Research Article

Vibration Analysis of Piezoelectric Composite Plate Resting on Nonlinear Elastic Foundations Using Sinc and Discrete Singular Convolution Differential Quadrature Techniques

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In this work, free vibration of the piezoelectric composite plate resting on nonlinear elastic foundations is examined. The three-dimensionality of elasticity theory and piezoelectricity is used to derive the governing equation of motion. By implementing two differential quadrature schemes and applying different boundary conditions, the problem is converted to a nonlinear eigenvalue problem. The perturbation method and iterative quadrature formula are used to solve the obtained equation. Numerical analysis of the proposed schemes is introduced to demonstrate the accuracy and efficiency of the obtained results. The obtained results are compared with available results in the literature, showing excellent agreement. Additionally, the proposed schemes have higher efficiency than previous schemes. Furthermore, a parametric study is introduced to investigate the effect of elastic foundation parameters, different materials of sensors and actuators, and elastic and geometric characteristics of the composite plate on the natural frequencies and mode shapes.

1. Introduction

The increase in the use of piezoelectric composite materials resting on elastic foundation structures, especially in aerospace, automotive, and marine environments, leads to the rise of difficulties in nonlinear vibrations in various modern engineering challenges. The elastic foundations play an important role that preserve the structural system under oscillations and avoid mechanical failures. The main role of nonlinear foundations is developing the accuracy of the model to describe the behavior of our system. The cubic nonlinearity is the most common type used in recent research [1–8].

Due to the mathematical complexity of such problems, only limited cases can analytically be solved. A number of exact methods of the vibration problems on elastic foundations are issued in [9–14]. Literature on the numerical solution of the research subject is sparse. Typical useful numerical methods, such as the finite element [15–17], the boundary element [18], differential quadrature [19], and state space [20, 21] techniques, are used to solve such

problems. One of the disadvantages of these methods is consuming too much time and effort to get the solution [22–26]. Additionally, there is no research in piezoelectric composite plate resting on nonlinear elastic foundations, whether analytically or numerically.

The sinc differential quadrature method (SDQM) [27–32] and the discrete singular convolution differential quadrature method (DSCDQM) [33–42] are more reliable than the polynomial-based DQM. The nonlinear partial differential equations are reduced to linear equations using perturbation [43] and iterative quadrature methods [44]. Previous DQ techniques are then employed to reduce the problem to the eigenvalue or bending problem. The same schemes are used for free vibration analysis of piezoelectric nanobeams [45, 46]. Recent studies have investigated the vibration analysis of composite plate resting on elastic foundations, using DSCDQM [47–58].

As far as the authors are aware, SDQM and DSCDQM have not been examined for vibration analysis of composite piezoelectric plate materials resting on nonlinear elastic foundations. Based on these versions, numerical schemes are

designed for free vibration of piezoelectric composites. The natural frequencies are obtained and compared with previous analytical and numerical frequencies. For each scheme, the convergence and efficiency are verified. Additionally, a parametric study is introduced to investigate the influence of elastic foundation parameters and elastic and geometric characteristics of the composite on the vibrated results.

2. Formulation of the Problem

Consider a three-dimensional piezoelectric composite with $(0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq h)$, where a , b , and h are the length, width, and total thickness of the composite, respectively. This composite is polarized in the z direction and consists of m layers with different types of materials. The plate rests on a three-parameter foundation model, as shown in Figure 1.

Based on the theory of elasticity and piezoelectricity, the equations of motion and the charge equation of electrostatic can be written as follows [59, 65–69]:

$$\sigma_{x,x} + \tau_{xy,y} + \tau_{xz,z} = \rho u_{,tt}, \quad (1)$$

$$\tau_{xy,x} + \sigma_{y,y} + \tau_{yz,z} = \rho v_{,tt}, \quad (2)$$

$$\tau_{xz,x} + \tau_{yz,y} + \sigma_{z,z} + k_1 w + k_3 w^3 - k_2 (w_{,xx} + w_{,yy}) = \rho w_{,tt}, \quad (3)$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{pmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ w_{,y} + v_{,z} \\ w_{,x} + u_{,z} \\ v_{,x} + u_{,y} \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & e_5 & 0 \\ 0 & 0 & 0 & e_4 & 0 & 0 \\ 0 & 0 & e_3 & 0 & 0 & 0 \end{bmatrix}^T \begin{pmatrix} \phi_{,x} \\ \phi_{,y} \\ \phi_{,z} \end{pmatrix}, \quad (5)$$

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_5 & 0 \\ 0 & 0 & 0 & e_4 & 0 & 0 \\ 0 & 0 & e_3 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ w_{,y} + v_{,z} \\ w_{,x} + u_{,z} \\ v_{,x} + u_{,y} \end{pmatrix} - \begin{bmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{bmatrix} \begin{pmatrix} \phi_{,x} \\ \phi_{,y} \\ \phi_{,z} \end{pmatrix}, \quad (6)$$

where C , e , and η are the components of the effective elastic, piezoelectric, and dielectric constants of the same

FIGURE 1: Piezoelectric composite (sensor is PZT-4 and actuator is $\text{Ba}_2\text{NaNb}_5\text{O}_{15}$) resting on nonlinear elastic foundations.

$$D_{x,x} + D_{y,y} + D_{z,z} = 0, \quad (4)$$

where $(\sigma_x, \sigma_y, \sigma_z)$, (u, v, w) , and (D_x, D_y, D_z) are the stresses, displacement, and induction field in the x , y , and z directions, respectively; $(\tau_{yz}, \tau_{xz}, \tau_{xy})$ are the shear stresses; ρ is the density of the material; and (k_1, k_2, k_3) are linear Winkler foundation, linear Pasternak foundation, and nonlinear Winkler foundation parameters, respectively.

The relation between mechanical and electric material properties is constitutive equations, which can be written as follows:

piezoelectric material, respectively, and ϕ is the electric potential.

For harmonic behavior, one can assume that

$$\begin{aligned} u(x, z, t) &= U e^{i\omega t}, \\ v(x, z, t) &= V e^{i\omega t}, \\ w(x, z, t) &= W e^{i\omega t}, \\ \phi(x, z, t) &= \Phi e^{i\omega t}, \end{aligned} \quad (7)$$

where ω is the natural frequency of the plate and $i = \sqrt{-1}$. U , W , and Φ are the amplitudes for u , w , and ϕ , respectively.

The elastic material constants can be determined as follows, using the reciprocal theorem [41]:

$$C_{11} = \frac{E_1(E_2 - \nu_{23}^2 E_3)}{E_2 - \nu_{12}^2 E_2^2/E_1 - \nu_{23}^2 E_3 - \nu_{13} E_2 E_3/E_1 (\nu_{13} - 2\nu_{12}\nu_{23})}, \quad (8)$$

$$C_{12} = C_{21} = \frac{\nu_{12} E_2 + \nu_{13} \nu_{23} E_3}{1 - \nu_{12}^2 E_2/E_1 - \nu_{23}^2 E_3/E_2 - \nu_{13} E_3/E_1 (\nu_{13} - 2\nu_{12}\nu_{23})}, \quad (9)$$

$$C_{13} = C_{31} = \frac{E_3(\nu_{13} + \nu_{12}\nu_{23})}{1 - \nu_{12}^2 E_2/E_1 - \nu_{23}^2 E_3/E_2 - \nu_{13} E_3/E_1 (\nu_{13} - 2\nu_{12}\nu_{23})}, \quad (10)$$

$$C_{22} = \frac{E_2(E_1 - \nu_{13}^2 E_3)}{E_1 - \nu_{12}^2 E_2 - \nu_{23}^2 E_1 E_3/E_2 - \nu_{13} E_3 (\nu_{13} - 2\nu_{12}\nu_{23})}, \quad (11)$$

$$C_{23} = C_{32} = \frac{E_3(\nu_{23} E_1 + \nu_{12}\nu_{13} E_2)}{E_1 - \nu_{12}^2 E_2 - \nu_{23}^2 E_1 E_3/E_2 - \nu_{13} E_3 (\nu_{13} - 2\nu_{12}\nu_{23})}, \quad (12)$$

$$C_{33} = \frac{E_3(E_1 - \nu_{12}^2 E_2)}{E_1 - \nu_{12}^2 E_2 - \nu_{23}^2 E_1 E_3/E_2 - \nu_{13} E_3 (\nu_{13} - 2\nu_{12}\nu_{23})}, \quad (13)$$

$$\begin{aligned} C_{44} &= G_{23}, \\ C_{55} &= G_{13}, \\ C_{66} &= G_{12}, \end{aligned} \quad (14)$$

where E_p , G_{pq} , and ν_{pq} ($p, q = 1, 2, 3$) are Young's moduli, shear moduli, and Poisson's ratios.

The isotropic material constants can be expressed as follows [56]:

$$\begin{aligned} C_{11} = C_{12} = C_{13} = C_{22} = C_{23} = C_{31} = C_{32} = C_{33} &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)} + 2 \frac{E}{2(1 + \nu)}, \\ C_{44} = C_{55} = C_{66} &= \frac{E}{2(1 + \nu)}. \end{aligned} \quad (15)$$

Substituting equations (5)–(16) into (1)–(4), the problem can be reduced to a quasistatic one, as follows:

$$C_{11}U_{,xx} + C_{66}U_{,yy} + C_{55}U_{,zz} + (C_{12} + C_{66})V_{,xy} + (C_{13} + C_{55})W_{,xz} + (e_1 + e_5)\Phi_{,xz} = -\rho\omega^2 U, \quad (16)$$

$$(C_{66} + C_{12})U_{,xy} + C_{66}^2 V_{,xx} + C_{11}V_{,yy} + C_{44}V_{,zz} + (C_{13} + C_{44})W_{,yz} + (e_2 + e_4)\Phi_{,yz} = -\rho\omega^2 V, \quad (17)$$

$$\begin{aligned} (C_{55} + C_{13})U_{,xz} + (C_{13} + C_{44})V_{,yz} + (C_{55} - k_2)W_{,xx} + (C_{44} - k_2)W_{,yy} \\ + C_{33}W_{,zz} + e_5\Phi_{,xx} + e_4\Phi_{,yy} + e_3\Phi_{,zz} + k_1W + k_3W^3 = -\rho\omega^2 W, \end{aligned} \quad (18)$$

$$(e_1 + e_5)U_{,xz} + (e_2 + e_4)V_{,yz} + e_5W_{,xx} + e_4W_{,yy} + e_3W_{,zz} - \eta_1\Phi_{,xx} - \eta_2\Phi_{,yy} - \eta_3\Phi_{,zz} = 0. \quad (19)$$

The boundary conditions can be described as follows:

(1) For a simply supported edge (S):

$$\begin{aligned} w = v = \sigma_x &= 0, \text{ at } x = 0, a, \\ w = u = \sigma_y &= 0, \text{ at } y = 0, b. \end{aligned} \quad (20)$$

(2) For a clamped edge (C):

$$w = v = u = 0, \text{ at } x = 0, a, y = 0, b. \quad (21)$$

(3) For a free edge (F):

$$\begin{aligned} \sigma_x = \tau_{xz} = \tau_{xy} &= 0, \text{ at } x = 0, a, \\ \sigma_y = \tau_{yz} = \tau_{xy} &= 0, \text{ at } y = 0, b. \end{aligned} \quad (22)$$

Mechanical and electrical boundary conditions at the lower and upper surfaces of the composite are as follows:

$$\begin{aligned} \sigma_z = \tau_{xz} = \tau_{zy} = D_z &= 0, \text{ at } z = 0, \\ \sigma_z = \tau_{xz} = \tau_{zy} = \Phi &= 0, \text{ at } z = h, \end{aligned} \quad (23)$$

To ensure the continuity between electric and elastic layers, the following conditions can be considered:

$$\begin{aligned} U(x, y, h_s^-) &= U(x, y, h_s^+), \\ V(x, y, h_s^-) &= V(x, y, h_s^+), \\ W(x, y, h_s^-) &= W(x, y, h_s^+), \\ \Phi(x, y, h_s^-) &= \Phi(x, y, h_s^+), \end{aligned} \quad (24)$$

$$\begin{aligned} U(x, y, (h_s + h_c)^-) &= U(x, y, (h_s + h_c)^+), \\ V(x, y, (h_s + h_c)^-) &= V(x, y, (h_s + h_c)^+), \\ W(x, y, (h_s + h_c)^-) &= W(x, y, (h_s + h_c)^+), \\ \Phi(x, y, (h_s + h_c)^-) &= \Phi(x, y, (h_s + h_c)^+). \end{aligned} \quad (25)$$

Additionally, the continuity conditions between different elastic materials are as follows:

$$\begin{aligned} U(x, y, h_k^-) &= U(x, y, h_k^+), \\ V(x, y, h_k^-) &= V(x, y, h_k^+), \\ W(x, y, h_k^-) &= W(x, y, h_k^+), \\ \Phi(x, y, h_k^-) &= \Phi(x, y, h_k^+). \end{aligned} \quad (26)$$

3. Method of Solution

Two differential quadrature techniques are employed to reduce the governing equations into a nonlinear eigenvalue problem, as follows.

3.1. Sinc Differential Quadrature Method (SDQM). A cardinal sine function is used as a shape function, such that the unknown ψ and its derivatives can be approximated as a weighted linear sum of nodal values, ψ_i , ($i = -N, N$), as follows [27–32]:

$$\psi(x_i) = \sum_{j=-N}^N \frac{\sin[\pi(x_i - x_j)/h_x]}{\pi(x_i - x_j)/h_x} \psi(x_j), \quad (i = -N, N), \quad h_x > 0, \quad (27)$$

$$\left. \frac{\partial \psi}{\partial x} \right|_{x=x_i} = \sum_{j=-N}^N A_{ij}^x \psi(x_j), \quad (i = -N, N), \quad (28)$$

$$\left. \frac{\partial^2 \psi}{\partial x^2} \right|_{x=x_i} = \sum_{j=-N}^N B_{ij}^x \psi(x_j), \quad (i = -N, N), \quad (29)$$

where ψ denotes U, V, W , and Φ ; N is the number of grid points; and h_x is the grid size. The weighting coefficients, A_{ij}^x and B_{ij}^x , can be determined by differentiating (28) as follows:

$$\begin{aligned} A_{ij}^x &= \begin{cases} \frac{(-1)^{i-j}}{h_x(i-j)}, & i \neq j, \\ 0, & i = j, \end{cases} \\ B_{ij}^x &= \begin{cases} \frac{-2(-1)^{i-j}}{h_x^2(i-j)^2}, & i \neq j, \\ \frac{-\pi^2}{3h_x^2}, & i = j. \end{cases} \end{aligned} \quad (30)$$

3.2. Discrete Singular Convolution Differential Quadrature Method (DSCDQM). A singular convolution can be defined as follows [33–42]:

$$F_{(t)} = (T * \eta)(t) = \int_{-\infty}^{\infty} T(t-x)\eta(x)dx, \quad (31)$$

where $T(t-x)$ is a singular kernel.

The DSC algorithm can be applied using many types of kernels. These kernels are applied as shape functions, such that the unknown ψ and its derivatives are approximated as a weighted linear sum of ψ_i , ($i = -N, N$) over a narrow bandwidth ($x - x_M, x + x_M$) [33–37].

Two kernels of DSC will be employed as follows:

(a) A Delta Lagrange kernel (DLK) can be used as a shape function, such that the unknown ψ and its

derivatives can be approximated as a weighted linear sum of nodal values, ψ_i , ($i = -N, N$), as follows:

$$\psi(x_i) = \sum_{j=-M}^M \frac{\prod_{k=-M}^M (x_i - x_k)}{(x_i - x_j) \prod_{j=-M, j \neq k}^M (x_j - x_k)} \psi(x_j), \quad (i = -N, N), M \geq 1, \quad (32)$$

$$\begin{aligned} \frac{\partial \psi}{\partial x} \Big|_{x=x_i} &= \sum_{j=-M}^M A_{ij}^x \psi(x_j), \\ \frac{\partial^2 \psi}{\partial x^2} \Big|_{x=x_i} &= \sum_{j=-M}^M B_{ij}^x \psi(x_j), \quad (i = -N, N), \end{aligned} \quad (33)$$

where $2M+1$ is the effective computational bandwidth. A_{ij}^x and B_{ij}^x are defined as follows:

$$\begin{aligned} A_{ij}^x &= \begin{cases} \frac{1}{(x_i - x_j)} \prod_{k=-M, k \neq i, j}^M \frac{(x_i - x_k)}{(x_j - x_k)}, & i \neq j, \\ -\sum_{j=-M, j \neq i}^M A_{ij}^x, & i = j, \end{cases} \\ B_{ij}^x &= \begin{cases} 2 \left(A_{ij}^x A_{ii}^x - \frac{A_{ij}^x}{(x_i - x_j)} \right), & i \neq j, \\ -\sum_{j=-M, j \neq i}^M B_{ij}^x, & i = j. \end{cases} \end{aligned} \quad (34)$$

- (b) A regularized Shannon kernel (RSK) can also be used as a shape function, such that the unknown ψ and its derivatives can be approximated as a weighted linear sum of nodal values, ψ_i , ($i = -N, N$), as follows:

$$\psi(x_i) = \sum_{j=-M}^M \left\langle \frac{\sin[\pi(x_i - x_j)/h_x]}{\pi(x_i - x_j)/h_x} e^{-((x_i - x_j)^2/2\sigma^2)} \right\rangle \psi(x_j), \quad (i = -N, N), \sigma = (r * h_x) > 0, \quad (35)$$

$$\begin{aligned} \frac{\partial \psi}{\partial x} \Big|_{x=x_i} &= \sum_{j=-M}^M A_{ij}^x \psi(x_j), \\ \frac{\partial^2 \psi}{\partial x^2} \Big|_{x=x_i} &= \sum_{j=-M}^M B_{ij}^x \psi(x_j), \quad (i = -N, N), \end{aligned} \quad (36)$$

where σ is a regularization parameter and r is a computational parameter. The weighting coefficients A_{ij}^x and B_{ij}^x can be defined as follows [33–42]:

$$\begin{aligned} A_{ij}^x &= \begin{cases} \frac{(-1)^{i-j}}{h_x(i-j)} e^{-h_x((i-j)^2/2\sigma^2)}, & i \neq j, \\ 0, & i = j, \end{cases} \\ B_{ij}^x &= \begin{cases} \left(\frac{2(-1)^{i-j+1}}{h_x^2(i-j)^2} + \frac{1}{\sigma^2} \right) e^{-h_x^2((i-j)^2/2\sigma^2)}, & i \neq j, \\ -\frac{1}{\sigma^2} - \frac{\pi^2}{3h_x^2}, & i = j. \end{cases} \end{aligned} \quad (37)$$

Similarly, one can approximate ψ_y , ψ_z , ψ_{yy} , and ψ_{zz} and calculate A_{ij}^y , A_{ij}^z , B_{ij}^y , and B_{ij}^z . Two methods are used to transform the governing equation to linear equations.

3.2.1. Perturbation Method. Assuming the solution of equations (17)–(27) to be a power series that can be solved by a perturbation method of the second order [43] by substituting $W = W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + \dots$ into (17)–(27) and equating the terms with the identical powers of ε , where ε is a perturbation parameter:

$$C_{11}U_{,xx} + C_{66}U_{,yy} + C_{55}U_{,zz} + (C_{12} + C_{66})V_{,xy} + (C_{13} + C_{55})(W_{0,xz} + \varepsilon W_{1,xz} + \varepsilon^2 W_{2,xz}) + (e_1 + e_5)\Phi_{,xz} = -\rho\omega^2 U, \quad (38)$$

$$(C_{66} + C_{12})U_{,xy} + C_{66}V_{,xx} + C_{11}V_{,yy} + C_{44}V_{,zz} + (C_{13} + C_{44})(W_{0,yz} + \varepsilon W_{1,yz} + \varepsilon^2 W_{2,yz}) + (e_2 + e_4)\Phi_{,yz} = -\rho\omega^2 V, \quad (39)$$

$$\begin{aligned} &(C_{55} + C_{13})U_{,xz} + (C_{13} + C_{44})V_{,yz} + (C_{55} - k_2)(W_{0,xx} + \varepsilon W_{1,xx} + \varepsilon^2 W_{2,xx}) + (C_{44} - k_2)(W_{0,yy} + \varepsilon W_{1,yy} + \varepsilon^2 W_{2,yy}) \\ &+ C_{33}(W_{0,zz} + \varepsilon W_{1,zz} + \varepsilon^2 W_{2,zz}) + e_5\Phi_{,xx} + e_4\Phi_{,yy} + e_3\Phi_{,zz} + k_1(W_0 + \varepsilon W_1 + \varepsilon^2 W_2) + k_3\varepsilon(W_0 + \varepsilon W_1 + \varepsilon^2 W_2)^3 \\ &= -\rho\omega^2(W_0 + \varepsilon W_1 + \varepsilon^2 W_2), \end{aligned} \quad (40)$$

$$\begin{aligned} &(e_1 + e_5)U_{,xz} + (e_2 + e_4)V_{,yz} + e_5(W_{0,xx} + \varepsilon W_{1,xx} + \varepsilon^2 W_{2,xx}) + e_4(W_{0,yy} + \varepsilon W_{1,yy} + \varepsilon^2 W_{2,yy}) + e_3(W_{0,zz} + \varepsilon W_{1,zz} + \varepsilon^2 W_{2,zz}) \\ &- \eta_1\Phi_{,xx} - \eta_2\Phi_{,yy} - \eta_3\Phi_{,zz} = 0. \end{aligned} \quad (41)$$

Equate the terms of ε^0 in (39)–(42):

$$C_{11}U_{,xx} + C_{66}U_{,yy} + C_{55}U_{,zz} + (C_{12} + C_{66})V_{,xy} + (C_{13} + C_{55})W_{0,xz} + (e_1 + e_5)\Phi_{,xz} = -\rho\omega^2U, \quad (42)$$

$$(C_{66} + C_{12})U_{,xy} + C_{66}V_{,xx} + C_{11}V_{,yy} + C_{44}V_{,zz} + (C_{13} + C_{44})W_{0,yz} + (e_2 + e_4)\Phi_{,yz} = -\rho\omega^2V, \quad (43)$$

$$(C_{55} + C_{13})U_{,xz} + (C_{13} + C_{44})V_{,yz} + (C_{55} - k_2)W_{0,xx} + (C_{44} - k_2)W_{0,yy} + C_{33}W_{0,zz} + e_5\Phi_{,xx} + e_4\Phi_{,yy} + e_3\Phi_{,zz} + k_1W_0 = -\rho\omega^2W_0, \quad (44)$$

$$(e_1 + e_5)U_{,xz} + (e_2 + e_4)V_{,yz} + e_5W_{0,xx} + e_4W_{0,yy} + e_3W_{0,zz} - \eta_1\Phi_{,xx} - \eta_2\Phi_{,yy} - \eta_3\Phi_{,zz} = 0. \quad (45)$$

The previous system is an eigenvalue problem, which is solved to obtain the natural frequencies and the bending deflection W_0 .

Equate the terms of ε^1 in (39)–(42):

$$C_{11}U_{,xx} + C_{66}U_{,yy} + C_{55}U_{,zz} + (C_{12} + C_{66})V_{,xy} + (C_{13} + C_{55})W_{1,xz} + (e_1 + e_5)\Phi_{,xz} = -\rho\omega^2U, \quad (46)$$

$$(C_{66} + C_{12})U_{,xy} + C_{66}V_{,xx} + C_{11}V_{,yy} + C_{44}V_{,zz} + (C_{13} + C_{44})W_{1,yz} + (e_2 + e_4)\Phi_{,yz} = -\rho\omega^2V, \quad (47)$$

$$(C_{55} + C_{13})U_{,xz} + (C_{13} + C_{44})V_{,yz} + (C_{55} - k_2)W_{1,xx} + (C_{44} - k_2)W_{1,yy} + C_{33}W_{1,zz} + e_5\Phi_{,xx} + e_4\Phi_{,yy} + e_3\Phi_{,zz} + (k_1 + \rho\omega^2)W_1 = -k_3W_0^3, \quad (48)$$

$$(e_1 + e_5)U_{,xz} + (e_2 + e_4)V_{,yz} + e_5W_{1,xx} + e_4W_{1,yy} + e_3W_{1,zz} - \eta_1\Phi_{,xx} - \eta_2\Phi_{,yy} - \eta_3\Phi_{,zz} = 0, \quad (49)$$

The previous system is a bending problem, which is solved to obtain W_1 .

Equate the terms of ε^2 in (39)–(42):

$$C_{11}U_{,xx} + C_{66}U_{,yy} + C_{55}U_{,zz} + (C_{12} + C_{66})V_{,xy} + (C_{13} + C_{55})W_{2,xz} + (e_1 + e_5)\Phi_{,xz} = -\rho\omega^2U, \quad (50)$$

$$(C_{66} + C_{12})U_{,xy} + C_{66}V_{,xx} + C_{11}V_{,yy} + C_{44}V_{,zz} + (C_{13} + C_{44})W_{2,yz} + (e_2 + e_4)\Phi_{,yz} = -\rho\omega^2V, \quad (51)$$

$$(C_{55} + C_{13})U_{,xz} + (C_{13} + C_{44})V_{,yz} + (C_{55} - k_2)W_{2,xx} + (C_{44} - k_2)W_{2,yy} + C_{33}W_{2,zz} + e_5\Phi_{,xx} + e_4\Phi_{,yy} + e_3\Phi_{,zz} + (k_1 + \rho\omega^2)W_2 = -3k_3W_0^2W_1, \quad (52)$$

$$(e_1 + e_5)U_{,xz} + (e_2 + e_4)V_{,yz} + e_5W_{2,xx} + e_4W_{2,yy} + e_3W_{2,zz} - \eta_1\Phi_{,xx} - \eta_2\Phi_{,yy} - \eta_3\Phi_{,zz} = 0. \quad (53)$$

The previous system is a bending problem, which is solved to obtain W_2 .

Finally, the series solution can be written as follows:

$$W_{\text{numerical}} = \lim_{\varepsilon \rightarrow 1} W = W_0 + W_1 + W_2. \quad (54)$$

The convergence condition [43] due to the perturbation method is set as follows:

$$\left| \frac{W_{i+1}}{W_i} \right| < 1, \quad \text{where } i = 0, 1. \quad (55)$$

3.2.2. Iterative Quadrature. We solved the following iterative system:

$$C_{11}U_{,xx} + C_{66}U_{,yy} + C_{55}U_{,zz} + (C_{12} + C_{66})V_{,xy} + (C_{13} + C_{55})W_{i+1,xz} + (e_1 + e_5)\Phi_{,xz} = -\rho\omega^2 U, \quad i = 0, 1, 2, \quad (56)$$

$$(C_{66} + C_{12})U_{,xy} + C_{66}V_{,xx} + C_{11}V_{,yy} + C_{44}V_{,zz} + (C_{13} + C_{44})W_{i+1,yz} + (e_2 + e_4)\Phi_{,yz} = -\rho\omega^2 V, \quad i = 0, 1, 2, \quad (57)$$

$$(C_{55} + C_{13})U_{,xz} + (C_{13} + C_{44})V_{,yz} + (C_{55} - k_2)W_{i+1,xx} + (C_{44} - k_2)W_{i+1,yy} + C_{33}W_{i+1,zz} + e_5\Phi_{,xx} + e_4\Phi_{,yy} + e_3\Phi_{,zz} + k_1W_{i+1} + k_3W_i^2W_{i+1} = -\rho\omega^2 W_{i+1}, \quad i = 0, 1, 2, \quad (58)$$

$$(e_1 + e_5)U_{,xz} + (e_2 + e_4)V_{,yz} + e_5W_{i+1,xx} + e_4W_{i+1,yy} + e_3W_{i+1,zz} - \eta_1\Phi_{,xx} - \eta_2\Phi_{,yy} - \eta_3\Phi_{,zz} = 0, \quad i = 0, 1, 2. \quad (59)$$

To get W_o , we solved the following eigenvalue problem:

$$C_{11} \sum_{l=-N_x}^{N_x} B_{il}^x U_{ljk} + C_{66} \sum_{m=-N_y}^{N_y} B_{jm}^y U_{imk} + C_{55} \sum_{n=-N_z}^{N_z} B_{kn}^z U_{ijn} + (C_{12} + C_{66}) \sum_{l=-N_x}^{N_x} A_{il}^x \sum_{m=-N_y}^{N_y} A_{jm}^y V_{lmk} + (C_{13} + C_{55}) \sum_{l=-N_x}^{N_x} A_{il}^x \sum_{n=-N_z}^{N_z} A_{kn}^z W_{0,ljn} + (e_1 + e_5) \sum_{l=-N_x}^{N_x} A_{il}^x \sum_{n=-N_z}^{N_z} A_{kn}^z \Phi_{ljn} = -\rho\omega^2 U, \quad (60)$$

$$(C_{12} + C_{66}) \sum_{l=-N_x}^{N_x} A_{il}^x \sum_{m=-N_y}^{N_y} A_{jm}^y U_{lmk} + C_{66} \sum_{l=-N_x}^{N_x} B_{il}^x V_{ljk} + C_{11} \sum_{m=-N_y}^{N_y} B_{jm}^y V_{imk} + C_{44} \sum_{n=-N_z}^{N_z} B_{kn}^z V_{ijn} + (C_{13} + C_{44}) \sum_{m=-N_y}^{N_y} A_{jm}^y \sum_{n=-N_z}^{N_z} A_{kn}^z W_{0,imn} + (e_2 + e_4) \sum_{m=-N_y}^{N_y} A_{jm}^y \sum_{n=-N_z}^{N_z} A_{kn}^z \Phi_{imn} = -\rho\omega^2 V, \quad (61)$$

$$(C_{55} + C_{13}) \sum_{l=-N_x}^{N_x} A_{il}^x \sum_{n=-N_z}^{N_z} A_{kn}^z U_{ljn} + (C_{13} + C_{44}) \sum_{m=-N_y}^{N_y} A_{jm}^y \sum_{n=-N_z}^{N_z} A_{kn}^z V_{imn} + (C_{55} - k_2) \sum_{l=-N_x}^{N_x} B_{il}^x W_{0,ljk} + (C_{44} - k_2) \sum_{m=-N_y}^{N_y} B_{jm}^y W_{0,imk} + C_{33} \sum_{n=-N_z}^{N_z} B_{kn}^z W_{ijn} + e_5 \sum_{l=-N_x}^{N_x} B_{il}^x \Phi_{ljk} + e_4 \sum_{m=-N_y}^{N_y} B_{jm}^y \Phi_{imk} + e_3 \sum_{n=-N_z}^{N_z} B_{kn}^z \Phi_{ijn} + k_1 W_{0,ijk} = -\rho\omega^2 W_0, \quad (62)$$

$$(e_1 + e_5) \sum_{l=-N_x}^{N_x} A_{il}^x \sum_{n=-N_z}^{N_z} A_{kn}^z U_{ljn} + (e_2 + e_4) \sum_{m=-N_y}^{N_y} A_{jm}^y \sum_{n=-N_z}^{N_z} A_{kn}^z V_{imn} + e_5 \sum_{l=-N_x}^{N_x} B_{il}^x W_{0,ljk} + e_4 \sum_{m=-N_y}^{N_y} B_{jm}^y W_{0,imk} + e_3 \sum_{n=-N_z}^{N_z} B_{kn}^z W_{0,ijn} - \eta_1 \sum_{l=-N_x}^{N_x} B_{il}^x \Phi_{ljk} - \eta_2 \sum_{m=-N_y}^{N_y} B_{jm}^y \Phi_{imk} - \eta_3 \sum_{n=-N_z}^{N_z} B_{kn}^z \Phi_{ijn} = 0. \quad (63)$$

The boundary conditions (21)–(27) can also be approximated using two DQMs as follows:

(1) Simply supported (S):

$$W_{ijk} = V_{ijk} = C_{11} \sum_{l=-N_x}^{N_x} A_{il}^x U_{ljk} + C_{12} \sum_{m=-N_y}^{N_y} A_{jm}^y V_{imk} + C_{13} \sum_{n=-N_z}^{N_z} A_{kn}^z W_{ijn} + e_1 \sum_{n=-N_z}^{N_z} A_{kn}^z \Phi_{ijn} = 0, \quad \text{at } x = 0, a, \quad (64)$$

$$W_{ijk} = U_{ijk} = C_{12} \sum_{l=-N_x}^{N_x} A_{il}^x U_{ljk} + C_{11} \sum_{m=-N_y}^{N_y} A_{jm}^y V_{imk} + C_{13} \sum_{n=-N_z}^{N_z} A_{kn}^z W_{ijn} + e_2 \sum_{n=-N_z}^{N_z} A_{kn}^z \Phi_{ijn} = 0, \quad \text{at } y = 0, b,$$

(2) Clamped (C):

$$W_{ijk} = V_{ijk} = U_{ijk} = 0, \quad \text{at } x = 0, a \text{ } y = 0, b. \quad (65)$$

(3) Free surface (F):

$$\begin{aligned} & C_{11} \sum_{l=-N_x}^{N_x} A_{il}^x U_{ljk} + C_{12} \sum_{m=-N_y}^{N_y} A_{jm}^y V_{imk} + C_{13} \sum_{n=-N_z}^{N_z} A_{kn}^z W_{ijn} + e_1 \sum_{n=-N_z}^{N_z} A_{kn}^z \Phi_{ijn} = 0, \\ & C_{55} \left(\sum_{l=-N_x}^{N_x} A_{il}^x W_{ljk} + \sum_{n=-N_z}^{N_z} A_{kn}^z U_{ijn} \right) + e_5 \sum_{l=-N_x}^{N_x} A_{il}^x \Phi_{ljk} = C_{66} \left(\sum_{l=-N_x}^{N_x} A_{il}^x V_{ljk} + \sum_{m=-N_y}^{N_y} A_{jm}^y U_{imk} \right) = 0, \quad \text{at } x = 0, a, \\ & C_{12} \sum_{l=-N_x}^{N_x} A_{il}^x U_{ljk} + C_{11} \sum_{m=-N_y}^{N_y} A_{jm}^y V_{imk} + C_{13} \sum_{n=-N_z}^{N_z} A_{kn}^z W_{ijn} + e_2 \sum_{n=-N_z}^{N_z} A_{kn}^z \Phi_{ijn} = 0, \\ & C_{44} \left(\sum_{m=-N_y}^{N_y} A_{jm}^y W_{imk} + \sum_{n=-N_z}^{N_z} A_{kn}^z V_{ijn} \right) + e_4 \sum_{m=-N_y}^{N_y} A_{jm}^y \Phi_{imk} = C_{66} \left(\sum_{l=-N_x}^{N_x} A_{il}^x V_{ljk} + \sum_{m=-N_y}^{N_y} A_{jm}^y U_{imk} \right) = 0, \quad \text{at } y = 0, b, \end{aligned} \quad (66)$$

Mechanical and electrical boundary conditions at the lower and upper surfaces of the composite are as follows:

$$\begin{aligned} & C_{13} \sum_{l=-N_x}^{N_x} A_{il}^x U_{ljk} + C_{13} \sum_{m=-N_y}^{N_y} A_{jm}^y V_{imk} + C_{33} \sum_{n=-N_z}^{N_z} A_{kn}^z W_{ijn} + e_3 \sum_{n=-N_z}^{N_z} A_{kn}^z \Phi_{ijn} = C_{55} \left(\sum_{l=-N_x}^{N_x} A_{il}^x W_{ljk} + \sum_{n=-N_z}^{N_z} A_{kn}^z U_{ijn} \right) \\ & + e_5 \sum_{l=-N_x}^{N_x} A_{il}^x \Phi_{ljk} = 0 \\ & C_{44} \left(\sum_{m=-N_y}^{N_y} A_{jm}^y W_{imk} + \sum_{n=-N_z}^{N_z} A_{kn}^z V_{ijn} \right) + e_4 \sum_{m=-N_y}^{N_y} A_{jm}^y \Phi_{imk} = e_1 \sum_{l=-N_x}^{N_x} A_{il}^x U_{ljk} + e_2 \sum_{m=-N_y}^{N_y} A_{jm}^y V_{imk} + e_3 \sum_{n=-N_z}^{N_z} A_{kn}^z W_{ijn} \\ & - \eta_3 \sum_{n=-N_z}^{N_z} A_{kn}^z \Phi_{ijn} = 0, \quad \text{at } z = 0, \quad (67) \\ & C_{13} \sum_{l=-N_x}^{N_x} A_{il}^x U_{ljk} + C_{13} \sum_{m=-N_y}^{N_y} A_{jm}^y V_{imk} + C_{33} \sum_{n=-N_z}^{N_z} A_{kn}^z W_{ijn} + e_3 \sum_{n=-N_z}^{N_z} A_{kn}^z \Phi_{ijn} = C_{55} \left(\sum_{l=-N_x}^{N_x} A_{il}^x W_{ljk} + \sum_{n=-N_z}^{N_z} A_{kn}^z U_{ijn} \right) \\ & + e_5 \sum_{l=1}^{N_x} A_{il}^x \Phi_{ljk} = 0 \\ & C_{44} \left(\sum_{m=-N_y}^{N_y} A_{jm}^y W_{imk} + \sum_{n=-N_z}^{N_z} A_{kn}^z V_{ijn} \right) + \sum_{m=-N_y}^{N_y} A_{jm}^y \Phi_{imk} = \Phi_{ijk} = 0, \quad \text{at } z = h. \end{aligned}$$

The continuity conditions between the interfaces of layers can be assumed as follows:

$$\begin{aligned} W_{ijk}^1 &= W_{ijk}^2, \\ V_{ijk}^1 &= V_{ijk}^2, \\ U_{ijk}^1 &= U_{ijk}^2, \\ \Phi_{ijk}^1 &= \Phi_{ijk}^2. \end{aligned} \quad (68)$$

We have solved the generalized eigenvalue problem [65]:

$$KX = \omega^2 MX, \quad (69)$$

where K is the coefficient matrix of the previous system, M is the mass matrix and can be diagonal with 0 diagonal elements, and ω is the free vibration frequencies squared.

For nontrivial solutions for (70), the following determinant should be 0:

$$|K - \omega^2 M| = 0. \quad (70)$$

Equation (71) gives the natural frequencies for the composite plate.

TABLE 1: Material property of composite piezoelectric plate [59, 61].

Material property	Young's moduli (GPa)			Shear moduli (GPa)			Poisson's ratios			Density (kg/m ³)
	E_2 7	E_1 $25E_2$	E_3 E_2	G_{12} $0.5E_2$	G_{13} G_{12}	G_{23} $0.2E_2$	ν_{12} 0.25	ν_{13} 0.03	ν_{23} 0.4	ρ 1600
Metal (aluminum, Al)		70	—	—	—	—		0.3		2702
Ceramic (alumina, Al ₂ O ₃)		380	—	—	—	—		0.3		3800
Material property	Effective elastic (GPa)									Density (kg/m ³)
	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}	
Sensor PZT-4	139	78	74	139	74	115	25.6	25.6		30.5
Sensor BaTiO ₃	166	77	78	166	78	162	43	43		44.5
Actuator Ba ₂ NaNb ₅ O ₁₅	239	104	5	247	52	135	65	66		76
Actuator PZT-5A	121	77	77	121	111	21	21	21		23
Material property	Piezoelectric constants (C/m ²)					Dielectric constants (F/m) * 10 ⁻⁹			Density (kg/m ³)	
	e_1	e_2	e_3	e_4	e_5	η_1	η_2	η_3	ρ	
Sensor PZT-4	-5.2	-5.2	15.1	12.7	12.7	6.5	6.5	5.6	7500	
Sensor BaTiO ₃	-4.4	-4.4	18.6	11.6	11.6	11.2	11.2	12.6	5700	
Actuator Ba ₂ NaNb ₅ O ₁₅	-0.4	-0.3	4.3	3.4	2.8	19.6	2.01	0.28	5300	
Actuator PZT-5A	-5.4	-5.4	15.8	3.4	3.4	8.11	8.11	7.34	2330	

4. Numerical Results

The present numerical results demonstrate the convergence and efficiency of each one of the proposed schemes for vibration analysis of the piezoelectric composite plate Al/Al₂O₃ resting on a nonlinear elastic foundation. For all results, the boundary conditions (65)–(69) are augmented in the governing equations (39)–(42).

For practical purposes, the field quantities are normalized as follows:

$$(U^*, V^*, W^*) = \frac{(U, V, W)}{h},$$

$$(\sigma_x^*, \sigma_y^*, \sigma_z^*, \tau_{xy}^*, \tau_{xz}^*, \tau_{yz}^*) = \frac{(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})}{E_2}, \quad (71)$$

$$C_{ij}^* = \frac{C_{ij}}{E_2},$$

$$x^* = \frac{x}{a},$$

$$y^* = \frac{y}{b}, \quad (72)$$

$$z^* = \frac{z}{h},$$

$$h = h_s + h_c + h_a,$$

$$h_a = h_s,$$

$$D = \frac{E_c h^3}{(12(1 - \nu^2))},$$

$$\overline{K}_1 = \frac{k_1 a^4}{D},$$

$$\overline{K}_2 = \frac{k_2 a^2}{D}, \quad (73)$$

$$\overline{K}_3 = \frac{k_3 a^4 h^2}{D},$$

where U^* , V^* , and W^* are the normalized amplitudes of displacements, σ_x^* , σ_y^* , and σ_z^* are the normalized amplitudes of stresses, τ_{xy}^* , τ_{xz}^* , and τ_{yz}^* are the normalized amplitudes of shear stresses, h_a and h_s are the thicknesses of the actuator and sensor, and D , $\overline{K}_1$, $\overline{K}_2$, and $\overline{K}_3$ are the flexural rigidity for the bottom layer of composite and the equivalent dimensionless elastic foundation constants.

The computational characteristics of each scheme are adapted to reach accurate results with error of order $\leq 10^{-8}$. The obtained dimensionless frequencies Ω , $\bar{\omega}$, and β are evaluated as follows: $\Omega = \omega h \sqrt{\rho/E}$, $\bar{\omega} = \omega b/\pi^2 \sqrt{\rho h/D}$, and $\beta = \omega a^2 \sqrt{\rho h/D}$, where ρ and E are the density and Young's modulus, respectively.

For the present results, material parameters for the composite are listed in Table 1.

For the sinc DQ scheme, the problem is solved over a regular grid, ranging from $3 \times 5 \times 5$ to $11 \times 5 \times 5$. Table 2 shows the convergence of the obtained results for the isotropic plate, which is in agreement with the exact results in [68] over grid size $\geq 7 \times 5 \times 5$.

For DSCDQ scheme based on Delta Lagrange kernel, the problem is also solved over a uniform grid ranging from $3 \times 5 \times 5$ to $11 \times 5 \times 5$. The bandwidth $2M+1$ ranges from 3 to 11.

Table 3 shows the convergence of the obtained fundamental frequency, which is in agreement with the exact results in [68] over grid size $\geq 7 \times 5 \times 5$ and bandwidth ≥ 5 . Table 4 shows that the obtained results are more accurate than those obtained using approximated theories [60, 69]. This table also shows that the execution time of DSCDQM-DLK was less than that of sinc DQM.

For the DSCDQ scheme based on the regularized Shannon kernel (RSK), the problem was also solved over a uniform grid ranging from $3 \times 5 \times 5$ to $9 \times 5 \times 5$. The bandwidth $2M+1$ ranged from 3 to 11 and the regularization parameter $\sigma = r h_x$ ranged from $1.0 h_x$ to $2.3 h_x$, where $h_x = 1/N - 1$. Table 5 shows the convergence of the obtained fundamental frequency to the exact results in [68] over grid

TABLE 2: Comparison between the normalized fundamental frequencies $\beta = \omega a^2 \sqrt{\rho_m h / D_m}$ using the sinc DQM, grid size N , and the previous exact and numerical frequencies for the simply supported square isotropic plate ($a/h = 5$, $h_c/h_p = 200$, $\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, and $\bar{K}_3 = 0$).

Results	Normalized frequencies N	β_1	β_2	β_3	β_4	β_5	β_6
Sinc DQM	3	55.9218	103.4553	124.1464	161.3903	177.5293	186.4058
	5	55.9721	103.5484	124.2581	161.5355	177.6891	186.5736
	7	56.0369	103.6682	124.2982	161.5391	177.6930	186.5777
	9	56.0369	103.6683	124.2982	161.5391	177.6930	186.5777
	11	56.0369	103.6683	124.2982	161.5391	177.6930	186.5777
	Analytical [68]	56.0359	—	—	—	—	—
	RHSDT [69]	56.0310	—	—	—	—	—
	NSDT [60]	56.0311	—	—	—	—	—
	Execution time (sec)	5.615225 over $N = 7 \times 5 \times 5$					

RHSDT: refined hyperbolic shear deformation theory; NSDT: new shear deformation theory; DQM: differential quadrature method.

TABLE 3: Comparison between the normalized fundamental frequency using DSCDQM-DLK, bandwidth $(2M+1)$, and grid size N for the simply supported square isotropic plate ($h_c/h_p = 200$, $a/h = 5$, $\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, and $\bar{K}_3 = 0$).

Fundamental frequency		DSCDQM-DLK				
Bandwidth	N	3	5	7	9	11
$2M+1=3$		55.9844	56.0849	56.0685	56.0685	56.0685
$2M+1=5$		55.9524	56.0349	56.0320	56.0320	56.0320
$2M+1=7$		56.1798	56.0650	56.0320	56.0320	56.0320
$2M+1=9$		55.9384	56.2877	56.0320	56.0320	56.0320
$2M+1=11$		55.9384	56.2877	56.0320	56.0320	56.0320

TABLE 4: Comparison between the normalized frequencies $\beta = \omega a^2 \sqrt{\rho_m h / D_m}$, using DSCDQM-DLK, grid sizes N , and the previous exact and numerical frequencies for the simply supported square isotropic plate ($2M+1=5$, $h_c/h_p = 200$, $a/h = 5$, $\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, and $\bar{K}_3 = 0$).

Normalized frequencies							
Results	N	β_1	β_2	β_3	β_4	β_5	β_6
DSCDQM-DLK	3	55.9524	103.5119	124.2142	161.4660	177.6126	186.4932
	5	56.0349	103.6085	124.3302	161.5049	177.6553	186.5380
	7	56.0320	103.6031	124.3237	161.4964	177.6444	186.5266
	9	56.0320	103.6031	124.3237	161.4964	177.6444	186.5266
Analytical [68]		56.0359	—	—	—	—	—
RHSDT [69]		56.0310	—	—	—	—	—
NSDT [60]		56.0311	—	—	—	—	—
Execution time (sec)	5.611001 over $N = 7 \times 5 \times 5$						

TABLE 5: Comparison between the normalized fundamental frequency β using the discrete singular convolution differential quadrature method on regularized Shannon kernel (DSCDQM-RSK), bandwidth $(2M+1)$, regularization parameter σ , and grid size N for the simply supported square isotropic plate ($h_c/h_p = 200$, $a/h = 5$, $\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, and $\bar{K}_3 = 0$).

Fundamental frequency	Regularization parameter	DSCDQM-RSK			
N	$2M+1$	$\sigma = 1 \times h_x$	$\sigma = 1.5 \times h_x$	$\sigma = 1.9 \times h_x$	$\sigma = 2.3 \times h_x$
3	3	55.9564	55.9468	55.9853	56.0821
	5	55.9561	55.8991	55.9780	56.0706
	7	55.9561	55.8991	55.9780	56.0706
5	3	55.8758	55.9715	56.0318	56.0318
	5	55.8758	55.9715	56.0318	56.0318
	7	55.8758	55.9715	56.0318	56.0318
7	3	55.8507	55.9392	56.0318	56.0318
	5	55.9563	55.9790	56.0318	56.0318
	7	55.9926	55.9917	56.0318	56.0318
9	3	55.9259	55.9857	56.0318	56.0318
	5	55.9823	55.9733	56.0318	56.0318
	7	55.9506	56.0193	56.0318	56.0318

TABLE 6: Comparison between the normalized frequencies $\beta = \omega a^2 \sqrt{\rho_m h / D_m}$ using DSCDQM-RSK, grid size, and the previous exact and numerical frequencies for the simply supported square isotropic plate ($2M + 1 = 3$, $\sigma = 1.9 * h_x$, $h_c/h_p = 200$, $a/h = 5$, $\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, and $\bar{K}_3 = 0$).

Normalized frequencies		β_1	β_2	β_3	β_4	β_5	β_6
Results	N						
DSCDQM-RSK	3	55.9853	103.5168	124.2201	161.4861	177.6347	186.5164
	5	56.0318	103.6027	124.3232	161.6201	177.7821	186.6712
	7	56.0318	103.6027	124.3232	161.6201	177.7821	186.6712
	9	56.0318	103.6027	124.3232	161.6201	177.7821	186.6712
Analytical [68]		56.0359	—	—	—	—	—
RHSDT [69]		56.0310	—	—	—	—	—
NSDT [60]		56.0311	—	—	—	—	—
Execution time (sec)		1.933552 over $N = 5 \times 5 \times 5$					

TABLE 7: Comparison between the normalized fundamental frequency $\beta = \omega a^2 \sqrt{\rho_m h / D_m}$ and the elastic foundation parameter for the simply supported squared plate ($h_c/h_p = 200$ and $a/h = 5$).

$\bar{K}_1, \bar{K}_2$	Nonlinear treatment methods		$\bar{K}_3 = 0$	$\bar{K}_3 = 50$	$\bar{K}_3 = 100$
0, 0	Perturbation		17.4590	17.4590	17.4590
	Execution time (sec)		0.480763	1.2346	1.2210
	Iterative quadrature		17.4590	17.4590	17.4590
	Execution time (sec)		0.480763	0.743408	0.818601
	Analytical [68]		17.5055	—	—
100, 10	Perturbation		24.2807	24.2807	24.2807
	Execution time (sec)		0.469989	1.1744	1.2081
	Iterative quadrature		24.2807	24.2807	24.2807
	Execution time (sec)		0.469989	0.629148	0.651756
	Analytical [68]		24.3074	—	—
1000, 100	Perturbation		56.0318	56.0318	56.0318
	Execution time (sec)		0.439478	1.1496	1.1351
	Iterative quadrature		56.0318	56.0318	56.0318
	Execution time (sec)		0.439478	0.620471	0.572076
	Analytical [68]		56.0359	—	—

TABLE 8: Comparison between the normalized fundamental frequency β , elastic foundation parameter, and different boundary conditions for the square plate ($h_c/h_p = 200$, $a/h = 5$, and $\bar{K}_3 = 50$).

$\bar{K}_1$	$\bar{K}_2$	Nonlinear treatment methods	SSSS	SCSC	SSSC	SSSF
0	0	Perturbation	19.7349	28.9157	23.6398	11.6883
		Iterative quadrature	19.7349	28.9157	23.6398	11.6883
	100	Perturbation	48.6329	54.6505	51.3365	37.1656
		Iterative quadrature	48.6329	54.6505	51.3365	37.1656
	1000	Perturbation	141.9431	146.7559	144.2712	111.7952
		Execution time (sec)	1.24152	1.2158	1.22158	1.187421
		Iterative quadrature	141.9425	146.7553	144.2706	111.7947
		Execution time (sec)	0.784509	0.70355	0.76487	0.69457
		Execution time (sec)	0.784509	0.70355	0.76487	0.69457
100	0	Perturbation	22.1402	30.6741	25.6818	15.3870
		Iterative quadrature	22.1402	30.6741	25.6818	15.3870
	100	Perturbation	49.6324	55.5689	52.2925	38.4718
		Iterative quadrature	49.6324	55.5689	52.2925	38.4718
	1000	Perturbation	142.2904	147.1665	144.7015	112.3010
		Execution time (sec)	1.24115	1.1187	1.24005	1.12571
		Iterative quadrature	142.3024	147.1790	144.7098	112.3075
		Execution time (sec)	0.77785	0.70254	0.75714	0.70478
		Execution time (sec)	0.77785	0.70254	0.75714	0.70478
1000	0	Perturbation	37.2791	42.8293	39.4889	33.7090
		Iterative quadrature	37.2791	42.8293	39.4889	33.7090
	100	Perturbation	57.9569	63.1485	60.8547	48.6837
		Iterative quadrature	57.9569	63.1485	60.8547	48.6837
	1000	Perturbation	145.3461	150.1200	147.6716	116.1315
		Execution time (sec)	1.14287	1.11276	1.01451	0.99475
		Iterative quadrature	145.4915	150.2702	147.6863	116.2471
		Execution time (sec)	0.75415	0.70125	0.74583	0.69573
		Execution time (sec)	0.75415	0.70125	0.74583	0.69573

TABLE 9: Comparison between the normalized frequencies $\beta = \omega a^2 \sqrt{\rho_c h / D_c}$, thickness ratio (a/h), elastic foundation parameter, and different boundary conditions for the square plate ($h_c/h_p = 200$, $\bar{K}_2 = 10$, and $\bar{K}_3 = 50$).

a/h	$\bar{K}_1$	SCSC			SCSS		
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
10/3	0	2.3150	3.4276	3.6742	2.1762	2.4572	3.5834
	10	2.3328	3.4280	3.6324	2.1905	2.4574	3.5909
	100	2.4834	3.4311	3.6325	2.3415	2.4616	3.7124
	1000	3.4298	3.4392	3.6326	3.3221	3.1212	3.8330
2	0	1.7629	2.2383	2.7214	1.7072	1.8167	2.6100
	10	1.7752	2.2883	2.7623	1.7132	1.8232	2.6112
	100	1.8589	2.2885	2.7628	1.7921	1.8672	2.6194
	1000	2.0977	2.3384	2.7456	2.0492	1.9688	2.6414
5	0	2.3897	4.7139	7.7506	2.2456	3.3940	7.5180
	10	2.4116	4.7240	7.7562	2.3392	3.9209	7.5247
	100	2.6001	4.8135	7.8054	2.4180	3.9989	7.5918
	1000	3.9708	5.5944	8.2594	2.4181	4.0015	7.5921

TABLE 10: Comparison between the normalized frequencies and the thickness h for the clamped plate ($a/h = 4$, $\bar{k}_1 = 1000$, $\bar{k}_2 = 100$, and $\bar{k}_3 = 50$).

Normalized frequencies H	β_1	β_2	β_3	β_4	β_5	β_6
0.05	79.648	164.334	211.562	237.039	301.984	327.423
0.06	79.918	164.853	211.828	237.510	302.594	327.996
0.07	79.918	164.854	211.853	237.512	302.594	328.010
0.075	79.918	164.854	211.853	237.512	302.594	328.010
0.08	79.918	164.854	211.853	237.512	302.594	328.010
0.09	79.918	164.854	211.853	237.512	302.594	328.010
0.1	79.918	164.854	211.853	237.512	302.594	328.010

FIGURE 2: Variation of the fundamental frequency (Ω) with Young's modulus (E_1/E_2), different boundary conditions, and different materials for the square plate ($\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, $\bar{K}_3 = 50$, $h_c/h_p = 45$, and $a/h = 10$). (a) Sensor is PZT-4 and actuator is $\text{Ba}_2\text{NaNb}_5\text{O}_{15}$; (b) sensor is BaTiO_3 and actuator is PZT-5A.

size $\geq 5 \times 5 \times 5$, bandwidth ≥ 3 , and regularization parameter $\sigma = 1.9 h_x$.

Table 6 shows that the obtained results from DQ schemes were more accurate than other models in the literature [60, 69]. Further, the execution time of this scheme is the

least. Therefore, the DSCDQM-RSK scheme is the best choice among the examined quadrature schemes for vibration analysis of piezoelectric composite materials resting on nonlinear elastic foundations. Furthermore, a parametric study is introduced to investigate the influence of foundation

FIGURE 3: Variation of the fundamental frequency (Ω) with shear modulus (G_{13}/G_{12}), different boundary conditions, and different materials of the square plate ($\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, $\bar{K}_3 = 50$, $h_c/h_p = 45$, and $a/h = 10$). (a) Sensor is PZT-4 and actuator is $\text{Ba}_2\text{NaNb}_5\text{O}_{15}$; (b) sensor is BaTiO_3 and actuator is PZT-5A.

FIGURE 4: Variation of the fundamental frequency (Ω) with the number of layers, different boundary conditions, and different materials of the square plate ($\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, $\bar{K}_3 = 100$, $h_c/h_p = 45$, and $a/h = 5$). (a) Sensor is PZT-4 and actuator is $\text{Ba}_2\text{NaNb}_5\text{O}_{15}$; (b) sensor is BaTiO_3 and actuator is PZT-5A.

parameters, elastic and geometric characteristics of the composite, and types of material on the values of natural frequencies and mode shapes. Tables 7 and 8 show the natural frequency for isotropic plates. These tables show the effect of elastic foundation parameters on the frequency for different boundary conditions. The natural frequency increased with the increase in foundation parameters $\bar{K}_1$ and $\bar{K}_2$. Additionally, the SCSC plate had the highest frequency and the SSSF had the lowest frequency. Moreover, the efficiency of the nonlinear treatment methods was

investigated. The execution time for the iterative quadrature method was less than the perturbation method, with a factor close to 1.5 until 1.9 for all boundary conditions used in Table 8.

Tables 8–10 and Figures 2–5 show that the natural frequencies increase with increasing side-to-thickness ratio (a/h), foundation parameters $\bar{K}_1$, $\bar{K}_2$, $\bar{K}_3$, total thickness, Young's modulus gradation ratio (E_1/E_2), shear modulus gradation ratio (G_{13}/G_{12}), and number of layers. The natural frequencies converge at a total thickness ($h = 0.075$) in the

FIGURE 5: Variation of the fundamental frequency (Ω) with Young's modulus (E_1/E_2), shear modulus (G_{13}/G_{12}), and different number of layers for the different materials (sensor is BaTiO₃ and actuator is PZT-5A) of the square plate ($\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, $\bar{K}_3 = 100$, $h_c/h_p = 45$, and $a/h = 10$).

TABLE 11: Comparison between the normalized frequencies and thickness h for the simply supported plate ($a/h = 4$, $\bar{k}_1 = 1000$, $\bar{k}_2 = 100$, and $\bar{k}_3 = 50$).

Normalized frequencies H	β_1	β_2	β_3	β_4	β_5	β_6
0.01	51.6122	92.90196	113.5468	144.5142	165.159	180.6427
0.02	51.7874	93.21732	113.9323	145.0047	165.7197	181.2559
0.03	51.7875	93.2175	113.9325	145.005	165.7200	181.2563
0.035	51.7875	93.2175	113.9325	145.005	165.7200	181.2563
0.04	51.7875	93.2175	113.9325	145.005	165.7200	181.2563
0.05	51.7875	93.2175	113.9325	145.005	165.7200	181.2563
0.07	51.7875	93.2175	113.9325	145.005	165.7200	181.2563

FIGURE 6: Variation of the fundamental frequency (Ω) with the composite layer thickness to piezoelectric thickness ratio (h_c/h_p), length-to-thickness ratio (a/h), and different boundary conditions for the square plate ($\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, $\bar{K}_3 = 50$, and $E_1/E_2 = 25$). (a) $a/h = 5$ and (b) $a/h = 10$.

FIGURE 7: Variation of the fundamental frequency (Ω) with the aspect ratio (a/b), different values of a/h , and different boundary conditions for the square plate ($\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, $\bar{K}_3 = 100$, $h_c/h_p = 25$, and $E_1/E_2 = 25$). (a) SSSS, (b) CSCS, (c) SCSC, and (d) CCCF.

CCCC plate and a total thickness ($h = 0.035$) in the SSSS plate, as shown in Tables 10 and 11. Figure 6 shows the natural frequencies decrease with the decrease of the piezoelectric layer thickness (h_c/h_p).

Furthermore, Figure 7 shows that the natural frequencies decreased with an increase in the aspect ratio (a/b) at different values (a/h). Moreover, the natural frequencies remained constant when $h_c/h_p > 20$ in all different edge conditions. Figures 2, 3, and 8 show the effect of the material type on the natural frequency. It can be seen that the sensor was more significant than the actuators. The natural frequency was almost unchanged when $a/h > 40$. Additionally,

the influence of the BaTiO_3 material was more affected by the natural frequencies than the PZT-4 material. Furthermore, Figures 9–12 show the first three mode shapes of normalized transverse displacement (w) and electric potential (ϕ). From the previous figures, it can be seen that the normalized transverse displacement was at a maximum for the CCCF plate and was at a minimum for the CCCC plate. Additionally, the electric potential was at a maximum for the CCCF plate and at a minimum for the CSCS plate. Furthermore, the previous figures show that the amplitude of displacement and electric potential increased with the increase of the nonlinear Winkler parameter $\bar{K}_3$.

FIGURE 8: Variation of the fundamental frequency (Ω) with the side-to-thickness ratio (a/h), different boundary conditions, and different materials for the square plate ($\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, $\bar{K}_3 = 100$, and $E_1/E_2 = 25$). (a) Sensor is BaTiO_3 and actuator is $\text{Ba}_2\text{NaNb}_5\text{O}_{15}$; (b) sensor is BaTiO_3 and actuator is PZT-5A.

FIGURE 9: Continued.

(c)

(d)

FIGURE 9: Variation of normalized mode shape deflection W with the time for the first three modes at different boundary conditions for the square plate ($\overline{K}_1 = 1000$, $\overline{K}_2 = 100$, $\overline{K}_3 = 0$, $h_c/h_p = 25$, $E_1/E_2 = 25$, and $a/h = 10$). (a) SSSS, (b) CCCC, (c) SCSC, and (d) CCCC.

(a)

(b)

FIGURE 10: Continued.

(c)

(d)

FIGURE 10: Variation of normalized mode shape deflection W with the time for the first three modes at different boundary conditions for the square plate ($\overline{K}_1 = 1000$, $\overline{K}_2 = 100$, $\overline{K}_3 = 100$, $h_c/h_p = 25$, $E_1/E_2 = 25$, and $a/h = 10$). (a) SSSS, (b) CCCC, (c) SCSC, and (d) CCCF.

(a)

(b)

FIGURE 11: Continued.

(c)

(d)

FIGURE 11: Variation of electric potential with the time for the first three modes at different boundary conditions for the square plate ($\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, $\bar{K}_3 = 0$, $h_c/h_p = 25$, $E_1/E_2 = 25$, and $a/h = 10$). (a) SSSS, (b) CCCC, (c) SCSC, and (d) CCF.

(a)

(b)

FIGURE 12: Continued.

FIGURE 12: Variation of electric potential with the time for the first three modes at different boundary conditions for the square plate ($\bar{K}_1 = 1000$, $\bar{K}_2 = 100$, $\bar{K}_3 = 100$, $h_c/h_p = 25$, $E_1/E_2 = 25$, and $a/h = 10$). (a) SSSS, (b) CCCC, (c) SCSC, and (d) CCCF.

5. Conclusions

Accurate quadrature schemes have been applied for free vibration analysis of piezoelectric composite plates with and without nonlinear elastic foundations for different types of boundary conditions. Numerical results were given to demonstrate the convergence, accuracy, and efficiency of the present methods. A MATLAB program was designed for each one, such that the maximum error (comparing with the previous exact results) was $\leq 10^{-8}$. Additionally, the execution time for each scheme was determined. It is concluded that the convergence speed for the discrete singular convolution differential quadrature method based on regularized Shannon kernel (DSCDQM-RSK) with grid size $\geq 5 \times 5 \times 5$, bandwidth $2M+1 \geq 3$, and regularization parameter $\sigma = 1.9 h_x$ is very fast and has the best accurate efficient results for the concerned problem (three-layered piezoelectric composite with total thickness = 0.075 in the CCCC plate and $h = 0.035$ in the SSSS plate). The efficiency of the iterative quadrature method is better than the perturbation technique. The influences of linear and nonlinear elastic foundation on the natural frequency and higher modes of frequency are investigated. The effect of the Pasternak parameter is larger than the effect of the linear and nonlinear Winkler coefficients on the natural frequencies. The natural frequency does not change when varying the nonlinear Winkler parameter. Additionally, the natural frequency converges at a larger value of the linear Winkler parameter. A parametric study is also introduced to investigate the influence of elastic and geometric characteristics of the composite and types of materials of the vibrated plate on the results. The nondimensional frequency for the material type BaTiO₃ is larger than that for the PZT-4 material. Further, the natural frequency decreases with the decrease in the thickness of the piezo material or with the

increase in the composite thickness until the value of ratio h_c/h_p closes to 10, so the natural frequency remains constant. The thinner composite has frequencies larger than thick composite. The composite plate with CCCC conditions has the minimum normalized transverse displacement, whereas it has the maximum value for CCCF conditions. Additionally, the electric potential is the maximum for the CCCF plate and has a minimum value for the CSCS plate. Moreover, the vertical displacement increases slightly when enlarging the nonlinear parameter. Finally, the aim is for these results to be useful for modern engineering smart structures.

Data Availability

The data in this paper are all open, and the data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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