Research Article

Approximate Controllability for a Kind of Fractional Neutral Differential Equations with Damping

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This paper gains several meaningful results on the mild solutions and approximate controllability for a kind of fractional neutral differential equations with damping (FNDED) and order belonging to $[1, 2]$ in Banach spaces. At first, a new expression for the mild solutions of FNDED via the (p, q) -regularized operator family and the technique of Laplace transform is acquired. Then, we consider the approximate controllability of FNDED by means of the approximate sequence method, and simultaneously, some applicable sufficient conditions are obtained.

1. Introduction

The primary description of the fractional-order derivative was proposed by Riemann and Liouville toward the end of the nineteenth century, but the notion of the arbitrary derivative and integral which generalized the classical integer-order derivative and integral was presented by Leibniz and Liouville in 1695. However, until the late 1960s when many phenomena on physics, engineering technology, and economics were more accurately described by fractional differential equations (FDEs), scientists began to show great interest on fractional order calculus. For example, they are widely adopted for nonlinear oscillations of earthquakes and in the fluid-dynamic traffic model. In practice, FDEs are deemed to optimize the traditional differential equation model. About the elementary theory of fractional differential and evolution systems, one can refer to Podlubny [1], Kilbas et al. [2], Zhou [3, 4], and [5–10] and the references cited therein.

In the particles' realistic movement, the resistance to motion is unavoidable, so it is suitable to add the damping character in mathematical models and control systems. Recently, a great deal of meaningful conclusions for the mathematical models with damping influence have been presented by the researchers [11–17].

As we all know, the controllability exerts a momentous effect on control theory and engineering technology. It lies in the fact that it is bound up with quadratic optimal control, observer design, and pole assignment. For this reason, the controllability has been actively investigated by many investigators, and an impressive progress has been made in recent years [7–12, 14–20]. Controllability of the deterministic systems in infinite dimensional spaces has been broadly investigated. In some results of the controllability for systems described by fractional differential models, the fixed-point and the approximate sequence method are felicitously used. Nonetheless, as demonstrated by Triggiani [21], for many parabolic partial differential systems, the conditions of complete controllability are very finite. The research to approximate controllability is more proper for the practical systems than to complete controllability. For the past few years, as regards differential dynamical systems in Banach spaces, several results are achieved about the approximate controllability [8, 19]. However, as far as we know, the approximate controllability of the fractional neutral differential equations with damping and order belonging to $[1, 2]$ is still relatively infrequent, so it is more interesting and necessary to study it.

2. Preliminaries and Notations

Inspired by the aforementioned analysis, the approximate controllability for a kind of fractional neutral differential equations with damping of order belonging to $[1, 2]$ in the Banach space is studied in our work. We acquire several

$$\begin{cases} {}^C D_t^p x'(t) + q {}^C D_t^p [x(t) - h(t, x_t)] = Ax(t) + f(t, x_t) + Bu(t), & t \in (0, b], \\ x(t) = \varphi(t), & t \in [-\tau, 0], \\ x'(0) = x_1, \end{cases} \quad (1)$$

where $p: 0 \leq p \leq 1$, ${}^C D_t^p$ denotes the Caputo fractional derivative, $q(q \geq 0)$, and the linear densely unbounded closed operator $A: D(A) \subseteq E \rightarrow E$ is the (p, q) -regularized family defined on the Banach space E . Here, the state $x(\cdot)$ is evaluated in Banach space E . Let U be a Banach space of admissible control functions. The variable $u(\cdot)$ takes a value in $L_2([0, b]; U)$, $B: U \rightarrow E$, which is linear and bounded. In addition, $f: [0, b] \times C([-\tau, 0]; E) \rightarrow E$ is nonlinear, which will be explained in detail later; $x_t \in L^2([-\tau, 0], E)$ and is denoted by $x_t(\vartheta) = \{x(t + \vartheta) | -\tau \leq \vartheta \leq 0\}$; $\varphi = \{\varphi(\vartheta) | -\tau \leq \vartheta \leq 0\} \in L^2([-\tau, 0], E)$. The relevant linear neutral system of FNDED (1) is

$$\begin{cases} {}^C D_t^p y'(t) + q {}^C D_t^p [y(t) - h(t, y_t)] = Ay(t) + Bv(t), & t \in (0, b], \\ x(t) = \varphi(t), & t \in [-\tau, 0], \\ x'(0) = x_1. \end{cases} \quad (2)$$

The structure of this paper is given as follows. In Section 2, we review several fundamental concepts and provide a new form of the mild solution for FNDED (1). Then, in Sections 3 and 4, we acquire several meaningful results for the existence and uniqueness of the mild solution and, furthermore, the approximate controllability for FNDED (1).

Let E be a Banach space with norm $\|\cdot\|_E$ and $L(E)$ be the space of all linear bounded operators on E . Let

sufficient conditions to pledge the approximate controllability of the FNDED via the contraction mapping theory and approximate sequence method. The FNDED and order in $[1, 2]$ is debated as follows:

$L_m([0, b]; E)$, $1 \leq m < \infty$, be the space of E -valued Bochner integrable function $f: [0, b] \rightarrow E$, and

$$\|f\|_{L_m} = \left(\int_0^b \|f\|_E^m dt \right)^{1/m}. \quad (3)$$

In the following, we recall some definitions to be adopted in the entire work.

Definition 1. (see [1, 2]). The Riemann–Liouville fractional integral of a function z and order α and from lower limit 0 can be denoted by

$$I_{0+}^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{z(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0. \quad (4)$$

Definition 2. (see [1, 2]). The Caputo derivative of order α for a function $z \in C([0, \infty), R)$ can be defined as

$${}^C D_{0+}^\alpha z(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\nu)^{-\alpha} z'(\nu) d\nu, \quad t > 0, 0 < \alpha < 1. \quad (5)$$

Remark 1 (see [1, 2]). If $z \in C([0, \infty), R)$, then

$${}^C D_{0+}^\alpha z(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\nu)^{-\alpha} z'(\nu) d\nu = I_{0+}^{1-\alpha} z'(t), \quad t > 0, 0 < \alpha < 1. \quad (6)$$

Lemma 1. (see [2]). Let $\alpha \in (n-1, n]$. If $z(\cdot) \in AC^n([a, b], E)$ or $z(\cdot) \in C^n([a, b], E)$, then the Caputo derivative

$$I_{0+}^\alpha {}^C D_{0+}^\alpha z(t) = z(t) - \sum_{i=0}^{n-1} \frac{z^{(i)}(0)}{i!} t^i, \quad t > 0. \quad (7)$$

Definition 3. (see [15, 16]). Let $q \geq 0$ and $0 \leq p \leq 1$. A is a linear closed operator, and its domain $D(A)$ is in Banach space E . We say that A is the generator of a (p, q) -regularized operator family $\{S_p(t)\}_{t \geq 0} \subset L(E)$ if the three formulas are established in the following:

- (a) $S_p(t)$ is strongly continuous on R_+ , and $S_p(0) = I$.
- (b) $S_p(t)D(A) \subset D(A)$ and $AS_p(t)x = S_p(t)Ax$ for all $x \in D(A)$, $t \geq 0$.
- (c) For every $x \in D(A)$, the following equation holds:

$$S_p(t)x = e^{-qt} + \frac{1}{\Gamma(p)} \int_0^t \int_0^\nu \mu^{(p-1)} e^{q(\mu-\nu)} AS_p(t-\nu)x d\mu d\nu, \quad t \geq 0, \quad (8)$$

where $S_p(t)$ is the (p, q) -regularized operator family generated by A . On the basis of the operator S_p , the two operators $P_p, Q_p: R_+ \rightarrow L(E)$ are reminded:

$$P_p(t) = \int_0^t S_p(\nu) d\nu, \quad t \geq 0, \quad (9)$$

$$Q_p(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\nu)^{(p-1)} S_p(\nu) d\nu, \quad t \geq 0. \quad (10)$$

The following basic statement is deduced from [22].

Lemma 2. (see [15, 16]). Let A be a densely linear closed operator in Banach space E . Then, A is the generator of a (p, q) -regularized operator family if and only if there is a strong continuous operator $S_p: R_+ \rightarrow L(E)$ and constant $\alpha \geq 0$ satisfying $\{\eta^p(\eta + q): \eta > \delta\} \subset \rho(A)$ and

$$H(\eta)x := \eta^p(\eta^p(\eta + q)I - A)^{-1}x = \int_0^\infty e^{-\eta t} S_p(t)x dt, \quad \eta > \delta, x \in E. \quad (11)$$

In view of Lemma 2, from (9) and (11), we have

$$\eta^{p-1}(\eta^p(\eta + q)I - A)^{-1}x = \int_0^\infty e^{-\eta t} P_p(t)x dt, \quad \eta > \delta, x \in E. \quad (12)$$

Combining (9) and (10), we obtain

$$(\eta^p(\eta + q)I - A)^{-1}x = \int_0^\infty e^{-\eta t} Q_p(t)x dt, \quad \eta > \delta, x \in E. \quad (13)$$

At first, we put forward a new form of the mild solution for the system in the following.

Lemma 3. Let $0 \leq p \leq 1$ and $g \in L^1(J, E)$; if x satisfies the equation

$$\begin{cases} {}^C D_t^p x'(t) + q {}^C D_t^p [x(t) - h(t, x_t)] = Ax(t) + g(t), & t \in (0, b], \\ x(t) = \varphi(t), & t \in [-\tau, 0], \\ x'(0) = x_1, \end{cases} \quad (14)$$

then x is denoted by the integral equation

$$\begin{aligned} x(t) = & S_p(t)\varphi(0) + P_p(t)[x_1 + q\varphi(0) - qh(0, x_0)] \\ & + q \int_0^t S_p(t-\nu)h(\nu, x_\nu) d\nu + \int_0^t Q_p(t-\nu)g(\nu) d\nu. \end{aligned} \quad (15)$$

Proof. Taking the Riemann–Liouville integral to both sides of the first equality of (14), we have

$$I_t^p D_t^p x'(t) + q I_t^p D_t^p [x(t) - h(t, x_t)] = I_t^p [Ax(t) + g(t)]. \quad (16)$$

From Lemma 1, we can get that

$$x'(t) - x'(0) + q[x(t) - h(t, x_t) - x(0) + h(0, x_0)] = I_t^p [Ax(t) + g(t)]. \quad (17)$$

That is,

$$\begin{aligned} x'(t) + qx(t) = & x'(0) + q\varphi(0) - qh(0, x_0) + qh(t, x_t) \\ & + I_t^p [Ax(t) + g(t)]. \end{aligned} \quad (18)$$

Let $\eta > 0$. Taking the Laplace transform for (18), we yield

$$\begin{aligned} -\varphi(0) + \eta X(\eta) + qX(\eta) = & \frac{1}{\eta} [x_1 + q\varphi(0) - qh(0, x_0)] \\ & + qH(\eta) + \frac{1}{\eta^p} [AX(\eta) + G(\eta)]. \end{aligned} \quad (19)$$

Hence, we obtain

$$\begin{aligned} X(\eta) = & \eta^p [\eta^p(\eta + q)I - A]^{-1} [\varphi(0) + \eta^{p-1} [\eta^p(\eta + q)I - A]^{-1} [x_1 + q\varphi(0) - qh(0, x_0)]] \\ & + \eta^p [\eta^p(\eta + q)I - A]^{-1} qH(\eta) + [\eta^p(\eta + q)I - A]^{-1} G(\eta). \end{aligned} \quad (20)$$

Then, applying Laplace inverse transform on (20) and combining with the result of Lemma 2 and the Laplace transform of the convolution, we achieve

$$\begin{aligned} x(t) = & S_p(t)\varphi(0) + P_p(t)[x_1 + q\varphi(0) - qh(0, x_0)] \\ & + q \int_0^t S_p(t-\nu)h(\nu, x_\nu) d\nu + \int_0^t Q_p(t-\nu)g(\nu) d\nu. \end{aligned} \quad (21)$$

This completes the proof.

Because of Lemma 3, we naturally deduce a new representation of the mild solution for (14). $\square$

Definition 4. A function $x(\cdot) \in C([- \tau, b], E)$ is reputedly a mild solution for the fractional neutral linear system with damping (14) if there exists $g \in L^1(J, E)$ satisfying

$$\begin{aligned} x(t) = & S_p(t)\varphi(0) + P_p(t)[x_1 + q\varphi(0) - qh(0, x_0)] \\ & + q \int_0^t S_p(t-\nu)h(\nu, x_\nu) d\nu + \int_0^t Q_p(t-\nu)g(\nu) d\nu, \quad t \in (0, b]. \end{aligned} \quad (22)$$

To gain the global existence of the mild solution for FNDED (1), we give hypothesis S(0).

S(0): A is the infinitesimal generator of an exponentially bounded (p, q) -regularized operator family $S_p(t)$ on E ,

namely, there exist two real numbers $M \geq 1$ and $\omega \geq 0$ satisfying

$$\|S_p(t)\| \leq Me^{\omega t}, \quad t \geq 0. \quad (23)$$

Remark 2. From hypothesis S(0) and formulas (9) and (10), we have

- (i) $\|S_p(t)\| \leq Me^{\omega t} \triangleq M_S, t \in [0, b]$.
- (ii) $\|P_p(t)\| = \left\| \int_0^t S_p(s) ds \right\| \leq Me^{\omega b} b \triangleq M_P, t \in [0, b]$.
- (iii) $\|Q_p(t)\| = \left\| \int_0^t g_p(t-s) S_p(s) ds \right\| \leq Me^{\omega b} b^p / (\Gamma(p+1)) \triangleq M_Q, t \in [0, b]$.

3. Existence and Uniqueness of Mild Solutions

In this segment, firstly, we consider the existence and uniqueness of mild solutions for FNDED (1). To this end, we impose the following conditions.

S(1): $h: [0, b] \times E \longrightarrow E$ is continuous and satisfies

$$\|h(t, x_t) - h(t, y_t)\|_E \leq n_h(\cdot) |x_t - y_t|, \quad (24)$$

for every $t \in [0, b]$, and $x_t, y_t \in L^2([-\tau, 0]; E)$, $n_h(\cdot) \in L_1(J, R^+)$, and $n_h(\cdot) \leq N_h \in R^+$.

S(2): for almost every $t \in [0, b]$, the nonlinear item $f(t, \cdot): E \longrightarrow E$ is continuous, and there is $n_f(\cdot) \in L_1(J, R^+)$ satisfying

$$\|f(t, x_t) - f(t, y_t)\|_E \leq n_f(\cdot) |x_t - y_t|, \quad (25)$$

for all $x_t, y_t \in L^2([-\tau, 0]; E)$ and $n_f(t) \leq N_f \in R^+$.

Theorem 1. Under conditions S(0)–S(2), the fractional neutral differential equations with damping (1) has one and only one mild solution belonging to the space $C([-\tau, b], E)$ with $x(t) = \varphi(t)$, $t \in [-\tau, b]$, and $x'(t) = x_1$ for every control item $u(\cdot) \in L_2([0, b]; U)$, provided that

$$b(b+\tau)(qM_S N_h + M_Q N_f) < 1. \quad (26)$$

Proof. Define the mapping $\Omega: C([-\tau, b], E) \longrightarrow C([-\tau, b], E)$ by

$$\left\{ \begin{array}{l} (\Omega x)(t) = S_p(t)\varphi(0) + P_p(t)[x_1 + q\varphi(0) - qh(0, x_0)] + q \int_0^t S_p(t-\nu)h(\nu, x_\nu) d\nu \\ + \int_0^t Q_p(t-\nu)[Bu(\nu) + f(\nu, x_\nu)] d\nu, \quad t \in [0, b], \\ x(t) = \varphi(t), \quad t \in [-\tau, 0], \\ x'(0) = x_1. \end{array} \right. \quad (27)$$

Let $M_h = \max_{0 \leq t \leq b} \|h(t, 0)\|_E$, $M_f = \max_{0 \leq t \leq b} \|f(t, 0)\|_E$, $\|B\|_E \leq M_B$, $r = [1 - (qM_S N_h + M_Q N_f)b(b+\tau)]^{-1} [(M_S + qM_P + qM_P N_h)\|\varphi\| + M_P \|x_1\| + q(M_S b + M_P)M_h + bM_Q M_f + b^{(1/2)}M_Q M_B \|u\|_{L_2}]$, and $B_r = \{x(\cdot) \in C([-\tau, b]; E): |x| \leq r, x(t) = \varphi(t), t \in [-\tau, 0]\}$, which is a closed and

bounded subset of $C([-\tau, b]; E)$. For each $y \in B_r$, we prove Ω has a fixed point in $C([-\tau, b]; E)$. Firstly, we prove that Ω maps B_r into itself. Obviously, $\Omega x_0 \in B_r$. For $t \in (0, b]$, we achieve

$$\begin{aligned} \|(\Omega x)(t)\| &\leq \|S_p(t)\varphi\| + \|P_p(t)[x_1 + q\varphi - qh(0, x_0)]\| + q \int_0^t \|S_p(t-\nu)h(\nu, x_\nu)\| d\nu \\ &\quad + \int_0^t \|Q_p(t-\nu)[Bu(\nu) + f(\nu, x_\nu)]\| d\nu \\ &\leq M_S \|\varphi\| + M_P [\|x_1\| + q\|\varphi\| + q\|h(0, x_0) - h(0, 0) + h(0, 0)\|] \\ &\quad + qM_S \int_0^t \|h(\nu, x_\nu) - h(\nu, 0) + h(\nu, 0)\| d\nu \\ &\quad + M_Q \int_0^t \|Bu(\nu)\| d\nu + M_Q \int_0^t \|f(\nu, x_\nu) - f(\nu, 0) + f(\nu, 0)\| d\nu \\ &\leq (M_S + qM_P + qM_P N_h)\|\varphi\| + M_P \|x_1\| + q(M_S b + M_P)M_h \\ &\quad + bM_Q M_f + b^{(1/2)}M_Q M_B \|u\|_{L_2} + (qM_S N_h + M_Q N_f)b(b+\tau)r \\ &= r. \end{aligned} \quad (28)$$

This is because

$$\begin{aligned}
 \int_0^t \|x_\nu\| d\nu &\leq \int_0^b \int_{-\tau}^0 \|x(\nu + \vartheta)\| d\vartheta d\nu \\
 &= \int_0^b \int_{\nu-\tau}^\nu \|x(\omega)\| d\omega d\nu \\
 &\leq b(b+\tau) \sup_{t \in [-\tau, b]} \|x(\omega)\| \\
 &= b(b+\tau)|x|,
 \end{aligned} \tag{29}$$

which means $\|(\Omega x)(t)\| \leq r$, $\Omega x \subseteq B_r$.

Next, we demonstrate that Ω is a contraction mapping on B_r . In fact, for $x(\cdot), y(\cdot) \in B_r$,

$$\begin{aligned}
 \|(\Omega x)(t) - (\Omega y)(t)\| &\leq q \int_0^t \|S_p(t-\nu)\| \|h(\nu, x_\nu) - h(\nu, y_\nu)\| d\nu \\
 &\quad + \int_0^t \|Q_p(t-s)\| \|f(s, x_s) - f(s, y_s)\| ds \\
 &\leq qM_S \int_0^t n_h(\nu) \|x_\nu - y_\nu\| d\nu + M_Q \int_0^t n_f(\nu) \|x_\nu - y_\nu\| d\nu \\
 &\leq b(b+\tau)(qM_S N_h + M_Q N_f) |x - y|.
 \end{aligned} \tag{30}$$

Since

$$\begin{aligned}
 \int_0^t \|x_\nu - y_\nu\| d\nu &\leq \int_0^b \int_{-\tau}^0 \|x(\nu + \vartheta) - y(\nu + \vartheta)\| d\vartheta d\nu \\
 &= \int_0^b \int_{\nu-\tau}^\nu \|x(\omega) - y(\omega)\| d\omega d\nu \\
 &\leq \int_0^b \int_{-\tau}^b \|x(\omega) - y(\omega)\| d\omega d\nu \\
 &\leq b(b+\tau) \sup_{t \in [-\tau, b]} \|x(\omega) - y(\omega)\| \\
 &= b(b+\tau)|x - y|,
 \end{aligned} \tag{31}$$

due to $b(b+\tau)(qM_S N_h + M_Q N_f) < 1$, Ω is a contraction mapping. Consequently, Ω has the unique fixed point belonging to the space $C([-\tau, b]; E)$. (27) is the unique mild solution of FNDED (1). $\square$

4. Approximate Controllability

In this segment, we acquire several appropriate sufficient conditions of the approximate controllability for FNDED (1) by virtue of the approximate technique and the iterative approach.

Define two continuous linear operators $\mathcal{L}$ and $\mathcal{J}$ from $L_2([0, b], E)$ to E by

$$\begin{aligned}
 \mathcal{L}v &= \int_0^b P_p(b-\nu)v(\nu)d\nu, \quad v(\cdot) \in L_2([0, b], E), \\
 \mathcal{J}v &= q \int_0^b S_p(b-\nu)v(\nu)d\nu, \quad v(\cdot) \in L_2([0, b], E).
 \end{aligned} \tag{32}$$

If we let the combination (x, u) be the mild solution for FNDED (1) with $u \in L_2([0, b], U)$, then we represent it as $x(t) = x(t; x_0, x_1, u)$, and the terminal item $x(b)$ can be written as

$$\begin{aligned}
 x(b) &= x(b; x_0, x_1, u) = S_p(b)\varphi(0) + P_p(b)[x_1 + q\varphi(0) - qh(0, x_0)] \\
 &\quad + q \int_0^b S_p(b-\nu) \cdot h(\nu, x_\nu(Bu)) d\nu + \int_0^b Q_p(b-\nu)[Bu(\nu) + f(\nu, x_\nu(Bu))] d\nu \\
 &\triangleq H(b, \varphi, x_1) + \mathcal{J}h(\nu, x_\nu(Bu)) + \mathcal{L}f(\nu, x_\nu(Bu)) + \mathcal{L}Bu(\nu),
 \end{aligned} \tag{33}$$

where $H(b; \varphi, x_1) = S_p(b)\varphi(0) + P_p(b)[x_1 + q\varphi(0) - qh(0, x_0)]$.

So, the reachable set $K_b(f)$ which is composed of all possible final states at time b is

$$K_b(f) = \{x(b; x_0, x_1, u); u \in U\}. \quad (34)$$

Thus, the approximate controllability for FNDED (1) means the set $K_b(f)$ is dense on space E . That is to say, the definition of approximate controllability is acquired.

Definition 5. Let $x_0, x_1 \in E$. We called the fractional neutral differential equations with damping (1) is approximately controllable on $[0, b]$ if, for any $\varepsilon > 0$ and $x_b \in E$, there is a control function $u_{N_\varepsilon} \in U$ satisfying

$$\|x_b - H(b; \varphi, x_1) - \mathcal{H}h(\nu, x_\nu(Bu_{N_\varepsilon})) - \mathcal{L}f(\nu, x_\nu(Bu_{N_\varepsilon})) - \mathcal{L}Bu_{N_\varepsilon}\| < \varepsilon. \quad (35)$$

And we add some of the postulated conditions.

S(3): for each given $\varepsilon > 0$ and $\nu(\cdot), w(\cdot) \in L_2([0, b]; E)$, there is $u(\cdot) \in U$ satisfying

$$\|\mathcal{F}w - \mathcal{L}v - \mathcal{L}Bu\|_E < \varepsilon, \quad (36)$$

where $\|Bu(\cdot)\|_{L_2(J, U)} \leq M_u \|\nu(\cdot)\|_{L_2(J, U)}$, and M_u is a positive real number irrelevant of $\nu(\cdot)$ and holds

$$(qM_S N_h + M_Q N_f + M_u M_Q N_f)b(b + \tau) < 1. \quad (37)$$

Because condition (37) implies (26), the existence is still met when inequality (26) is changed into (37) in Theorem 1. Next, to demonstrate the conclusion of the approximate controllability remains true for (1), we give the following two lemmas.

Lemma 4. Let $(x_1, u_1), (x_2, u_2)$ be two pairs relevant to FNDED (1). Then, in terms of conditions S(0)–S(3), the following result holds:

$$|x_t(Bu_1) - x_t(Bu_2)| \leq \frac{M_Q b^{(1/2)}}{1 - (qM_S N_h + M_Q N_f)b(b + \tau)} \|Bu_1 - Bu_2\|_{L_2}. \quad (38)$$

Proof. The mild solution $x(t) = x(t; x_0, x_1, u)$ of FNDED (1) in E satisfies

$$x_t(Bu) = \begin{cases} \varphi(t), & t \in [-\tau, 0], \\ S_p(t)\varphi(0) + P_p(t)[x_1 + q\varphi(0) - qh(0, x_0)] \\ + q \int_0^t S_p(t-\nu)h(\nu, x_\nu) d\nu + \int_0^t Q_p(t-\nu)g(\nu) d\nu, & t \in [0, b]. \end{cases} \quad (39)$$

We define $y(\cdot, \varphi): [-\tau, b] \rightarrow E$ as

$$y(t, \varphi) = \begin{cases} \varphi(t), & t \in [-\tau, 0], \\ S_p(t)\varphi(0) + P_p(t)[x_1 + q\varphi(0) - qh(0, x_0)], & t \in [0, b]. \end{cases} \quad (40)$$

Let $x_t = y_t + z_t$, $t \in [-\tau, b]$. Apparently, $x(\cdot)$ meets (1) if and only if $z_0 = 0$ and $t \in [-\tau, 0]$, and for each $t \in [0, b]$, we acquire

$$\begin{aligned} z(t) &= q \int_0^t S_p(t-\nu)h(\nu, x_\nu) d\nu + \int_0^t Q_p(t-\nu)[Bu(\nu) + f(\nu, x_\nu)] d\nu \\ &= q \int_0^t S_p(t-\nu)h(\nu, z_\nu(Bu) + y_\nu) d\nu + \int_0^t Q_p(t-\nu)[Bu(\nu) + f(\nu, z_\nu(Bu) + y_\nu)] d\nu. \end{aligned} \quad (41)$$

Now, let us take $x_t(Bu_1), x_t(Bu_2) \in E$ and $u_1(\cdot), u_2(\cdot) \in U$; then,

$$\begin{aligned} &|x_t(Bu_1) - x_t(Bu_2)| \\ &= |[z_t(Bu_1) + y_t] - [z_t(Bu_2) + y_t]| \\ &= |z_t(Bu_1) - z_t(Bu_2)| \\ &\leq q \int_0^t \|S_p(t-\nu)\| \cdot \|h(\nu, x_\nu(Bu_1)) - h(\nu, x_\nu(Bu_2))\| d\nu \\ &\quad + \int_0^t \|Q_p(t-\nu)\| \cdot \|Bu_1(\nu) - Bu_2(\nu)\| d\nu \\ &\quad + \int_0^t \|Q_p(t-\nu)\| \cdot \|f(\nu, x_\nu(Bu_1)) - f(\nu, x_\nu(Bu_2))\| d\nu \\ &\leq qM_S N_h \int_0^t \|x_\nu(Bu_1) - x_\nu(Bu_2)\| d\nu + b^{(1/2)} M_Q \|Bu_1(\nu) - Bu_2(\nu)\|_{L_2} \\ &\quad + M_Q N_f \int_0^t \|x_\nu(Bu_1) - x_\nu(Bu_2)\| d\nu \\ &= M_Q b^{(1/2)} \|Bu_1(\nu) - Bu_2(\nu)\|_{L_2} + (qM_S N_h + M_Q N_f) \int_0^t \|z_\nu(Bu_1) - z_\nu(Bu_2)\| d\nu \\ &\leq M_Q b^{(1/2)} \|Bu_1(\nu) - Bu_2(\nu)\|_{L_2} + (qM_S N_h + M_Q N_f)b(b + \tau) \|z_\nu(Bu_1) - z_\nu(Bu_2)\|. \end{aligned} \quad (42)$$

Recombining the formula, we get

$$\begin{aligned} & |x_t(Bu_1) - x_t(Bu_2)| \\ &= |z_t(Bu_1) - z_t(Bu_2)| \\ &\leq \frac{M_Q b^{(1/2)}}{1 - (qM_S N_h + M_Q N_f)b(b + \tau)} \|Bu_1 - Bu_2\|_{L_2}. \end{aligned} \quad (43)$$

This completes the proof. $\square$

Theorem 2. *If hypotheses S(0)–S(3) hold, the fractional neutral differential equations with damping (1) is approximately controllable on J .*

Proof. A is the infinitesimal generator of a (p, q) -regularized operator family in a Banach space, and its domain is dense in Banach space E . So, in accordance with the definition of approximate controllability, it is only needed to demonstrate that $D(A) \subset K_b(\varphi, x_1)$. Next, we should prove that, for every $\varepsilon > 0$ and $x_b \in D(A)$, there is $u_{N_\varepsilon} \in U$ with

$$\|x_b - H(b, \varphi, x_1) - \mathcal{J}h(\nu, x_{\nu}(Bu_{N_\varepsilon}(\nu))) - \mathcal{L}f(\nu, x_{\nu}(Bu_{N_\varepsilon}(\nu))) - \mathcal{L}Bu_{N_\varepsilon}(\nu)\| < \varepsilon, \quad (44)$$

where $x_\nu(t) = x(t; \varphi, x_1, u_{N_\varepsilon})$ satisfies

$$\begin{aligned} x_\nu(t) &= H(t; \varphi, x_1) + q \int_0^t S_p(t - \nu) h(\nu, x_\nu(Bu_{N_\varepsilon}(\nu))) d\nu \\ &\quad + \int_0^t Q_p(t - \nu) [Bu_{N_\varepsilon}(\nu) + f(\nu, x_\nu(Bu(\nu)))] d\nu, \quad t \in [0, b]. \end{aligned} \quad (45)$$

By the theory of the fractional resolvent, we deduce that $H(b; \varphi, x_1) \in D(A)$ for $\varphi(t), x_1 \in E$, which implies that $x_b - H(b; \varphi, x_1) \in D(A)$ for $x_b \in D(A)$. It is evident that there is one $\nu \in L_1([0, b], E)$ such that

$$\mathcal{L}\nu = x_b - H(b; \varphi, x_1). \quad (46)$$

Condition (S3) suggests for each $\varepsilon > 0$ and $u_1 \in U$, there is $u_2 \in U$ such that

$$\|x_b - H(b; \varphi, x_1) - \mathcal{J}h(\nu, x_\nu(Bu_1)) - \mathcal{L}f(\nu, x_\nu(Bu_1)) - \mathcal{L}Bu_2\| < \frac{\varepsilon}{3^2}. \quad (47)$$

Furthermore, for $u_2 \in U$, we determine $w_2 \in U$ again by condition S(3) and (36) with the following two properties:

$$\begin{aligned} & \|\mathcal{J}[h(\nu, x_\nu(Bu_2)) - h(\nu, x_\nu(Bu_1))] + \mathcal{L}[f(\nu, x_\nu(Bu_2)) - f(\nu, x_\nu(Bu_1))] - \mathcal{L}Bw_2\| < \frac{\varepsilon}{3^3}, \\ & \|Bw_2\|_{L_2} \leq M_u \|f(\nu, x_\nu(Bu_2)) - f(\nu, x_\nu(Bu_1))\|_{L_2} \\ & \leq M_u N_f \left(\int_0^b \|x_\nu(Bu_2) - x_\nu(Bu_1)\|_{L_1}^2 d\nu \right)^{(1/2)} \\ & \leq M_u N_f b^{(1/2)} (b + \tau) |x_\nu(Bu_2) - x_\nu(Bu_1)| \\ & \leq \frac{M_Q M_u N_f b(b + \tau)}{1 - (qM_S N_h + M_Q N_f)b(b + \tau)} \|Bu_1 - Bu_2\|_{L_2}, \end{aligned} \quad (48)$$

where $x_2(t) = x(t; x_0, x_1, u_2)$, $t \in (0, b]$.

Thus, we may define $u_3 = u_2 - w_2$ in U and derive the following inequality:

$$\begin{aligned} & \|x(b) - H(b; \varphi, x_1) - \mathcal{J}h(\nu, x_\nu(Bu_2)) - \mathcal{L}f(\nu, x_\nu(Bu_2)) - \mathcal{L}Bu_3\| \\ & \leq \|x(b) - H(b; \varphi, x_1) - \mathcal{J}h(\nu, x_\nu(Bu_1)) - \mathcal{L}f(\nu, x_\nu(Bu_1)) - \mathcal{L}Bu_2\| \\ & \quad + \|\mathcal{L}Bw_2 - \mathcal{J}[h(\nu, x_\nu(Bu_2)) - h(\nu, x_\nu(Bu_1))] - \mathcal{L}[f(\nu, x_\nu(Bu_2)) - f(\nu, x_\nu(Bu_1))]\| \\ & \leq \frac{\varepsilon}{3^2} + \frac{\varepsilon}{3}. \end{aligned} \quad (49)$$

By the same way, we obtain the sequence $\{u_n: n \geq 1\} \subset U$ satisfying

$$\begin{aligned} & \|x(b) - H(b; \varphi, x_1) - \mathcal{J}h(\nu, x_\nu(Bu_n)) - \mathcal{L}f(\nu, x_\nu(Bu_n)) - \mathcal{L}Bu_{n+1}\| \\ & \leq \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}} \right) \varepsilon, \\ & \|Bu_{n+1} - Bu_n\|_{L_2} \leq \frac{M_Q M_u N_f b(b + \tau)}{1 - (q M_S N_h + M_Q N_f) b(b + \tau)} \|Bu_n - Bu_{n-1}\|_{L_2}, \end{aligned} \quad (50)$$

where $x_n(t) = x(t; x_0, x_1, u_n)$, $t \in (0, b]$ and $n = 1, 2, \dots$. Because hypothesis S(3) holds, we can acquire that the sequence $\{Bu_n: n \geq 1\}$ is a Cauchy sequence in $L_2([0, b], E)$, and in this way, we can gain a function item $\nu \in L_2([0, b], E)$ satisfying

$$\lim_{n \rightarrow \infty} Bu_n = \nu. \quad (51)$$

On account of the mapping $\mathcal{L}: L_2([0, b], E) \rightarrow E$ is linear-continuous, for any given $\varepsilon > 0$, we can find a real integer number N_ε satisfying

$$\|L(Bu_{N_\varepsilon+1}) - L(Bu_{N_\varepsilon})\| < \frac{\varepsilon}{3}. \quad (52)$$

Consequently, by inequalities (50) and (52), we derive

$$\begin{aligned} & \|x(b) - H(b; \varphi, x_1) - \mathcal{J}h(\nu, x_\nu(Bu_{N_\varepsilon})) - \mathcal{L}f(\nu, x_\nu(Bu_{N_\varepsilon})) - \mathcal{L}Bu_{N_\varepsilon}\| \\ & \leq \|x(b) - H(b; \varphi, x_1) - \mathcal{J}h(\nu, x_\nu(Bu_{N_\varepsilon})) - \mathcal{L}f(\nu, x_\nu(Bu_{N_\varepsilon})) - \mathcal{L}Bu_{N_\varepsilon+1}\| \\ & \quad + \|\mathcal{L}Bu_{N_\varepsilon+1} - \mathcal{L}Bu_{N_\varepsilon}\| \\ & \leq \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}} \right) \varepsilon + \frac{\varepsilon}{3} \leq \varepsilon, \end{aligned} \quad (53)$$

where $x_{n_\varepsilon}(t) = x(t; x_0, x_1, u_{n_\varepsilon})$, $t \in (0, b]$.

This means that $\xi \in \overline{K_b(f)}$. Thus, $\overline{K_b(0)} \subset K_b(f)$; therefore, the fractional neutral differential equations with damping (1) is approximately controllable on $[0, b]$. $\square$

Theorem 3. Postulate the range of operator B is denoted by $R(B)$ and it is dense in $L_2([0, b]; E)$. If conditions S(0)–S(3) are true, the fractional neutral differential equations with damping (1) is approximately controllable on $[0, b]$.

Proof. Because $R(B)$ is dense in $L_2([0, b]; E)$, for every function $j(\cdot) \in L_2([0, b]; E)$ and $\iota > 0$, there is $Bu(\cdot) \in R(B)$, where $u(\cdot) \in U$, satisfying

$$\|Bu(\cdot) - j(\cdot)\|_{L_2([0, b], E)} < \iota \|j(\cdot)\|_{L_2([0, b], E)}. \quad (54)$$

Now, we have

$$\begin{aligned} \|\mathcal{L}p - \mathcal{L}Bu\| & \leq M_Q \int_0^b \|j(s) - Bu(s)\| ds \\ & \leq M_Q b^{(1/2)} \|j(s) - Bu(s)\|_{L_2([0, b], E)} \\ & \leq M_Q b^{(1/2)} \iota \|j(\cdot)\|_{L_2([0, b], E)} \\ & < \varepsilon. \end{aligned} \quad (55)$$

Thus, from (50), we have

$$\begin{aligned} \|Bu(\cdot)\|_{L_2([0, b], E)} & = \|Bu(\cdot) - j(\cdot) + j(\cdot)\|_{L_2([0, b], E)} \\ & \leq \|Bu(\cdot) - j(\cdot)\|_{L_2([0, b], E)} + \|j(\cdot)\|_{L_2([0, b], E)} \\ & \leq \iota \|j(\cdot)\|_{L_2([0, b], E)} + \|j(\cdot)\|_{L_2([0, b], E)} \\ & = (\iota + 1) \|j(\cdot)\|_{L_2([0, b], E)}. \end{aligned} \quad (56)$$

It indicates that condition S(3) holds if we select $\iota > 0$. Therefore, FNDED (1) is approximately controllable on $[0, b]$ by using Theorem 2. $\square$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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