P_h^n v\right) - (A^n u_h^n, P_h^n v)\\ &= ((I-P_h^n)\partial_t u_h, v) + (P_h^n f^n, v)\\ &\quad + \left(\frac{(I^n u_h^{n-1} - u_h^{n-1})}{\tau_n}, P_h^n v\right) - B^n(w^n, v),\end{aligned} \tag{61}$$

for all $v \in H_0^1(\Omega)$. Here, P_h^n stands for the L^2-projection operator onto V_h^n. Thus,

$$\begin{aligned}&(\partial_t e, v) + B^n(\varphi, v)\\ &\quad = (\partial_t (u_h - u), v) + B^n(w - u, v)\\ &\quad = (\partial_t u_h, v) + B^n(w, v) - (\partial_t u, v) - B^n(u, v)\\ &\quad = (\partial_t u_h, v) + B^n(w, v) - (f, v)\\ &\quad = ((I-P_h^n)\partial_t u_h, v) + (P_h^n f^n, v) + \left(\left(I^n u_h^{n-1} - u_h^{n-1}\right)/\tau_n, P_h^n v\right)\\ &\qquad - B^n(w^n, v) + B^n(w, v) - (f, v),\end{aligned} \tag{62}$$

where in the last line we have used (61).

In addition, for each $t \in (t_{n-1}, t_n]$ $(n = 1, 2, \ldots, N)$, it follows from (56)–(58) that

$$\begin{aligned}B^n(w - w^n, v) &= B^n\left(l_n(t) w^n + l_{n-1}(t) w^{n-1} - w^n, v\right)\\ &= \frac{t - t^n}{\tau_n} B^n\left(w^{n-1} - w^n, v\right)\\ &= \frac{t - t^n}{\tau_n}\left(A^n I^n u_h^{n-1} - A^n u_h^n, v\right).\end{aligned} \tag{63}$$

On the contrary, we have

$$\begin{aligned}((I-P_h^n)\partial_t u_h, v) &= \left((I-P_h^n)\frac{u_h^n - u_h^{n-1}}{\tau_n}, v\right)\\ &= \frac{-1}{\tau_n}\left((I-P_h^n) u_h^{n-1}, v\right)\\ &= \frac{-1}{\tau_n}\left(u_h^{n-1}, v\right) + \frac{1}{\tau_n}\left(P_h^n u_h^{n-1}, v\right).\end{aligned} \tag{64}$$

We also have

$$\begin{aligned}\left(\frac{(I^n u_h^{n-1} - u_h^{n-1})}{\tau_n}, P_h^n v - v\right) &= \frac{-1}{\tau_n}\left(u_h^{n-1}, P_h^n v - v\right)\\ &= \frac{-1}{\tau_n}\left(u_h^{n-1}, P_h^n v\right) + \frac{1}{\tau_n}\left(u_h^{n-1}, v\right).\end{aligned} \tag{65}$$

Substituting (63)–(65) into (62) gives the desired result. We then define the error estimators as follows:

(1) We define the time-stepping estimator as

$$\alpha_n = \frac{C_P}{\sqrt{3}}\left\| I^n u_h^{n-1} - u_h^n \right\|_{\mathcal{T}_n}, \tag{66}$$

with C_P satisfying

$$\left\| P_h^n v \right\|_{\mathcal{T}_n} \le C_P \|v\|_{\mathcal{T}_n}, \quad \text{for all } v \in H_0^1(\Omega). \tag{67}$$

(2) Set the data approximation estimator in time to be

$$\beta_n = \left(\int_{t_{n-1}}^{t_n} \frac{\left\| P_h^n f^n - f(s) \right\|_{H^{-1}(\Omega)}^2}{\tau_n} \mathrm{d}s\right)^{1/2}. \tag{68}$$

(3) The estimator with respect to mesh-change is given by

$$\zeta_n = \frac{\left\| I^n u_h^{n-1} - u_h^{n-1} \right\|_{H^{-1}(\Omega)}}{\tau_n}. \tag{69}$$

(4) The nonconforming part of parabolic estimator is defined as

$$\xi_n = \frac{\left\| u_d^n - u_{d+}^{n-1} \right\|_{H^{-1}(\Omega)}}{\tau_n}, \tag{70}$$

and the nonconforming part of elliptic estimator is given by

$$\theta_n = \left(\left\| u_d^n \right\|_{\mathcal{T}_n}^2 + \left\| u_{d+}^{n-1} \right\|_{\mathcal{T}_n}^2\right)^{1/2}. \tag{71}$$

(5) The space error estimator is

$$\rho_n = \mathscr{E}\left(u_h^n, A^n u_h^n, \mathcal{T}_n\right), \tag{72}$$

where $\mathscr{E}$ is stated in (47) in Theorem 3. Similarly, we define

$$\rho_{n-1} = \mathscr{E}\left(I^n u_h^{n-1}, A^n I^n u_h^{n-1}, \mathcal{T}_n\right). \tag{73}$$

(6) We also define

$$\epsilon_n = \frac{\left\|u_d^n - u_{d+}^n\right\|_0}{\tau_n}, \tag{74}$$

which can be understood as the nonconforming parabolic part estimator of higher order.

We now state the main result of this section, which is largely based on the work [18]. For the sake of completeness, we sketch a proof. □

Theorem 4. *Let u and $u_h(t)$ be solutions of (1) and (51), respectively. For each $m = 1, 2, \ldots, N$, we have*

$$\left(\int_0^{t_m} \|u(s) - u_h(s)\|_{\mathcal{T}_n}^2 \mathrm{d}s\right)^{1/2} \le \left\|u(0) - u_h^c(0)\right\|_0 + \sqrt{2}\eta_{e,m} + 3\eta_{p,m} + \left(\frac{1}{2}\sum_{n=1}^{m}\theta_n^2\tau_n\right)^{1/2} + \sqrt{3}\sum_{n=1}^{m-1}\epsilon_n\tau_n. \tag{75}$$

Here, $\eta_{e,m}$ refers to the elliptic error estimator which is defined as

$$\eta_{e,m} = \left(\sum_{n=1}^{m}\left(\rho_{n-1}^2 + \rho_n^2\right)\tau_n\right)^{1/2}, \tag{76}$$

and $\eta_{p,m}$ is parabolic error estimator defined by

$$\eta_{p,m} = \left(\sum_{n=1}^{m}\left(\alpha_n + \beta_n + \zeta_n + \xi_n\right)^2\tau_n\right)^{1/2}. \tag{77}$$

Proof. Selecting $v = e_c$ in (60) and noting that $e = e_c + u_h^d$ and $\varphi = e_c + \chi_c$, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_c\|_0^2 + \|\varphi\|_{\mathcal{T}_n}^2 &= (\partial_t e_c, e_c) + B^n(\varphi,\varphi)\\ &= -\left(\partial_t u_h^d, e_c\right) + B^n(\varphi,\chi_c) + l_{n-1}(t)\\ &\quad\cdot\left(A^n I^n u_h^{n-1} - A^n u_h^n, e_c\right)\\ &\quad+\left(\frac{\left(I^n u_h^{n-1} - u_h^{n-1}\right)}{\tau_n} + P_h^n f^n - f, e_c\right).\end{aligned} \tag{78}$$

Integrating (78) on $[0, t_m]$ implies that

$$\begin{aligned}&\frac{1}{2}\left\|e_c(t_m)\right\|_0^2 + \frac{1}{2}\int_0^{t_m}\|\varphi(s)\|_{\mathcal{T}_n}^2\mathrm{d}s\\ &= \frac{1}{2}\left\|e_c(0)\right\|_0^2 + \int_0^{t_m} B^n(\varphi,\chi_c)\mathrm{d}s + \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} l_{n-1}(s)\left(A^n I^n u_h^{n-1} - A^n u_h^n, e_c\right)\mathrm{d}s\\ &\quad+\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\left(\frac{\left(I^n u_h^{n-1} - u_h^{n-1}\right)}{\tau_n} + P_h^n f^n - f, e_c\right)\mathrm{d}s - \int_0^{t_m}\left(\partial_t u_h^d, e_c\right)\mathrm{d}s\\ &\quad+\frac{1}{2}\sum_{n=1}^{m-1}\left(\left\|u(t_n) - u_{c+}^n\right\|_0^2 - \left\|e_c(t_n)\right\|_0^2\right)\\ &\equiv \mathbb{M}_0 + \mathbb{M}_1 + \mathbb{M}_2 + \mathbb{M}_3 + \mathbb{M}_4 + \mathbb{M}_5.\end{aligned} \tag{79}$$

The first term $\mathbb{M}_1$ can be bounded by

$$\mathbb{M}_1 = \int_0^{t_m} B^n(\varphi,\chi_c)\mathrm{d}s \le \int_0^{t_m}\|\varphi(s)\|_{\mathcal{T}_n}\left\|\chi_c(s)\right\|_{\mathcal{T}_n}\mathrm{d}s. \tag{80}$$

To estimate $\mathbb{M}_2$, it follows from (67) that

$$\begin{aligned}\left(A^n I^n u_h^{n-1} - A^n u_h^n, e_c\right) &= \left(A^n I^n u_h^{n-1} - A^n u_h^n, P_h^n e_c\right)\\ &= B^n\left(I^n u_h^{n-1} - u_h^n, P_h^n e_c\right)\\ &\le \left\|I^n u_h^{n-1} - u_h^n\right\|_{\mathcal{T}_n}\left\|P_h^n e_c\right\|_{\mathcal{T}_n}\\ &\le C_P\left\|I^n u_h^{n-1} - u_h^n\right\|_{\mathcal{T}_n}\left\|e_c\right\|_{\mathcal{T}_n},\end{aligned} \tag{81}$$

this together with $e_c = \varphi - \chi_c$ implies that

$$\begin{aligned}\mathbb{M}_2 &\le \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} l_{n-1}(s)C_P\left\|I^n u_h^{n-1} - u_h^n\right\|_{\mathcal{T}_n}\left\|e_c\right\|_{\mathcal{T}_n}\mathrm{d}s\\ &\le \sum_{n=1}^{m}\left(\int_{t_{n-1}}^{t_n} l_{n-1}^2(s)C_P^2\left\|I^n u_h^{n-1} - u_h^n\right\|_{\mathcal{T}_n}^2\mathrm{d}s\right)^{1/2}\left(\int_{t_{n-1}}^{t_n}\left\|e_c\right\|_{\mathcal{T}_n}^2\mathrm{d}s\right)^{1/2}\\ &= \sum_{n=1}^{m}\alpha_n\sqrt{\tau_n}\left(\int_{t_{n-1}}^{t_n}\left\|e_c\right\|_{\mathcal{T}_n}^2\mathrm{d}s\right)^{1/2}\\ &\le \sum_{n=1}^{m}\alpha_n\sqrt{\tau_n}\left(\left(\int_{t_{n-1}}^{t_n}\|\varphi\|_{\mathcal{T}_n}^2\mathrm{d}s\right)^{1/2} + \left(\int_{t_{n-1}}^{t_n}\left\|\chi_c\right\|_{\mathcal{T}_n}^2\mathrm{d}s\right)^{1/2}\right).\end{aligned} \tag{82}$$

For the term $\mathbb{M}_3$, we first have

$$\begin{aligned}\mathbb{M}_3 &= \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\left(\frac{\left(I^n u_h^{n-1}-u_h^{n-1}\right)}{\tau_n}+P_h^n f^n - f, e_c\right)\mathrm{d}s \\ &\leq \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\left\|\frac{\left(I^n u_h^{n-1}-u_h^{n-1}\right)}{\tau_n}\right\|_{H^{-1}(\Omega)}\left\|\nabla e_c\right\|_0 \mathrm{d}s \\ &\quad + \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\left\|P_h^n f^n - f\right\|_{H^{-1}(\Omega)}\left\|\nabla e_c\right\|_0 \mathrm{d}s \\ &\equiv \mathbb{M}_{31}+\mathbb{M}_{32}.\end{aligned} \tag{83}$$

We then estimate $\mathbb{M}_{31}$ as follows:

$$\begin{aligned}\mathbb{M}_{31} &= \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\left\|\frac{\left(I^n u_h^{n-1}-u_h^{n-1}\right)}{\tau_n}\right\|_{H^{-1}(\Omega)}\left\|\nabla e_c\right\|_0 \mathrm{d}s \\ &\leq \sum_{n=1}^{m}\left(\int_{t_{n-1}}^{t_n}\frac{\left\|I^n u_h^{n-1}-u_h^{n-1}\right\|_{H^{-1}(\Omega)}^2}{\tau_n^2}\mathrm{d}s\right)^{1/2} \\ &\quad \cdot\left(\int_{t_{n-1}}^{t_n}\left\|\nabla e_c\right\|_0^2 \mathrm{d}s\right)^{1/2} \\ &= \sum_{n=1}^{m}\zeta_n\sqrt{\tau_n}\left(\int_{t_{n-1}}^{t_n}\left\|\nabla e_c\right\|_0^2 \mathrm{d}s\right)^{1/2}.\end{aligned} \tag{84}$$

Similarly, the term $\mathbb{M}_{32}$ can be bounded by

$$\begin{aligned}\mathbb{M}_{32} &= \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\tau_n^{-1/2}\left\|P_h^n f^n - f\right\|_{H^{-1}(\Omega)}\tau_n^{1/2}\left\|\nabla e_c\right\|_0 \mathrm{d}s \\ &\leq \sum_{n=1}^{m}\left(\int_{t_{n-1}}^{t_n}\frac{\left\|P_h^n f^n - f(s)\right\|_{H^{-1}(\Omega)}^2}{\tau_n}ds\right)^{1/2} \\ &\quad \cdot\sqrt{\tau_n}\left(\int_{t_{n-1}}^{t_n}\left\|\nabla e_c\right\|_0^2 \mathrm{d}s\right)^{1/2} \\ &= \sum_{n=1}^{m}\beta_n\sqrt{\tau_n}\left(\int_{t_{n-1}}^{t_n}\left\|\nabla e_c\right\|_0^2 \mathrm{d}s\right)^{1/2}.\end{aligned} \tag{85}$$

Substituting (84) and (85) into (83) yields

$$\begin{aligned}\mathbb{M}_3 \leq \sum_{n=1}^{m}\left(\zeta_n+\beta_n\right)\sqrt{\tau_n}&\left(\left(\int_{t_{n-1}}^{t_n}\left\|\nabla\varphi\right\|_0^2 \mathrm{d}s\right)^{1/2}\right. \\ &\left.+\left(\int_{t_{n-1}}^{t_n}\left\|\nabla\chi_c\right\|_0^2 \mathrm{d}s\right)^{1/2}\right).\end{aligned} \tag{86}$$

For the term $\mathbb{M}_4$, we have

$$\begin{aligned}\mathbb{M}_4 &\leq \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\left\|\partial_t u_h^d\right\|_{H^{-1}(\Omega)}\left\|\nabla e_c\right\|_0 \mathrm{d}s \\ &\leq \sum_{n=1}^{m}\xi_n\sqrt{\tau_n}\left(\left(\int_{t_{n-1}}^{t_n}\left\|\nabla\varphi\right\|_0^2 \mathrm{d}s\right)^{1/2}+\left(\int_{t_{n-1}}^{t_n}\left\|\nabla\chi_c\right\|_0^2 \mathrm{d}s\right)^{1/2}\right).\end{aligned} \tag{87}$$

Table 1: Numerical results for $t=1$ with $\tau=h^2$, for Example 1.

h	$\|\nabla_h(u^n-u_h^n)\|_0$	Order	$\|u^n-u_h^n\|_0$	Order
1/2	$9.013994e-2$	—	$1.833267e-2$	—
1/4	$4.754053e-2$	0.9230	$5.642120e-3$	1.7001
1/8	$2.375353e-2$	1.0010	$1.396086e-3$	2.0148
1/16	$1.184181e-2$	1.0043	$3.369024e-4$	2.0510
1/32	$5.915072e-3$	1.0014	$8.210111e-05$	2.0369

Observing that

$$\begin{aligned}\left\|u(t_n)-u_{c+}^n\right\|_0^2-\left\|e_c(t_n)\right\|_0^2 &= \left\|u_d^n-u_{d+}^n\right\|_0^2 \\ &\quad + 2\left(u_d^n-u_{d+}^n, e_c(t_n)\right),\end{aligned} \tag{88}$$

we then have

$$\begin{aligned}\mathbb{M}_5 &= \frac{1}{2}\sum_{n=1}^{m-1}\left(\left\|u(t_n)-u_{c+}^n\right\|_0^2-\left\|e_c(t_n)\right\|_0^2\right) \\ &\leq \frac{1}{2}\sum_{n=1}^{m-1}\left(\epsilon_n^2\tau_n^2+2\epsilon_n\tau_n\max_{1\leq k\leq m-1}\left\|e_c(t_k)\right\|_0\right) \\ &\leq \frac{3}{2}\left(\sum_{n=1}^{m-1}\epsilon_n\tau_n\right)^2+\frac{1}{4}\max_{1\leq k\leq m-1}\left\|e_c(t_k)\right\|_0^2.\end{aligned} \tag{89}$$

Adapting similar techniques as in the proof of Theorem 5.7 in [18], we can obtain the desired estimate (75). □

5. Numerical Experiments

In this section, we give some numerical tests to validate our theoretical analysis in Theorem 2. The numerical scheme (9) shows that, at each time step $1\leq n\leq N$, we shall solve the following linear system: Given $u_h^0=P_h^0 u_0$, find $u_h^n\in V_h$ such that

$$\tau a_h\left(u_h^n, v_h\right)+\left(u_h^n, v_h\right)=\tau\left(f^n, v_h\right)+\left(u_h^{n-1}, v_h\right), \quad \forall v_h\in V_h. \tag{90}$$

In view of this expression, we then implemented our numerical experiments by MATLAB package.

Example 1. We consider problem (1) with $\Omega=(0,1)\times(0,1)$, $T=1$. We choose the source term f such that the exact solution is given by $u(x,t)=\sin(\pi t/2)x(1-x)y(1-y)$.

The numerical results for the WOPSIP method are stated in Table 1 and Figure 1. The convergence orders shown in Table 1 (which are also the slopes of the line in Figure 1) are consistent with the theoretical results in Theorem 2.

Example 2. We consider problem (1) with $\Omega=(0,1)\times(0,1)$, $T=1$. The exact solution is $u(x,t)=t^2\sin(\pi x)\sin(\pi y)$, and the source term f can be computed accordingly.

The convergence orders stated in Table 2 (see also the slopes of the line in Figure 2) agree with the theoretical analysis in Theorem 2.

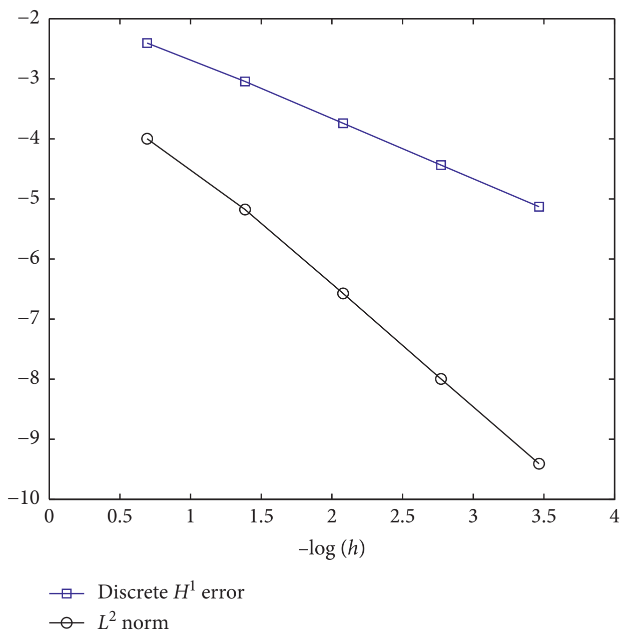

Figure 1: The convergence rate for $t = 1$ with $\tau = h^2$ for Example 1.

Table 2: Numerical results for $t = 1$ with $\tau = h^2$ for Example 2.

h	$\Vert\nabla_h (u^n - u_h^n)\Vert_0$	Order	$\Vert u^n - u_h^n\Vert_0$	Order
1/2	1.259907	—	1.386570e − 1	—
1/4	0.665702	0.9204	4.937369e − 2	1.4897
1/8	0.328806	1.0176	1.259566e − 2	1.9708
1/16	0.163091	1.0116	3.053313e − 3	2.0445
1/32	0.081347	1.0035	7.4407982e − 4	2.0368

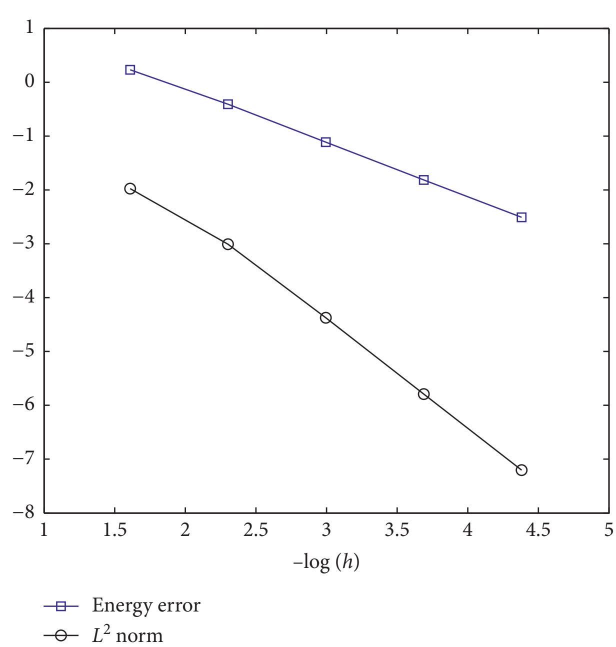

Figure 2: The convergence rate for $t = 1$ with $\tau = h^2$ for Example 2.

6. Conclusion and Future Work

A weakly overpenalized symmetric interior penalty method is proposed and analyzed for solving the heat equation. Optimal a priori error estimates in the energy norm are established. Moreover, we derive a posteriori error estimators which are key indicators to design adaptive algorithms. We only present some numerical tests to validate the a priori error estimate. A posteriori error estimators stated in Section 4 include elliptic and parabolic terms. In particular, the error estimators with respect to the mesh-change operator I^n, such as α_n and ς_n, that need to be carefully investigated in the implementation procedure. Thus, the implementation of adaptive algorithms based on the proposed error estimators will be addressed in the further work.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

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