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Research Article

On the Solution of Quadratic Nonlinear Integral Equation with Different Singular Kernels

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All the previous authors discussed the quadratic equation only with continuous kernels by different methods. In this paper, we introduce a mixed nonlinear quadratic integral equation (MQNLIE) with singular kernel in a logarithmic form and Carleman type. An existence and uniqueness of MQNLIE are discussed. A quadrature method is applied to obtain a system of nonlinear integral equation (NLIE), and then the Toeplitz matrix method (TMM) and Nystrom method are used to have a nonlinear algebraic system (NLAS). The Newton–Raphson method is applied to solve the obtained NLAS. Some numerical examples are considered, and its estimated errors are computed, in each method, by using Maple 18 software.

1. Introduction

Integral equations of various types and kinds play an important role in several mathematical problems modelling. Analytical solutions of integral equations, however, neither exist nor simple to find, so several numerical methods have been developed for finding the solutions of integral equations. The quadratic equation provides an important tool for modelling many numerical phenomena, bio-mathematical problems, and process engineering. In genetics, it represents the reproduction equation, through which events affecting the cells can be predicted. Gripenberg [1] studied the existence and the uniqueness of a bounded continuous solution to the following integral equation of product type:

$$x(t) = k\left[p(t) + \int_0^t A(t-s)x(s)\mathrm{d}s\right]\left[g(t) + \int_0^t B(t-s)x(s)\mathrm{d}s\right], \tag{1}$$

which arises in the study of the spread of an infectious disease that does not induce permanent immunity.

Abdou and Basseem [2] used Chebyshev polynomial in solving mixed integral equation in position and time using spectral relationships. Javidi and Golbabai [3] solved NLFIE by the modified homotopy perturbation method. Alipanah and Esmaeili [4] used radial basis function to find a solution of two-dimensional FIE. Bernstein's method is used to solve VIE by Maleknejad et al. [5]. The Toeplitz matrix method is used to solve NLIE of Hammerstein by Abdou et al. [6]. Orsi [7] used the product Nystrom method to get the solution of NVIE when its kernel takes a logarithmic form and Carleman function. The degenerate kernel method is discussed in three-dimensional NLIE by Basseem [8, 9]. Guoqiang et al. [10] obtained numerically the solution of two-dimensional NVIE by collocation and iterated collocation methods. Brunner et al. [11] introduced a class of methods to obtain numerically the solution of Abel integral equation. Abdou and Raad [12] used the Adomian decomposition method for solving quadratic NLIE. The radial basis function method with collocation scheme for solving quadratic integral equation of Urysohn's type is described by Avazzadeh [13]. Assaria et al. used meshless methods for solving NLIE (see [14–17]).

In this paper, a new problem in a product type of mixed integral equation with singular kernel is considered. The existence and uniqueness of its solution are discussed.

The quadratic method is applied to obtain a NLS of FIE, and then the Toeplitz matrix method or Nystrom method is used to obtain a NLAS which is solved numerically by the Newton–Raphson method.

Consider the QNLE

$$\gamma(x,t,\varphi(x,t)) - \lambda\int_0^t V(t,\tau)\varphi(x,t)\mathrm{d}\tau\int_{-1}^1 k(|x-y|)\varphi(y,t)\mathrm{d}y = f(x,t), \tag{2}$$

where $V(|t,\tau|)$ is a continuous function of time, belongs to the class $C([0,T],[0,T]), t,\tau\in[0,T]$, $T<1$. The singular kernel of position $k(|x-y|)$ takes many different forms. The given function $f(x,t)$ is in the space $C[0,T]\times L_2[-1,1]$. $\gamma(x,t,\varphi(x,t))$ is a given nonlinear function of the unknown function $\varphi(x,t)$. The constant λ has many physical meanings.

2. Existence and Uniqueness

In order to guarantee the existence of a unique solution of equation (2), assume

(1) The discontinuous kernel of equation (2) verifies

$$\left\{\int_{-1}^1\int_{-1}^1 k^2(x-y)\mathrm{d}x\mathrm{d}y\right\}^{(1/2)} = C. \tag{3}$$

(2) The positive kernel of time is continuous and satisfies

$$\max_{0\le t,\tau\le T} V(|t-\tau|)\le M,\quad \forall t,\tau\in[0,T]. \tag{4}$$

(3) The given continuous function $f(x,t)\in L_2[-1,1]\times C[0,T]$ and its norm is defined as

$$\|f(x,t)\| = \max_{0\le t\le T}\int_0^t f(x,\tau)\mathrm{d}\tau\left[\int_{-1}^1 f^2(x,t)\mathrm{d}x\right]^{(1/2)}\le H. \tag{5}$$

(4) The function $\gamma(x,t,\varphi(x,t))$ satisfies

(i) $\|\gamma(x,t,\varphi(x,t))\| = \max_{0\le t\le T}\int_0^t \gamma(x,\tau,\varphi(x,\tau))\mathrm{d}\tau\ [\int_{-1}^1 \gamma^2(x,t,\varphi(x,t))\mathrm{d}x]^{1/2}\le E$ and $\|\varphi\| = D$, where D and E are constants.

(ii) For any two functions φ_1 and φ_2, $\gamma(x,t,\varphi(x,t))$ satisfies Lipchitz condition which is

$$\left\|\gamma(x,t,\varphi_1(x,t)) - \gamma(x,t,\varphi_2(x,t))\right\|\le N\left\|\varphi_1(x,t)-\varphi_2(x,t)\right\|. \tag{6}$$

3. Integral Operator of MQNIE

Eq. (2) can be written in the operator form as

$$\Gamma = \Upsilon\varphi + F, \tag{7}$$

where

$$\Upsilon\varphi = \lambda\int_0^t V(t,\tau)\,\varphi(x,t)\mathrm{d}\tau\int_{-1}^1 k(|x-y|)\,\varphi(y,t)\mathrm{d}y. \tag{8}$$

Γ and F are the Nemytskii operator generated by the functions $\gamma(x,t,\varphi_2(x,t))$ and $f(x,t)$, respectively.

Theorem 1. *The solution of equation (2) exists and is unique under the condition $|\lambda|<(1/MC)$.*

The proof of this theorem can be deduced after the following discussion.

Lemma 1. *The operator Y is bounded.*

Proof. Since

$$\|\Upsilon\varphi\|\le|\lambda|\int_0^t |V(t,\tau)\varphi(x,\tau)|\mathrm{d}\tau\int_{-1}^1 |k(|x-y|)\varphi(y,t)|\mathrm{d}y, \tag{9}$$

applying Cauchy–Schwarz inequality, we have

$$\|\Upsilon\varphi\|\le|\lambda|MC\,D. \tag{10}$$

□

Lemma 2. *The operator Υ is continuous.*

Proof. We assume two functions φ_n and φ_m satisfy equation (2), and then we get

$$\left\|\Upsilon\varphi_n - \Upsilon\varphi_m\right\|\le|\lambda|\int_0^t |V(t,\tau)|\left|\varphi_n(x,t)-\varphi_m(x,t)\right|\mathrm{d}\tau \cdot\int_{-1}^1 |k(|x-y|)|\left|\varphi_n(x,t)-\varphi_m(x,t)\right|\mathrm{d}y. \tag{11}$$

Applying Cauchy–Schwarz inequality, we have

$$\left\|\Upsilon\varphi_n - \Upsilon\varphi_m\right\| \le |\lambda| MC \left\|\varphi_n(x,t) - \varphi_m(x,t)\right\|, \tag{12}$$

whenever $\|\varphi_n(x,t) - \varphi_m(x,t)\| \longrightarrow 0$, one can deduce $\|\Upsilon\varphi_n - \Upsilon\varphi_m\| \longrightarrow 0$, which proves the continuity of the operator.

Moreover, under the condition $|\lambda| < (1/MC)$, the operator Υ is a contraction mapping, and by fixed point theorem, equation (2) has a unique solution in the space $L_2[-1,1] \times C[0,T]$. □

4. Quadratic Numerical Method (See [18])

To obtain a system of NLIE, divide the time interval $[0,T]$as

$$0 = t_0 < t_1 < t_2 < \ldots < t_l = T. \tag{13}$$

Let $t = t_i$, then equation (2) becomes

$$\gamma(x,t_i,\varphi(x,t_i)) - \lambda \int_0^{t_i} V(t_i,\tau)\varphi(x,t)\mathrm{d}\tau \int_{-1}^{1} k(|x-y|)\varphi(y,t_i)\mathrm{d}y = f(x,t_i). \tag{14}$$

Applying the quadrature rule, equation (14) reduces to

$$\gamma(x,t_i,\varphi(x,t_i)) - \lambda \sum_{j=0}^{i} \omega_j V_{i,j}\varphi_j(x) \int_{-1}^{1} k(|x-y|)\varphi_i(y)\mathrm{d}y = f_i(x), \tag{15}$$

where

$$\omega_j = \begin{cases} \dfrac{h}{2}, & j = 0, i, \\ h, & \text{otherwise.} \end{cases} \tag{16}$$

Using the notation

$$\begin{aligned} \varphi_i(x) &= \varphi(x,t_i), \\ V_{i,j} &= V(t_i,t_j), \\ \gamma_i(x,\varphi_i(x)) &= \gamma(x,t_i,\varphi_i(x)), \end{aligned} \tag{17}$$

we get

$$\gamma_i(x,\varphi_i(x)) - \lambda \int_{-1}^{1} k(|x-y|)\varphi_i(y)\mathrm{d}y \left[\frac{h}{2}V_{i,i}\varphi_i(x) + \sum_{j=0}^{i-1} \omega_j V_{i,j}\varphi_j(x)\right] = f_i(x), \tag{18}$$

which is the system of NLIE can be solved by two different methods, namely, Toeplitz matrix method and Nystrom method.

4.1. Algebraic System of NLIE. Consider

$$\Gamma\gamma(\varphi_i,x) = \lambda U\varphi_i(x) \cdot V\varphi_i(x) + f_i(x), \tag{19}$$

where

$$U\varphi_i(x) = \int_{-1}^{1} k|x-y|\varphi_i(y)\mathrm{d}y \tag{20}$$

and

$$V\varphi_i(x) = \sum_{j=0}^{i} \omega_j V_{i,j}\varphi_j(x). \tag{21}$$

In order to guarantee the existence of a unique solution of an algebraic system of NIE, we assume the following conditions:

(i) $\max_i f_i(x) \le H^*$
(ii) $\sum_{j=0}^{i} \max_j |\omega_j V_{i,j}| \le M^*$
(iii) $\max_i |\gamma(x,\varphi_i(x))| \le E^*$

Hence, formula (19) has a unique solution under condition $\lambda < (1/CM^*)$.

Definition 1. The estimate local error$R^{(1)}$is determined by the following relation:

$$\begin{aligned} R^{(1)} = \Bigg| &\int_{-1}^{1} k(|x-y|)\varphi(y,t)\mathrm{d}y \int_0^t V(t,\tau)\varphi(x,\tau)\mathrm{d}\tau \\ &- \int_{-1}^{1} k(|x-y|)\varphi_i(y)\mathrm{d}y \sum_{j=0}^{i} \omega_j V_{i,j}\varphi_j(x)\Bigg|. \end{aligned} \tag{22}$$

5. Toeplitz Matrix Method (See [6])

We apply the TMM to have a nonlinear algebraic equation. For this, consider $h = (1/N)$; therefore,

$$\int_{-1}^{1} k(|x-y|)\varphi_i(y)\mathrm{d}y = \sum_{n=-N}^{N-1}\int_{nh}^{nh+h} k(|x-y|)\varphi_i(y)\mathrm{d}y \tag{23}$$

and

$$\int_{nh}^{nh+h} k(|x-y|)\varphi_i(y)\mathrm{d}y \approx A_n(x)\varphi(nh) + B_n(x)\varphi(nh+h) + R. \tag{24}$$

The functions $A_n(x)$ and $B_n(x)$ are arbitrary functions to be determined, and R is the error term. In order to obtain the values of two functions, assume $\varphi(y) = \{1, y\}$. This yields a set of two equations in terms of two unknown functions. After ignoring the error term, equation (18) becomes

$$\begin{aligned}\gamma_i(x,\varphi_i(x)) - \frac{h|\lambda|}{2}\sum_{n=-N}^{N}\xi_n(x)\varphi_i(nh)V_{i,i}\varphi_i(x)\\ = |\lambda|\sum_{n=-N}^{N}\xi_n(x)\varphi_i(nh)\sum_{j=0}^{i-1}\omega_j V_{i,j}\varphi_j(x) + f_i(x).\end{aligned} \tag{25}$$

Let $x = mh$, then using the following notation:

$$\begin{aligned}\varphi_i(x) &= \varphi_i(mh) = \varphi_{im},\\ \xi_n(x) &= \xi_{nm},\\ f_i(x) &= f_{im},\\ \gamma_i(x,\varphi_i(x)) &= \gamma_{im}(\varphi_{im}),\end{aligned} \tag{26}$$

equation (25) becomes

$$\begin{aligned}\gamma_{im}(\varphi_{im}) - \frac{h|\lambda|V_{ii}}{2}\sum_{n=-N}^{N}\xi_{nm}\varphi_{in}\varphi_{im}\\ = |\lambda|\sum_{n=-N}^{N}\xi_{nm}\varphi_{in}\sum_{j=0}^{i-1}\omega_j V_{i,j}\varphi_{jm} + f_{im},\end{aligned} \tag{27}$$

where

$$\xi_{nm} = \begin{cases} A_{-n}(mh), & n=-N,\\ A_n(mh) + B_{n-1}(mh), & -N<n<N,\\ B_{n-1}(mh), & n=N.\end{cases} \tag{28}$$

Equation (27) represents that the NLAS can be solved using the Newton–Raphson method.

Definition 2. The Toeplitz matrix method is said to be convergent of order r in the interval $[-1, 1]$, if and only if, for sufficient large N, there exists a constant $D>0$ independent of N such that

$$\left\|\varphi(x) - \varphi_N(x)\right\| \le DN^{-r}. \tag{29}$$

Definition 3. The estimate local error $R^{(2)}$ is determined by the following relation:

$$R^{(2)} = \left|\int_{-1}^{1} k(|x-y|)\varphi_i(y)\mathrm{d}y\sum_{j=0}^{i}\omega_j V_{i,j}\varphi_j(x) - \sum_{n=-N}^{N}\xi_{nm}\varphi_{in}\sum_{j=0}^{i}\omega_j V_{i,j}\varphi_{jm}\right|. \tag{30}$$

5.1. Existence and Uniqueness of NLAS. In order to guarantee the existence of a unique solution of a NLAS, we assume the following conditions:

(i) $\sup_{n,m}|\sum_{n=-N}^{N}\xi_{nm}| \le C^*$

(ii) $\sup_{i,m}|f_{im}| \le H^{**}$

(iii) $\sup_{i,n}|\gamma_i(nh,\varphi(nh))| \le E^{**}$

Definition 4. The estimate local error $R^{(T)}$ is determined by the following relation:

$$R^{(T)} = \left|\int_{-1}^{1} k(|x-y|)\varphi(y,t)\mathrm{d}y\int_0^t V(t,\tau)\varphi(x,t)\mathrm{d}\tau - \sum_{n=-N}^{N}\xi_{nm}\varphi_{in}\sum_{j=0}^{i}\omega_j V_{i,j}\varphi_{jm}\right|, \tag{31}$$

where $R^{(T)} \le R^{(1)} + R^{(2)}$.

6. Nystrom Method (See [7])

Here, by using the product integration, we approximate the integral part of equation (18) by a suitable Lagrange interpolation polynomial. For this, let $x = x_m$, and the integral part can be written as

$$\int_{-1}^{1} k(x_m,y)\varphi_i(y)\mathrm{d}y = \sum_{n=0}^{N}\mathfrak{I}_{m,n}\varphi_i(y_n) = \sum_{n=0}^{((N-2)/2)}\int_{y_{2n}}^{y_{2n+2}} k(x_m,y)\varphi_i(y)\mathrm{d}y, \tag{32}$$

where $x_m = y_m = -1 + mh, m = 1, 2, 3, \ldots, N$with $h = 2/N$ and N is even number.

Approximate the nonsingular part of the integral $\varphi(y)$ over each interval $[y_{2n}, y_{2n+2}]$ by the Lagrange interpolation polynomial at the points $2n$, $2n+1$, and $2n+2$. Therefore, equation (32) becomes

$$\int_{-1}^{1} k(x_m, y)\varphi_i(y)\mathrm{d}y = \sum_{n=0}^{N-2/2} \int_{y_{2n}}^{y_{2n+2}} k(x_m, y)\left[\frac{(y_{2n+1}-y)(y_{2n+2}-y)}{(y_{2n+1}-y_{2n})(y_{2n+2}-y_{2n})}\varphi_i(y_{2n}) + \frac{(y_{2n}-y)(y_{2n+2}-y)}{(y_{2n}-y_{2n+1})(y_{2n}-y_{2n+1})}\varphi_i(y_{2n+1}) + \frac{(y_{2n}-y)(y_{2n+1}-y)}{(y_{2n}-y_{2n+2})(y_{2n+1}-y_{2n+2})}\varphi_i(y_{2n+2})\right]\mathrm{d}y. \tag{33}$$

Comparing equations (32) and (33), we deduce

$$\begin{aligned}
\mathfrak{I}_{m,0} &= \frac{1}{2h^2}\int_{y_0}^{y_2} k(x_m, y)(y_1 - y)(y_2 - y)\mathrm{d}y,\\
\mathfrak{I}_{m,2n+1} &= \frac{1}{h^2}\int_{y_{2n}}^{y_{2n+2}} k(x_m, y)(y - y_{2n})(y_{2n+2} - y)\mathrm{d}y,\\
\mathfrak{I}_{m,2n} &= \frac{1}{2h^2}\left[\int_{y_{2n}}^{y_{2n+2}} k(x_m, y)(y_{2n+1} - y)(y_{2n+2} - y)\mathrm{d}y + \int_{y_{2n-2}}^{y_{2n}} k(x_m, y)(y - y_{2n-1})(y - y_{2n-2})\mathrm{d}y\right],\\
\mathfrak{I}_{m,N} &= \frac{1}{2h^2}\int_{y_{N-2}}^{y_N} k(x_m, y)(y - y_{N-2})(y - y_{N-1})\mathrm{d}y.
\end{aligned} \tag{34}$$

Introduce the following notations:

$$\begin{aligned}
\alpha_n(y_m) &= \frac{1}{2h^2}\int_{y_{2n-2}}^{y_{2n}} k(x_m, y)(y - y_{2n-2})(y - y_{2n-1})\mathrm{d}y,\\
\beta_n(y_m) &= \frac{1}{2h^2}\int_{y_{2n-2}}^{y_{2n}} k(x_m, y)(y_{2n-1} - y)(y_{2n} - y)\mathrm{d}y,
\end{aligned} \tag{35}$$

and

$$\zeta_n(y_m) = \frac{1}{2h^2}\int_{y_{2n-2}}^{y_{2n}} k(x_m, y)(y - y_{2n-2})(y_{2n} - y)\mathrm{d}y. \tag{36}$$

Then,

$$\begin{aligned}
\mathfrak{I}_{m,0} &= \beta_1(x_m),\\
\mathfrak{I}_{m,2n+1} &= 2\zeta_{n+1}(x_m),\\
\mathfrak{I}_{m,2n} &= \alpha_n(x_m) + \beta_{n+1}(x_m),\\
\mathfrak{I}_{m,N} &= \alpha_{(N/2)}(x_m).
\end{aligned} \tag{37}$$

By substituting in equation (18), we get

$$\gamma_i m(\varphi_{im}) - \lambda\sum_{n=0}^{N}\mathfrak{I}_{m,n}\varphi_i(y_n)\left[\frac{h}{2}V_{i,i}\varphi_{im} + \sum_{j=0}^{i-1}\omega_j V_{i,j}\varphi_{jm}\right] = f_{im}, \tag{38}$$

where equation (38) represents the NAS in which its existence and uniqueness can be easily shown as in the previous section.

Definition 5. The Nystrom method is said to be convergent of order r in the interval $[-1, 1]$, if and only if, for sufficient large N, there exists a constant $K > 0$ independent of N such that

$$\left\|\varphi(x) - \varphi_N(x)\right\| \leq KN^{-r}. \tag{39}$$

7. Numerical Examples

7.1. Example 1. Consider the equation

$$\gamma(\varphi, x, t) - \lambda\int_0^t V(t, \tau)\varphi(x, \tau)\mathrm{d}\tau \cdot \int_{-1}^{1} k|x - y|\varphi(y, t)\mathrm{d}y\mathrm{d}t = f(x, t), \tag{40}$$

where $f(x,t)$ is given by putting $\varphi(x,t) = x^2t^2$ as an exact value with $\gamma = \varphi^2, V(t,\tau) = (t-\tau)^2$, and

$$k(|x-y|) = \begin{cases} \ln|y-x|, \\ |x-y|^\nu, \quad 0 < \nu < 1. \end{cases} \tag{41}$$

(1) The following table is selected among a large amount of data to compare between the exact solution and its numerical solution in the case of logarithmic kernel for both of the previous methods in some points in the region $x \in [0,1]$ and for different values of time$T = \{0.009, 0.02, 0.8\}$.

(2) In the following table, we compare between the Toeplitz matrix method and Nystrom method for different ν and fixed time $T = 0.3$ in Carleman kernel form.

7.2. Example 2. In the next example, the Nystrom method and Toeplitz matrix method are used with fixed time $T = 0.4$ and the position interval is divided with $N = 2, 4, 8, 16, 32$ points. The rate of errors is evaluated using the formula

$$\text{Rate} = \log_2 \frac{\text{Error}(2N)}{\text{Error}(N)}. \tag{42}$$

The negative sign means that by increasing N, the error decreases (see Table 3).

$$e^{\varphi(x,t)} - \lambda \int_0^t (t-\tau)^2 \varphi(x,\tau)d\tau \cdot \int_{-1}^1 \ln|x-y|\varphi(y,t)dydt = f(x,t), \tag{43}$$

where $f(x,t)$is given by setting $\phi(x,t) = xt$ as an exact value.

8. General Conclusion

From the above tables and our numerical results, we can deduce the following:

(1) The estimated error increases by time, where its mean errors by using Toeplitz and Nystrom methods, when $T = 0.02$, are 1.876×10^{-12} and 2.191×10^{-12}, respectively, while, its mean errors when $T = 0.8$ are 8.565×10^{-6} and 3.192×10^{-5}, respectively.

(2) The Toeplitz matrix method is comparatively better than the Nystrom method for different kernels (see Tables 1 and 2).

(3) By increasing N, the error is extremely stable in both methods, but in the Toeplitz matrix method, the error almost decreases by increasing in N, where the convergence rate with +ve sign means the increasing of errors, while its −ve sign means the errors decreasing (see Table 3).

(4) In Carleman kernel form, the estimated error decreases in small values of ν, where its mean errors take 1.816×10^{-8} when $\nu = 0.07$, while it takes 5.925×10^{-8} when $\nu = 0.47$ by using the Toeplitz matrix method (see Table 2).

Table 1: Comparison between the Toeplitz matrix method and Nystrom method in different time when kernel takes logarithmic form.

		Toeplitz matrix method		Nystrom method	
T	x	φ	Error	φ	Error
0.009	−0.87	0.0000613	2.630×10^{-15}	0.0000613	4.950×10^{-15}
	−0.37	0.0000111	2.157×10^{-15}	0.0000111	8.298×10^{-15}
	0.13	0.0000015	1.637×10^{-7}	0.0000015	1.642×10^{-7}
	0.63	0.0000321	1.219×10^{-14}	0.0000321	4.992×10^{-15}
	0.9	0.0000656	5.469×10^{-15}	0.0000656	1.139×10^{-14}
0.02	−0.87	0.0003028	1.179×10^{-13}	0.0003028	4.435×10^{-13}
	−0.37	0.0000548	5.055×10^{-14}	0.0000548	2.952×10^{-13}
	0.13	0.0000068	9.062×10^{-12}	0.0000068	9.582×10^{-12}
	0.63	0.0001588	1.417×10^{-13}	0.0001588	3.315×10^{-13}
	0.9	0.0003240	6.330×10^{-15}	0.0003240	3.040×10^{-13}
0.8	−0.87	0.4844307	1.468×10^{-5}	0.4844625	4.647×10^{-5}
	−0.37	0.0876232	7.192×10^{-6}	0.0876449	2.889×10^{-5}
	0.13	0.0108114	4.647×10^{-6}	0.0108321	1.606×10^{-5}
	0.63	0.2540051	1.094×10^{-5}	0.2540541	3.807×10^{-5}
	0.9	0.5183947	5.367×10^{-6}	0.5184301	3.009×10^{-5}

Table 2: Errors in the Toeplitz matrix method and Nystrom method in different values of ν in Carleman kernel form.

		Toeplitz matrix method	Nystrom method
ν	x	Error	Error
0.07	−0.87	2.665×10^{-8}	3.198×10^{-8}
	−0.37	2.218×10^{-8}	2.205×10^{-8}
	0.13	1.439×10^{-8}	1.734×10^{-8}
	0.63	1.204×10^{-8}	1.563×10^{-8}
	0.9	1.556×10^{-8}	2.184×10^{-8}
0.17	−0.87	4.191×10^{-8}	5.629×10^{-8}
	−0.37	2.924×10^{-8}	2.716×10^{-8}
	0.13	9.517×10^{-9}	1.726×10^{-8}
	0.63	9.449×10^{-10}	1.125×10^{-8}
	0.9	1.209×10^{-8}	3.042×10^{-8}
0.47	−0.87	1.352×10^{-7}	2.110×10^{-7}
	−0.37	6.733×10^{-8}	4.547×10^{-8}
	0.13	1.876×10^{-8}	1.748×10^{-8}
	0.63	6.492×10^{-8}	3.365×10^{-10}
	0.9	1.004×10^{-8}	1.268×10^{-7}

Table 3: Convergence rate in both methods with fixed time $T = 0.4$.

	Toeplitz matrix method		Nystrom method	
N	Mean error	Rate	Mean error	Rate
2	6.641×10^{-6}	—	7.144×10^{-6}	—
4	6.298×10^{-6}	−0.0765067	1.079×10^{-5}	0.5956999
8	6.482×10^{-6}	0.0415453	1.110×10^{-5}	0.0408648
16	5.691×10^{-6}	−0.1877568	1.102×10^{-5}	−0.0104355
32	5.556×10^{-6}	−0.0346356	1.096×10^{-5}	−0.0078764

Data Availability

The authors confirm that the data supporting the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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