Research Article

Fixed-Time Synchronization of Neural Networks with Discrete Delay

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In this paper, we establish a new fixed-time stability theorem, which provides a novel fixed-time stability criterion and a novel upper bound estimate formula for the settling time. Numerical simulations show that the upper bound estimate for the settling time in this paper is tighter than those given in the existing fixed-time stability theorems. By designing a simple feedback controller, the fixed-time synchronization of neural networks with discrete delay is investigated based on the fixed-time stability theorem established in this paper. A numerical example is included to validate the effectiveness of the obtained theoretical results.

1. Introduction

Neural networks can be described by differential equations [1], and they usually present complex dynamic behaviors [2]. The stability and synchronization of neural networks can be applied in many fields, such as associative memory [3], secure communication [4], and image encryption [5]. In recent years, considerable results have been reported about the stability and synchronization of neural networks [6–12].

Different from infinite-time synchronization (including asymptotic synchronization and exponential synchronization), finite-time synchronization [13] requires that the trajectories of the error system tend to the origin after some finite time, which is called the settling time. Considering the control cost, finite-time synchronization is more feasible and valuable than infinite-time synchronization in the engineering fields. Recently, the finite-time synchronization of neural networks has been intensively studied [14–22]. In [14], the finite-time synchronization of neural networks with mixed delays and perturbations was investigated. In [19], the

finite-time synchronization of Cohen–Grossberg neural networks was studied via delayed feedback control.

The drawback of finite-time synchronization is that the settling time varies with the initial conditions of the drive-response systems. Since different initial conditions lead to different settling times, the corresponding settling times will be confusing when we consider many different initial conditions. Moreover, if the initial conditions of the studied dynamical systems are unknown beforehand, it is even impossible to estimate the settling times. Therefore, it will be desirable that finite-time synchronization can be achieved in a fixed time interval, irrespective of the initial conditions of the studied systems.

In 2012, a new concept named “fixed-time stability” was introduced by Polyakov [23]. Fixed-time stability is a kind of special finite-time stability actually, and its settling time is bounded by a fixed constant for any initial conditions of the studied systems. As we know, the synchronization problem of the drive-response systems can be converted into the stability problem of the error system, so the related fixed-time stability

theorems [23, 24] can be used to investigate the fixed-time synchronization of nonlinear systems [25–28]. The fixed-time synchronization of nonlinear systems has important applications in secure communication and image encryption. Take secure communication for example: when can the ciphertext signals be decrypted correctly without considering the specific initial values of the drive-response systems? The solution to this problem must be attributed to the fixed-time synchronization control of the drive-response systems. That is to say, for the drive-response systems, the upper bound estimate for the settling time of the fixed-time synchronization can be used to adjust the start time of correct decryption.

As typical nonlinear systems, neural networks usually present some strange dynamic behaviors, so the fixed-time synchronization of neural networks can also be applied in many fields, such as secure communication and image encryption. Based on the fixed-time stability theorems in [23, 24, 29], there have been some publications about the fixed-time synchronization of neural networks [29–34]. It should be pointed out that the upper bound estimate formulas for the settling time given in [23, 24, 29] are very inaccurate. Owing to the limitations of theoretical analysis, it is difficult to obtain the least upper bound estimate for the settling time. However, from an application point of view, it is important to derive a tighter upper bound estimate for the settling time.

Inspired by the aforementioned discussion, in this paper, we establish a new fixed-time stability theorem, which provides a novel fixed-time stability criterion and a novel upper bound estimate formula for the settling time. Numerical simulations show that the upper bound estimate for the settling time in this paper is tighter than those given in [23, 24, 29]. By utilizing the established fixed-time stability theorem, the fixed-time synchronization of neural networks with discrete delay is studied. A numerical example is included to show the effectiveness of the obtained theoretical results.

2. Preliminaries

Consider the following neural network with discrete delay:

$$\dot{\xi}_i(t) = -q_i \xi_i(t) + \sum_{j=1}^n a_{ij} f_j(\xi_j(t)) + \sum_{j=1}^n b_{ij} f_j(\xi_j(t - \tau(t))) + I_i, \quad (1)$$

$i = 1, 2, \dots, n$, where $\xi_i(t)$ expresses the state of neuron; $q_i > 0$ denotes the rate of neuron self-inhibition; a_{ij} and b_{ij} denote connection weights; $f_j(\cdot)$ represents the activation function; discrete delay $\tau(t)$ satisfies $0 \leq \tau(t) \leq \tau$, where τ is a positive constant; and I_i stands for the external input. The initial condition of system (1) is assumed to be $\xi(\nu) = \varphi(\nu) \in C([- \tau, 0], R^n)$, where $\xi(\nu) = (\xi_1(\nu), \xi_2(\nu), \dots, \xi_n(\nu))^T$.

In this paper, system (1) is the drive system, and the corresponding response system is

$$\begin{aligned} \dot{\eta}_i(t) = & -q_i \eta_i(t) + \sum_{j=1}^n a_{ij} f_j(\eta_j(t)) \\ & + \sum_{j=1}^n b_{ij} f_j(\eta_j(t - \tau(t))) + I_i + u_i(t), \end{aligned} \quad (2)$$

$i = 1, 2, \dots, n$, where $u_i(t)$ stands for the controller. Let $\eta(t) = ((\eta_1(t), \eta_2(t), \dots, \eta_n(t))^T$; the initial condition of MNN (2) is $\eta(\nu) = \phi(\nu) \in C([- \tau, 0], R^n)$.

The synchronization errors between systems (1) and (2) are defined by $e_i(t) = \eta_i(t) - \xi_i(t)$, $i = 1, 2, \dots, n$; then, the error system should be

$$\dot{e}_i(t) = -q_i e_i(t) + F_i(t) + u_i(t), \quad (3)$$

$i = 1, 2, \dots, n$, where

$$\begin{aligned} F_i(t) = & \sum_{j=1}^n a_{ij} f_j(\eta_j(t)) - \sum_{j=1}^n a_{ij} f_j(\xi_j(t)) \\ & + \sum_{j=1}^n b_{ij} f_j(\eta_j(t - \tau(t))) - \sum_{j=1}^n b_{ij} f_j(\xi_j(t - \tau(t))). \end{aligned} \quad (4)$$

Let $e(t) = (e_1(t), e_2(t), \dots, e_n(t))^T$; then, the initial condition of system (3) should be $e(\nu) = \phi(\nu) - \varphi(\nu)$, $\nu \in [- \tau, 0]$.

Throughout this paper, the following assumptions will be needed:

A1. For $\forall \mu, \nu \in R$, there exist constants $l_j > 0$ such that

$$|f_j(\mu) - f_j(\nu)| \leq l_j |\mu - \nu|, \quad j = 1, 2, \dots, n. \quad (5)$$

A2. For $\forall j \in \{1, 2, \dots, n\}$, activation function $f_j(\cdot)$ satisfies $|f_j(\cdot)| \leq M_j$, where $M_j > 0$ is a constant.

Lemma 1. $|F_i(t)| \leq \sum_{j=1}^n |a_{ij}| \cdot l_j |e_j(t)| + 2 \sum_{j=1}^n |b_{ij}| M_j$, $i = 1, 2, \dots, n$.

Proof. Based on assumptions **A1** and **A2**, the proof is obvious. $\square$

Lemma 2 (see [35]). If $y_1, y_2, \dots, y_n \geq 0, 0 < \rho \leq 1, \sigma > 1$, then we have

$$\begin{aligned} \sum_{i=1}^n y_i^\rho & \geq \left(\sum_{i=1}^n y_i \right)^\rho, \\ \sum_{i=1}^n y_i^\sigma & \geq n^{1-\sigma} \left(\sum_{i=1}^n y_i \right)^\sigma. \end{aligned} \quad (6)$$

Definition 1. The origin of system (3) can achieve finite-time stability, if there is a constant $T(e(0)) > 0$ such that $\lim_{t \rightarrow T} (e(0)) \|e(t)\| = 0$ and $\|e(t)\| \equiv 0$ for $\forall t > T(e(0))$, where $T(e(0))$ is called the settling time.

Definition 2. The origin of system (3) can achieve fixed-time stability, if two conditions can be satisfied: (a) the origin of system (3) can achieve finite-time stability; (b) for any $e(0)$, there is a fixed constant $T_{\max} > 0$ such that $T(e(0)) \leq T_{\max}$.

Lemma 3 (see [23]). Suppose that $V(\cdot): R^n \longrightarrow R_+ \cup \{0\}$ is a continuous radically unbounded function and satisfies the following:

- (1) $V(z) = 0 \iff z = 0$
- (2) $\dot{V}(e(t)) \leq -\alpha V^\rho(e(t)) - \beta V^\sigma(e(t))$ for any $e(t)$, where $\alpha, \beta > 0, 0 < \rho < 1$ and $\sigma > 1$

Then, the origin of system (3) can achieve fixed-time stability, and

$$T_{\max}^1 = \frac{1}{\alpha(1-\rho)} + \frac{1}{\beta(\sigma-1)}. \quad (7)$$

Lemma 4 (see [29]). Suppose that $V(\cdot): R^n \longrightarrow R_+ \cup \{0\}$ is a continuous radically unbounded function and satisfies the following:

- (1) $V(z) = 0 \iff z = 0$
- (2) $\dot{V}(e(t)) \leq -\alpha V^\rho(e(t)) - \beta V^\sigma(e(t))$ for any $e(t)$, where $\alpha, \beta > 0, 0 < \rho < 1$ and $\sigma > 1$

Then, the origin of system (3) can achieve fixed-time stability, and

$$T_{\max}^2 = \frac{1}{\alpha} \cdot \left(\frac{\alpha}{\beta}\right)^{(1-\rho)/(\sigma-\rho)} \left(\frac{1}{1-\rho} + \frac{1}{\sigma-1}\right). \quad (8)$$

Lemma 5 (see [24]). Suppose that $V(\cdot): R^n \longrightarrow R_+ \cup \{0\}$ is a continuous radically unbounded function and satisfies the following

- (1) $V(z) = 0 \iff z = 0$
- (2) $\dot{V}(e(t)) \leq -\alpha V^\rho(e(t)) - \beta V^\sigma(e(t))$ for any $e(t)$, where $\alpha, \beta > 0, \rho = 1 - (1/2\delta)$ and $\sigma = 1 + (1/2\delta)$ ($\delta > 1$)

Then, the origin of system (3) can achieve fixed-time stability, and

$$T_{\max}^3 = \frac{\pi\delta}{\sqrt{\alpha\beta}}. \quad (9)$$

Lemma 6 (see [36]). Suppose $V(\cdot): R^n \longrightarrow R_+ \cup \{0\}$ is C -regular. If $V(\cdot)$ satisfies

$$\dot{V}(e(t)) \leq -\alpha V^\rho(e(t)), \quad (10)$$

for any $e(t)$, where $\alpha > 0$ and $0 < \rho < 1$ are two constants, then the origin of system (3) can achieve finite-time stability, and the settling time $T(e(0))$ is given by

$$T(e(0)) = \frac{V^{1-\rho}(e(0))}{\alpha(1-\rho)}. \quad (11)$$

3. Main Results

First, we derive a new fixed-time stability theorem, which provides a novel fixed-time stability criterion and a novel upper bound estimate formula for the settling time.

Theorem 1. Suppose that $V(\cdot): R^n \longrightarrow R_+ \cup \{0\}$ is a continuous radically unbounded function and satisfies the following:

- (1) $V(z) = 0 \iff z = 0$
- (2) $\dot{V}(e(t)) \leq -\alpha V^\rho(e(t)) - \beta V^\sigma(e(t)) - \gamma V(e(t))$ for any $e(t)$, where $\alpha, \beta, \gamma > 0, 0 < \rho < 1$ and $\sigma > 1$

Then, the origin of system (3) can achieve fixed-time stability, and

$$T_{\max}^4 = \frac{1}{\gamma(1-\rho)} \ln\left(1 + \frac{\gamma}{\alpha}\right) + \frac{1}{\beta(\sigma-1)}. \quad (12)$$

Proof. It is obvious that

$$\begin{aligned} \dot{V}(e(t)) &\leq -\alpha V^\rho(e(t)) - \beta V^\sigma(e(t)) - \gamma V(e(t)) \\ &\leq -\alpha V^\rho(e(t)) - \beta V^\sigma(e(t)). \end{aligned} \quad (13)$$

Based on Lemma 3, the origin of system (3) can achieve fixed-time stability.

Let $W(s) = V^{1-\rho}(s)$; then, we have

$$\dot{V}(s) = \frac{1}{1-\rho} \dot{W}(s) V^\rho(s). \quad (14)$$

Therefore,

$$\begin{aligned} \frac{1}{1-\rho} \dot{W}(e(t)) V^\rho(e(t)) \\ \leq -\alpha V^\rho(e(t)) - \beta V^\sigma(e(t)) - \gamma V(e(t)). \end{aligned} \quad (15)$$

Since $V(s) = W^{1/(1-\rho)}(s)$, it follows that

$$\begin{aligned} \dot{W}(e(t)) &\leq (1-\rho) \left[-\alpha - \beta W^{(\sigma-\rho)/(1-\rho)}(e(t)) - \gamma W^{1-\rho}(e(t)) \right] \\ &= (1-\rho) \left[-\alpha - \beta W^{(\sigma-\rho)/(1-\rho)}(e(t)) - \gamma W(e(t)) \right]. \end{aligned} \quad (16)$$

This means

$$\frac{dt}{dW(e(t))} \geq \frac{1}{(1-\rho) \left[-\alpha - \beta W^{(\sigma-\rho)/(1-\rho)}(e(t)) - \gamma W(e(t)) \right]}. \quad (17)$$

So we can obtain that

$$\begin{aligned} T(e(0)) &\leq \int_{W(e(0))}^0 \frac{1}{(1-\rho) \left(-\alpha - \beta w^{(\sigma-\rho)/(1-\rho)} - \gamma w \right)} dw \\ &= \frac{1}{1-\rho} \int_0^{W(e(0))} \frac{1}{\alpha + \beta w^{(\sigma-\rho)/(1-\rho)} + \gamma w} dw. \end{aligned} \quad (18)$$

Two cases will be considered separately:

- (1) If $0 \leq V(e(0)) \leq 1$, we have $0 \leq W(e(0)) = V^{1-\rho}(e(0)) \leq 1$. Then,

$$T(e(0)) \leq \frac{1}{1-\rho} \int_0^{W(e(0))} \frac{1}{\alpha + \gamma w} dw$$

$$= \frac{1}{\gamma(1-\rho)} \ln\left(1 + \frac{\gamma}{\alpha} W(e(0))\right) \leq \frac{1}{\gamma(1-\rho)} \ln\left(1 + \frac{\gamma}{\alpha}\right). \quad (19)$$

(2) If $V(e(0)) > 1$, we have $W(e(0)) = V^{1-\rho}(e(0)) > 1$.
Then,

$$\begin{aligned} T(e(0)) &\leq \frac{1}{1-\rho} \left[\int_0^1 \frac{1}{\alpha + \beta w^{(\sigma-\rho)/(1-\rho)} + \gamma w} dw + \int_1^{W(e(0))} \frac{1}{\alpha + \beta w^{(\sigma-\rho)/(1-\rho)} + \gamma w} dw \right] \\ &\leq \frac{1}{1-\rho} \left[\int_0^1 \frac{1}{\alpha + \gamma w} dw + \int_1^{W(e(0))} \frac{1}{\beta w^{(\sigma-\rho)/(1-\rho)}} dw \right] \\ &= \frac{1}{\gamma(1-\rho)} \ln\left(1 + \frac{\gamma}{\alpha}\right) + \frac{1}{\beta(1-\rho)} \left[\frac{1-\rho}{\sigma-1} - \frac{(1-\rho)W^{(1-\rho)/(\sigma-1)}(e(0))}{\sigma-1} \right] \\ &\leq \frac{1}{\gamma(1-\rho)} \ln\left(1 + \frac{\gamma}{\alpha}\right) + \frac{1}{\beta(\sigma-1)}. \end{aligned} \quad (20)$$

Then, we have

$$T_{\max}^4 = \frac{2}{\gamma(1-\rho)} \ln\left(1 + \frac{\gamma}{\alpha}\right) + \frac{2}{\beta(\sigma-1)}. \quad (21)$$

Remark 1. Since $\ln(1 + (\gamma/\alpha)) < (\gamma/\alpha)$, we have $(1/(\gamma(1-\rho))) \ln(1 + (\gamma/\alpha)) + (1/(\beta(\sigma-1))) < (1/(\alpha(1-\rho))) + (1/(\beta(\sigma-1)))$. Therefore, compared with Lemma 3, Theorem 1 can give $T_{\max}$ a more accurate estimate. Moreover, if $\alpha \ll \gamma$, $\ln(1 + (\gamma/\alpha)) \ll (\gamma/\alpha)$.

Next, we investigate the fixed-time synchronization of systems (1) and (2) by means of the following controller:

$$\begin{aligned} u_i(t) &= -k_i e_i(t) - q_i \text{sign}(e_i(t)) - r_i \text{sign}(e_i(t)) |e_i(t)|^\xi \\ &\quad - s_i \text{sign}(e_i(t)) |e_i(t)|^\eta, \end{aligned} \quad (22)$$

$i = 1, 2, \dots, n$, where k_i and q_i will be determined later, r_i and s_i are arbitrary positive constants, $0 < \xi < 1$, and $\eta > 1$.

Theorem 2. Suppose assumptions A1 and A2 hold. If control gains k_i and q_i satisfy $k_i > \sum_{j=1}^n ((|a_{ij}|l_j + |a_{ji}|l_i)/2) - q_i$, $q_i \geq 2 \sum_{j=1}^n |b_{ij}|M_j$, $i = 1, 2, \dots, n$, systems (1) and (2) can realize fixed-time synchronization under controller (22). In addition,

$$T_{\max}^4 = \frac{2}{\gamma(1-\xi)} \ln\left(1 + \frac{\gamma}{\alpha}\right) + \frac{2}{\beta(\eta-1)}, \quad (23)$$

where $\alpha = \min_i \{r_i\} 2^{(\xi+1)/2}$, $\beta = \min_i \{s_i\} n^{(1-\eta)/2} \cdot 2^{(\eta+1)/2}$, and $\gamma = \min_i \{2q_i + 2k_i - \sum_{j=1}^n (|a_{ij}|l_j + |a_{ji}|l_i)\}$.

Proof. We choose the following Lyapunov function:

$$V(e(t)) = \frac{1}{2} e^T(t) e(t) = \frac{1}{2} \sum_{i=1}^n (e_i(t))^2. \quad (24)$$

The derivative of $V(e(t))$ is

$$\begin{aligned} \dot{V}(e(t)) &= \sum_{i=1}^n e_i(t) \dot{e}_i(t) \\ &= \sum_{i=1}^n e_i(t) \cdot [-q_i e_i(t) + F_i(t) + u_i(t)] \\ &\leq -\sum_{i=1}^n q_i (e_i(t))^2 + \sum_{i=1}^n |e_i(t)| \left[\sum_{j=1}^n |a_{ij}| \cdot l_j |e_j(t)| + 2 \sum_{j=1}^n |b_{ij}| M_j \right] \\ &\quad - \sum_{i=1}^n k_i (e_i(t))^2 - \sum_{i=1}^n q_i |e_i(t)| - \sum_{i=1}^n r_i |e_i(t)|^{\xi+1} - \sum_{i=1}^n s_i |e_i(t)|^{\eta+1} \end{aligned}$$

$$\begin{aligned}
&\leq -\sum_{i=1}^n \left(\varrho_i + k_i - \sum_{j=1}^n \frac{|a_{ij}|l_j + |a_{ji}|l_i}{2} \right) (e_i(t))^2 + \sum_{i=1}^n \left(2 \sum_{j=1}^n |b_{ij}|M_j - q_i \right) |e_i(t)| \\
&\quad - \sum_{i=1}^n r_i |e_i(t)|^{\xi+1} - \sum_{i=1}^n s_i |e_i(t)|^{\eta+1},
\end{aligned} \tag{25}$$

where Lemma 1 has been used.

Since $q_i \geq 2 \sum_{j=1}^n |b_{ij}|M_j$, $i = 1, 2, \dots, n$, we obtain that

$$\begin{aligned}
\dot{V}(e(t)) &\leq -\min_i \{r_i\} \sum_{i=1}^n (|e_i(t)|^2)^{(\xi+1)/2} - \min_i \{s_i\} \sum_{i=1}^n (|e_i(t)|^2)^{(\eta+1)/2} \\
&\quad - \min_i \left\{ \varrho_i + k_i - \sum_{j=1}^n \frac{|a_{ij}|l_j + |a_{ji}|l_i}{2} \right\} \sum_{i=1}^n (e_i(t))^2 \\
&\leq -\min_i \{r_i\} \left(\sum_{i=1}^n |e_i(t)|^2 \right)^{(\xi+1)/2} - \min_i \{s_i\} n^{(1-\eta)/2} \sum_{i=1}^n (|e_i(t)|^2)^{(\eta+1)/2} \\
&\quad - \min_i \left\{ \varrho_i + k_i - \sum_{j=1}^n \frac{|a_{ij}|l_j + |a_{ji}|l_i}{2} \right\} \sum_{i=1}^n (e_i(t))^2 \\
&= -\min_i \{r_i\} 2^{(\xi+1)/2} V^{(\xi+1)/2}(e(t)) - \min_i \{s_i\} n^{(1-\eta)/2} \cdot 2^{(\eta+1)/2} V^{(\eta+1)/2}(e(t)) \\
&\quad - 2 \min_i \left\{ \varrho_i + k_i - \sum_{j=1}^n \frac{|a_{ij}|l_j + |a_{ji}|l_i}{2} \right\} V(e(t)),
\end{aligned} \tag{26}$$

where Lemma 2 has been used.

Let $\alpha = \min_i \{r_i\} 2^{(\xi+1)/2}$, $\beta = \min_i \{s_i\} n^{(1-\eta)/2} \cdot 2^{(\eta+1)/2}$, and $\gamma = \min_i \{2\varrho_i + 2k_i - \sum_{j=1}^n (|a_{ij}|l_j + |a_{ji}|l_i)\}$. Then, we have

$$\dot{V}(e(t)) \leq -\alpha V^{(\xi+1)/2}(e(t)) - \beta V^{(\eta+1)/2}(e(t)) - \gamma V(e(t)). \tag{27}$$

It is obvious that $0 < ((\xi+1)/2) < 1$ and $((\eta+1)/2) > 1$. Since $k_i > \sum_{j=1}^n ((|a_{ij}|l_j + |a_{ji}|l_i)/2) - \varrho_i$, we have $\gamma > 0$. Based on Theorem 1, the origin of system (3) can achieve fixed-time stability. In addition,

$$\begin{aligned}
T_{\max}^4 &= \frac{1}{\gamma(1 - ((\xi+1)/2))} \ln\left(1 + \frac{\gamma}{\alpha}\right) + \frac{1}{\beta(((\eta+1)/2) - 1)} \\
&= \frac{2}{\gamma(1 - \xi)} \ln\left(1 + \frac{\gamma}{\alpha}\right) + \frac{2}{\beta(\eta - 1)}.
\end{aligned} \tag{28}$$

□

Corollary 1. Suppose assumptions A1 and A2 hold. If control gains k_i and q_i satisfy $k_i > \sum_{j=1}^n ((|a_{ij}|l_j + |a_{ji}|l_i)/2) - \varrho_i$, $q_i \geq 2 \sum_{j=1}^n |b_{ij}|M_j$, $i = 1, 2, \dots, n$, systems (1) and (2)

can realize fixed-time synchronization under controller (22). In addition,

$$T_{\max}^1 = \frac{2}{\alpha(1 - \xi)} + \frac{2}{\beta(\eta - 1)}, \tag{29}$$

where $\alpha = \min_i \{r_i\} 2^{(\xi+1)/2}$ and $\beta = \min_i \{s_i\} n^{(1-\eta)/2} \cdot 2^{(\eta+1)/2}$.

Proof. Similarly, we can prove that

$$\begin{aligned}
\dot{V}(e(t)) &\leq -\alpha V^{(\xi+1)/2}(e(t)) - \beta V^{(\eta+1)/2}(e(t)) - \gamma V(e(t)) \\
&\leq -\alpha V^{(\xi+1)/2}(e(t)) - \beta V^{(\eta+1)/2}(e(t)),
\end{aligned} \tag{30}$$

where $\alpha = \min_i \{r_i\} 2^{(\xi+1)/2}$, $\beta = \min_i \{s_i\} n^{(1-\eta)/2} \cdot 2^{(\eta+1)/2}$, and $\gamma = \min_i \{2\varrho_i + 2k_i - \sum_{j=1}^n (|a_{ij}|l_j + |a_{ji}|l_i)\} > 0$.

Based on Lemma 3, the origin of system (3) can achieve fixed-time stability. In addition,

$$\begin{aligned}
T_{\max}^1 &= \frac{1}{\alpha(1 - ((\xi+1)/2))} + \frac{1}{\beta(((\eta+1)/2) - 1)} \\
&= \frac{2}{\alpha(1 - \xi)} + \frac{2}{\beta(\eta - 1)}.
\end{aligned} \tag{31}$$

□

Corollary 2. Suppose assumptions **A1** and **A2** hold. If control gains k_i and q_i satisfy $k_i > \sum_{j=1}^n ((|a_{ij}|l_j + |a_{ji}|l_i)/2) - \varrho_i$, $q_i \geq 2 \sum_{j=1}^n |b_{ij}|M_j$, $i = 1, 2, \dots, n$, systems (1) and (2) can realize fixed-time synchronization under controller (22). In addition,

$$T_{\max}^2 = \frac{1}{\alpha} \cdot \left(\frac{\alpha}{\beta}\right)^{(1-\xi)/(\eta-\xi)} \left(\frac{2}{1-\xi} + \frac{2}{\eta-1}\right), \quad (32)$$

where $\alpha = \min_i \{r_i\} 2^{(\xi+1)/2}$ and $\beta = \min_i \{s_i\} n^{(1-\eta)/2} \cdot 2^{(\eta+1)/2}$.

Proof. Similarly, we can prove that

$$\dot{V}(e(t)) \leq -\alpha V^{(\xi+1)/2}(e(t)) - \beta V^{(\eta+1)/2}(e(t)), \quad (33)$$

where $\alpha = \min_i \{r_i\} 2^{(\xi+1)/2}$ and $\beta = \min_i \{s_i\} n^{(1-\eta)/2} \cdot 2^{(\eta+1)/2}$.

Based on Lemma 4, the origin of system (3) can achieve fixed-time stability. In addition,

$$T_{\max}^2 = \frac{1}{\alpha} \cdot \left(\frac{\alpha}{\beta}\right)^{(1-\xi)/(\eta-\xi)} \left(\frac{2}{1-\xi} + \frac{2}{\eta-1}\right). \quad (34)$$

Now, we consider the following controller:

$$\begin{aligned} u_i(t) = & -k_i e_i(t) - q_i \text{sign}(e_i(t)) - r_i \text{sign}(e_i(t)) |e_i(t)|^\xi \\ & - s_i \text{sign}(e_i(t)) |e_i(t)|^\eta, \end{aligned} \quad (35)$$

$i = 1, 2, \dots, n$, where k_i and q_i will be determined later, r_i and s_i are arbitrary positive constants, $\xi = 1 - (1/\delta)$, and $\eta = 1 + (1/\delta)$, $\delta > 1$. $\square$

Corollary 3. Suppose assumptions **A1** and **A2** hold. If control gains k_i and q_i satisfy $k_i > \sum_{j=1}^n ((|a_{ij}|l_j + |a_{ji}|l_i)/2) - \varrho_i$, $q_i \geq 2 \sum_{j=1}^n |b_{ij}|M_j$, $i = 1, 2, \dots, n$, systems (1) and (2) can realize fixed-time synchronization under controller (35). In addition,

$$T_{\max}^3 = \frac{\pi\delta}{\sqrt{\alpha\beta}} \quad (36)$$

where $\alpha = \min_i \{r_i\} 2^{(\xi+1)/2}$ and $\beta = \min_i \{s_i\} n^{(1-\eta)/2} \cdot 2^{(\eta+1)/2}$.

Proof. Similarly, we can prove that

$$\dot{V}(e(t)) \leq -\alpha V^{(\xi+1)/2}(e(t)) - \beta V^{(\eta+1)/2}(e(t)), \quad (37)$$

where $\alpha = \min_i \{r_i\} 2^{(\xi+1)/2}$, $\beta = \min_i \{s_i\} n^{(1-\eta)/2} \cdot 2^{(\eta+1)/2}$, $((\xi+1)/2) = 1 - (1/2\delta)$, $((\eta+1)/2) = 1 + (1/2\delta)$, and $\delta > 1$.

Based on Lemma 5, the origin of system (3) can achieve fixed-time stability. In addition,

$$T_{\max}^3 = \frac{\pi\delta}{\sqrt{\alpha\beta}} \quad (38)$$

To study the finite-time synchronization of systems (1) and (2), we design the following controller:

$$u_i(t) = -k_i e_i(t) - q_i \text{sign}(e_i(t)) - r_i \text{sign}(e_i(t)) |e_i(t)|^\xi, \quad (39)$$

$i = 1, 2, \dots, n$, where k_i and q_i will be determined later, r_i is an arbitrary positive constant, and $0 < \xi < 1$. $\square$

Corollary 4. Suppose assumptions **A1** and **A2** hold. If control gains k_i and q_i satisfy $k_i > \sum_{j=1}^n ((|a_{ij}|l_j + |a_{ji}|l_i)/2) - \varrho_i$, $q_i \geq 2 \sum_{j=1}^n |b_{ij}|M_j$, $i = 1, 2, \dots, n$, systems (1) and (2) can realize finite-time synchronization under controller (36). In addition,

$$T(e(0)) = \frac{1}{\alpha(1-\xi)} \left(\frac{1}{2} \sum_{i=1}^n (e_i(0))^2 \right)^{(1-\xi)/2}, \quad (40)$$

where $\alpha = \min_i \{r_i\} 2^{(\xi+1)/2}$.

Proof. Similarly, we can prove that

$$\dot{V}(e(t)) \leq -\alpha V^{(\xi+1)/2}(e(t)), \quad (41)$$

where $\alpha = \min_i \{r_i\} 2^{(\xi+1)/2}$, $0 < ((\xi+1)/2) < 1$.

Based on Lemma 6, the origin of system (3) can achieve finite-time stability. In addition,

$$\begin{aligned} T(e(0)) &= \frac{V^{1-((\xi+1)/2)}(e(0))}{\alpha(1-((\xi+1)/2))} = \frac{2V^{(1-\xi)/2}(e(0))}{\alpha(1-\xi)} \\ &= \frac{1}{\alpha(1-\xi)} \left(\frac{1}{2} \sum_{i=1}^n (e_i(0))^2 \right)^{(1-\xi)/2}. \end{aligned} \quad (42)$$

$\square$

Remark 2. Based on the fixed-time stability theorems in [23, 24, 29], there have been some publications about the fixed-time synchronization of neural networks [29–34]. However, the upper bound estimate formulas for the settling time given in [23, 24, 29] are very inaccurate. In this paper, we establish a new fixed-time stability theorem, which provides a novel upper bound estimate formula for the settling time. Numerical simulations show that the upper bound estimate for the settling time in this paper is tighter than those given in [23, 24, 29].

Remark 3. In the controllers designed in this paper, the switching item $\text{sign}(e_i(t))$ is used to deal with the sign problems in theoretical derivation. In theoretical derivation, $|e_i(t)|$ is desirable, while $e_i(t)$ is not desirable. In this case, since $\text{sign}(e_i(t))e_i(t) = |e_i(t)|$, $\text{sign}(e_i(t))$ can play a key role. It should be pointed out that $\text{sign}(e_i(t))$ is a discontinuous term, which may lead to chattering phenomenon. Alternatively, the continuous term $e_i(t)/(|e_i(t)| + \gamma)$ can be used to replace $\text{sign}(e_i(t))$ in engineering applications, where $\gamma > 0$ is a sufficiently small constant. However, in strict theoretical analysis, we need not do this kind of replacement, and this kind of replacement usually leads to problems.

4. Numerical Simulations

In this section, a numerical example is provided to validate the obtained theoretical results.

Example 1. Consider the following neural network:

$$\begin{aligned} \dot{\xi}_i(t) = & -q_i \xi_i(t) + \sum_{j=1}^2 a_{ij} f_j(\xi_j(t)) \\ & + \sum_{j=1}^2 b_{ij} f_j(\xi_j(t - \tau(t))) + I_i, \end{aligned} \quad (43)$$

$i = 1, 2$, where $q_1 = 0.9$, $q_2 = 1.1$, $a_{11} = -0.5$, $a_{12} = 2$, $a_{21} = 0.1$, $a_{22} = 1.5$, $b_{11} = -1.2$, $b_{12} = 1$, $b_{21} = 0.1$, $b_{22} = -1.5$, $I_1 = \sin t$, and $I_2 = \cos t$.

Let $f_1(\mu) = f_2(\mu) = (|\mu + 1| - |\mu - 1|)/2$ and $\tau(t) = 1.2$; we have $M_1 = M_2 = 1$, $l_1 = l_2 = 1$, and $\tau = 1.2$. The initial condition of system (43) is $\varphi(\nu) = (1.5, -0.5)^T$, where $\nu \in [-1.2, 0]$.

The corresponding response system is

$$\begin{aligned} \dot{\eta}_i(t) = & -q_i \eta_i(t) + \sum_{j=1}^2 a_{ij} f_j(\eta_j(t)) \\ & + \sum_{j=1}^2 b_{ij} f_j(\eta_j(t - \tau(t))) + I_i + u_i(t), \end{aligned} \quad (44)$$

$i = 1, 2$, where $u_i(t)$, $i = 1, 2$, are controllers. Suppose the initial conditions of system (44) are $\phi(\nu) = (-0.6, 1.8)^T$, $(-8, 9)^T$, $(-50, 60)^T$, and $\nu \in [-1.2, 0]$, respectively. The synchronization errors between systems (43) and (44) without control inputs are given in Figure 1.

Choose $k_1 = 3$, $k_2 = 4$, $q_1 = 4.4$, $q_2 = 3.2$, $r_1 = r_2 = 1$, $s_1 = s_2 = 1$, $\xi = (2/3)$, and $\eta = (4/3)$; then, we have $\alpha = 2^{(5/6)}$, $\beta = 2$, $\gamma = 2.7$. It can be verified that $k_i > \sum_{j=1}^n (|a_{ij}|l_j + |a_{ji}|l_i)/2 - q_i$ and $q_i \geq 2 \sum_{j=1}^n |b_{ij}|M_j$, $i = 1, 2$.

Based on Theorem 2, systems (43) and (44) can realize fixed-time synchronization under controller (22) and $T_{\max}^4 \approx 4.6486$. The synchronization errors between systems (43) and (44) under controller (22) are presented in Figure 2. Although there are three kinds of different synchronization errors, one can see that all the synchronization errors tend to 0 within $T_{\max}^4$.

Remark 4. Based on Corollary 1 and Corollary 2, systems (43) and (44) can realize fixed-time synchronization under controller (22), and $T_{\max}^1 \approx 6.3674$ and $T_{\max}^2 \approx 6.3568$. Furthermore, systems (43) and (44) can realize fixed-time synchronization under controller (35) according to Corollary 3 ($\delta = 3$), and $T_{\max}^3 \approx 4.9926$. In fact, if $\xi = (2/3)$ and $\eta = (4/3)$, controller (35) is the same as controller (22). Obviously, numerical example shows that $T_{\max}^4$ is smaller than $T_{\max}^1$, $T_{\max}^2$, and $T_{\max}^3$ (see Figure 2).

Remark 5. Based on Corollary 4, by choosing the same parameters as those in Example 1, systems (43) and (44) can realize finite-time synchronization under controller (36). It should be pointed out that the settling time $T(e(0))$ is related to $e(0)$, which depends on the initial values of systems (43) and (44). Therefore, the settling time $T(e(0))$ cannot be bounded by a unified upper bound. If $\|e(0)\|$ is large enough, the settling time $T(e(0))$ will be larger than $T_{\max}^4$. In contrast, for arbitrary $e(0)$, systems (43) and (44) can realize

FIGURE 1: The synchronization errors between systems (43) and (44) without control inputs, where the initial condition of system (43) is $\varphi(\nu) = (1.5, -0.5)^T$ and the initial conditions of system (44) are $\phi(\nu) = (-0.6, 1.8)^T$, $(-8, 9)^T$, $(-50, 60)^T$, and $\nu \in [-1.2, 0]$, respectively.

FIGURE 2: The synchronization errors between system (22) under controller (22), where the initial condition of system (43) is $\varphi(\nu) = (1.5, -0.5)^T$, and the initial conditions of system (44) are $\phi(\nu) = (-0.6, 1.8)^T$, $(-8, 9)^T$, $(-50, 60)^T$, and $\nu \in [-1.2, 0]$, respectively.

fixed-time synchronization under controller (22) within $T_{\max}^4$.

Remark 6. In Example 1, if we choose some larger k_i and keep the other controller parameters unchanged, the obtained $T_{\max}^4$ can be smaller. However, $T_{\max}^1$, $T_{\max}^2$, and $T_{\max}^3$ still remain unchanged in this case.

5. Conclusions

In this paper, the fixed-time synchronization of neural networks with discrete delay is investigated by utilizing the newly developed fixed-time stability theorem, which can give the settling time a tighter upper bound estimate compared with the existing fixed-time stability theorems. The settling time of fixed-time synchronization/stability is bounded by a fixed constant, irrespective of the initial conditions of the considered systems. The obtained fixed-time synchronization criteria can be verified easily, and

numerical simulations are provided to demonstrate the validity of the theoretical results. In the future, we will study the application of neural networks in associative memory.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

Julian Shen and Wei Wei are co-first authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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