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Research Article

Design of Static Output Feedback Controller for Fractional-Order T-S Fuzzy System

Zhile Xia

Department of Mathematics, TaiZhou University, ZheJiang, TaiZhou 318000, China

Correspondence should be addressed to Zhile Xia; zhilexia@163.com

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This paper studies the design of fuzzy static output feedback controllers for two kinds of fractional-order T-S fuzzy systems. The fractional order α satisfies $0 < \alpha < 1$ and $1 \leq \alpha < 2$. Based on the fractional order theory, matrix decomposition technique, and projection theorem, four new sufficient conditions for the asymptotic stability of the system and the corresponding controller design methods are given. All the results can be expressed by linear matrix inequalities, and the relationship between fuzzy subsystems is also considered. These have great advantages in solving the results and reducing the conservatism. Finally, a simulation example is given to show the effectiveness of the proposed method.

1. Introduction

Compared with the integer-order system, the fractional-order system has many advantages, for example, the integer order is a special case of the fractional-order system; the fractional-order system has global properties, and the system state depends not only on the current time but also on the past time; it can accurately describe the memory and genetic properties in physics and engineering. Therefore, in recent decades, fractional-order control systems have also been widely concerned, and important results have been obtained. For example, Chen et al. [1–3] made a comprehensive introduction and in-depth study of fractional-order systems. Specifically, the paper [1] considered the robust stability and stabilization of fractional-order linear systems with polyhedral uncertainties. The results obtained are not only applicable to the fractional order α satisfying $0 < \alpha < 1$ but also to the fractional order α satisfying $1 \leq \alpha < 2$. For the fractional-order neural network system with time-varying delay, the literature [2] solved a challenging problem: the delay-dependent stability and stabilization of the fractional-order delay system. The literature [3] studied the stability and synchronization of the delayed neural network, and a series of very important results were given, including delay-independent stability criterion, measurable algebra criterion, and synchronization criterion. Gallegos and Duarte-Mermoud [4] investigated the stability of fractional-order and integral-order coupled systems, in which the concepts of dissipativity and passivity were extended, and the theorems of small gain and passivity for correlated systems were obtained. Liu et al. [5] used the fractional filtering method to study the backstepping controller design of the actuator fault fuzzy neural network system with completely unknown parameters and modes. Combined with the fractional adaptive law, the tracking error and compensation tracking error converged to a small enough area. Based on Lyapunov functions and the comparison principle, Jia et al. [6] designed delayed state feedback control and coupling state feedback control for fractional-order memristor-based neural networks with time delay. To sum up, the control problem of the fractional-order system is a hot issue at present, attracting a large number of researchers to participate in it.

In addition, the fuzzy theory opens up a new way to deal with the fuzzy uncertainty in nature with strict mathematical methods and proposes the method of the membership function to describe the degree of the element set [7]. When the membership degree is 1, it means that the element belongs to the set completely, when the membership degree is 0, it means that the element does not belong to the set at all, and when the membership degree belongs to the interval $(0, 1)$, it means that the element belongs to the set but does not completely belong to the set. Obviously, this is a generalization of the classical set theory. Fuzzy control is based on the fuzzy theory. By using expert language, fuzzy control can achieve good and effective control for some complex nonlinear control objects which are difficult to establish the mathematical model accurately [8, 9]. So far, two main fuzzy models have been proposed. The first model is based on the input-output model proposed in document [9]. The disadvantage is that there is a lot of useful information which is not used. The second model is the T-S fuzzy model proposed by Japanese scholars Takagi and Sugeno in 1985 [10, 11]. In [12–14], it is proved that this kind of fuzzy system has the ability of universal approximation. Some recent literature studies are listed to show that the T-S fuzzy system has been widely studied. For example, based on the T-S fuzzy model, in [15], the design of the fuzzy filter for the discrete-time nonlinear networked system is studied, and a new method is proposed to ensure that the filtering error system is dissipative by using the method of event triggering and quantization. For continuous-time nonlinear time-delay systems, by using repetitive control, Rathinasamy et al. [16] proposed an improved output feedback controller design method. This method ensures that the system can track the given reference signal within the allowable error range. Li et al. [17] studied the finite-time H_∞ control problem, and the sufficient conditions based on LMI were given. The research results of the output feedback can be found in the literature studies [18–28]. There are also a large number of references related to the T-S fuzzy model, which we cannot be enumerated here.

Compared with many research results of the integer-order T-S fuzzy system, the research results of the fractional-order T-S fuzzy system are less. For the fractional-order T-S fuzzy system, Zheng et al. [29] studied the stabilization of the chaotic system by using the adaptive method. When the order of the fractional order considered satisfies $1 \le \alpha < 2$ and $0 < \alpha < 1$, respectively, the stability conditions based on LMI were proposed in [30, 31]. Huang et al. [32] studied the stabilization of the state feedback that can be applied to both the following situations: $1 \le \alpha < 2$ and $0 < \alpha < 1$. However, in the practical engineering field, the state of the system is often difficult to obtain directly. At this time, the state observer or output feedback control methods need to be considered. Based on the T-S fuzzy model, Duan et al. [33] studied the design problem of the nonfragile observer by using singular value decomposition of the matrix and put forward sufficient conditions of linear matrix inequalities, which guarantee that the corresponding closed-loop system is stochastic asymptotically stable; when the fuzzy antecedent variables are unmeasurable, Duan and Li [34] considered the design problem of dynamic output feedback controllers and proposed sufficient conditions for the system to be asymptotically stable; Karthick et al. [35] designed a dynamic output feedback controller by means of interference suppression and quantization; and Lin et al. and Ji et al. [36, 37] studied the design of the fuzzy static output feedback controller. The design of the static fuzzy output feedback controller will increase the fuzzy relation in the stabilization condition by a certain multiple because the product term of coefficient matrices $B_iF_jC_k$ appears, while the design state feedback only contains the product term of coefficient matrices $B_iF_j\,(i, j, k = 1, 2, \ldots, r)$, where r is an integer, indicating the number of fuzzy rules. For the traditional T-S fuzzy system, Chaibi et al. [38] used the matrix decomposition technique to effectively separate the coefficient term $B_iF_jC_k$ and effectively reduced the fuzzy relation of one layer and proposed the design method based on the linear matrix inequality. In this paper, we try to extend the research results of Chaibi et al. [38] to the fractional-order T-S fuzzy system.

In conclusion, this paper studies the fuzzy static output feedback control of the fractional-order T-S fuzzy system. The order includes two cases: $0 < \alpha < 1$ and $1 \le \alpha < 2$. According to the theory of fractional differential correlation and the projection theorem, several new stabilization design methods are given. All the results are expressed by the linear matrix inequality. The main contributions of this paper are as follows: (1) the control method of the fuzzy static output feedback is extended from the integer-order fuzzy system to the fractional-order fuzzy system, so it is more practical; (2) the condition of system stabilization and controller design methods are all expressed by LMIs. Therefore, LMI toolbox of Matlab can be used to program and calculate; (3) output matrix $C_1, C_2, \ldots, C_r$ of the system can still be completely different; (4) for some similar Lyapunov matrices, only symmetry or antisymmetry is required, no further diagonal is required, and stability conditions are relaxed; and (5) the stabilization conditions are also considered. Finally, a numerical simulation example is given to show the effectiveness of the proposed method.

Notations. Throughout this paper, $\mathbb{R}^n$ denotes the n–dimensional Euclidean space; $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices; I and O represent the identity matrix and the zero matrix with appropriate dimensions; $P > 0\,(P < 0)$ represents the positive definite (negative definite) matrix; the right superscript T denotes the transposition of the matrix; the symbol $\mathrm{sym}\{S\}$ means $S + S^T$; the diagonal block matrix is denoted by $\mathrm{diag}\{A_1, A_2, \ldots, A_n\}$; and the symbol $*$ shows the symmetric part of the block matrix.

2. Basic Definition and Lemmas

Several definitions of fractional-order differential are given in monograph [39], among which Caputo and Riemann–Liouville fractional-order differential are widely used. In this paper, we adopt the definition of Caputo fractional-order differential.

Definition 1 (see Podlubny [39]). The Caputo-type fractional derivative of order $\alpha > 0$ for a function $f(t)$ is defined as follows:

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} \mathrm{d}\tau, \qquad (1)$$

where $\Gamma(\cdot)$ is the gamma function defined as $\Gamma(q) = \int_0^\infty t^{q-1} e^{-t} \mathrm{d}t$ and the integer m satisfies the condition $m - 1 < \alpha \le m$.

Next, we give the following lemmas that will be used in this paper.

Lemma 1 (see Zhang et al. [40, 41]). *Let $A \in \mathbb{R}^{n\times n}$ be a real matrix. The fractional-order system $D^{\alpha}x(t) = Ax(t)$ with $0 < \alpha < 1$ is asymptotically stable if and only if there exist real symmetric positive definite matrices P_{11} and P_{21} and skew-symmetric matrices P_{12} and P_{22} such that*

$$\begin{gathered}\begin{bmatrix} P_{11} & P_{12} \\ -P_{12} & P_{11} \end{bmatrix} > 0, \\ \begin{bmatrix} P_{21} & P_{22} \\ -P_{22} & P_{21} \end{bmatrix} > 0, \\ \sum_{i=1}^{2}\sum_{j=1}^{2} \operatorname{sym}\left(\theta_{ij} \otimes \left(AP_{ij}\right)\right) < 0,\end{gathered} \tag{2}$$

where $\theta_{11} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, $\theta_{12} = \begin{bmatrix} b & a \\ -a & b \end{bmatrix}$, $\theta_{21} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, *and* $\theta_{22} = \begin{bmatrix} -b & a \\ -a & -b \end{bmatrix}$, $a = \sin\theta_1, b = \cos\theta_1,\ \theta_1 = \pi\alpha/2$.

Lemma 2 (see Chilali et al. [42]). *Let $A \in \mathbb{R}^{n\times n}$ be a real matrix. The fractional-order system $D^{\alpha}x(t) = Ax(t)$ with $1 \le \alpha < 2$ is asymptotically stable if and only if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} c\left(AP + PA^{T}\right) & d\left(AP - PA^{T}\right) \\ * & c\left(AP + PA^{T}\right) \end{bmatrix} < 0, \tag{3}$$

where $c = \sin\theta_2$ and $d = \cos\theta_2$, $\theta_2 = \pi - \pi\alpha/2$.

Lemma 3 (see Chaibi et al. [38]). *For matrices T, Q, U, and W with proper dimensions, scalar ξ, if inequality*

$$\begin{bmatrix} T & \xi Q + W^{T}U \\ * & -\operatorname{sym}(\xi U) \end{bmatrix} < 0 \tag{4}$$

holds, then

$$T + \operatorname{sym}(QW) < 0. \tag{5}$$

Lemma 4 (see Gahinet and Apkarian [43]). *Given a symmetric matrix $Z_0 \in R^{m\times m}$ and two matrices X, Y of column m, there exists a matrix Z such that the LMI*

$$Z_0 + \operatorname{sym}\left(X^{T}ZY\right) < 0 \tag{6}$$

holds if and only if the following two projection inequalities are satisfied:

$$\begin{gathered} X_{\perp}^{T}Z_0X_{\perp} < 0, \\ Y_{\perp}^{T}Z_0Y_{\perp} < 0, \end{gathered} \tag{7}$$

where $X_{\perp}$ and $Y_{\perp}$ are arbitrary matrices whose columns form bases of the null bases of X and Y, respectively.

3. Problem Description and Formation

Consider the following fractional-order T-S fuzzy systems.

Plant rule i: if $\theta_1(t)$ is $\mu_{i1}(t)$, and . . ., and $\theta_p(t)$ is $\mu_{ip}(t)$, then

$$\begin{cases} D^{\alpha}x(t) = A_i x(t) + B_i u(t), \\ y(t) = C_i x(t), \quad i = 1, 2, \ldots, r, \end{cases} \tag{8}$$

where the order α satisfies $0 < \alpha < 1$ or $1 \le \alpha < 2$; r is the number of if-then rules; $\theta_j(t)$ and $\mu_{ij}(t)\,(j = 1, 2, \ldots, p)$ are the premise variables and the fuzzy sets, respectively; $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^l$, and $y(t) \in \mathbb{R}^m$ are the state, the measurable output, and the controller, respectively; and A_i, B_i, and C_i are matrices with appropriate dimensions.

Using fuzzy reasoning technology, the final output of fuzzy system (8) is

$$\begin{cases} D^{\alpha}x(t) = \sum_{i=1}^{r} h_i(\theta(t))\left(A_i x(t) + B_i u(t)\right), \\ y(t) = \sum_{i=1}^{r} h_i(\theta(t))C_i x(t), \end{cases} \tag{9}$$

where $\theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_p(t)]$ and $h_i(\theta(t)) = \omega_i(\theta(t))/\sum_{i=1}^{r}\omega_i(\theta(t))$, $\omega_i(\theta(t)) = \Pi_{j=1}^{p}\mu_{ij}(\theta_j(t))$; $\mu_{ij}(\theta_j(t))$ is the membership degree of $\theta_j(t)$ in fuzzy set $\mu_{ij}(t)$.

According to the definition of the membership function, we can easily get the following properties:

$$\begin{gathered} \omega_i(\theta(t)) \ge 0, \\ 0 \le h_i(\theta(t)) \le 1, \quad i = 1, 2, \ldots, r; \ \sum_{i=1}^{r} h_i(\theta(t)) = 1. \end{gathered} \tag{10}$$

For system (8), we will design the following fuzzy static output feedback controller:

$$u(t) = \sum_{i=1}^{r} h_i(\theta(t))F_i y(t), \tag{11}$$

where $F_i\,(i = 1, 2, \ldots, r)$ are the matrices with proper dimensions to be designed.

By bringing controller (11) into system (8), we can get the closed-loop system:

$$D^{\alpha}x(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(\theta(t))h_j(\theta(t))h_k(\theta(t))\left(A_i + B_iF_jC_k\right)x(t). \tag{12}$$

The main purpose of this paper is to design static fuzzy controller (11) for system (8) so that the corresponding closed-loop system (12) is asymptotically stable.

4. Design of the Fuzzy Static Output Feedback Controller

In this section, the design methods of the fuzzy static output feedback controller for fractional-order fuzzy system (8) are given, and the corresponding closed-loop system is guaranteed to be asymptotically stable. Now, we can prove the following results.

Theorem 1. *Fractional-order closed-loop system (12) with $0 < \alpha < 1$ is asymptotically stable if there is scalar $\xi > 0$, proper dimensional matrices $U, G_{1i}, G_{2i}, G_{3i}, G_{4i}, K_i\,(i = 1, 2, \ldots, r)$,* *symmetric matrices P_{11}, P_{21}, and skew-symmetric matrices P_{21}, P_{22} such that the following LMIs are satisfied:*

$$\begin{bmatrix} P_{11} & P_{12} \\ -P_{12} & P_{11} \end{bmatrix} > 0, \qquad \begin{bmatrix} P_{21} & P_{22} \\ -P_{22} & P_{21} \end{bmatrix} > 0, \tag{13}$$

$$\Omega_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{14}$$

$$\Omega_{ij} + \Omega_{ji} < 0, \quad i = 1, 2, \ldots, r-1, \; j = i+1, \ldots, r. \tag{15}$$

In addition, the controller gain matrices can be designed as

$$F_i = K_i U^{-1}, \quad i = 1, 2, \ldots, r, \tag{16}$$

where

$$\Omega_{ij} = \begin{bmatrix} -\mathrm{sym}(G_{1i}) & \Gamma_1 + G_{1i}A_j^T - G_{2i}^T & 0 & \Gamma_2 & \\ * & \mathrm{sym}\left(G_{2i}A_j^T\right) & -\Gamma_2^T & 0 & \xi Q_{ij} + W_{ij}^T \\ * & * & -\mathrm{sym}(G_{3i}) & \Gamma_1 + G_{3i}A_j^T - G_{4i}^T & \\ * & * & * & \mathrm{sym}\left(G_{4i}A_j^T\right) & \\ * & * & * & * & -\mathrm{sym}(\xi \mathbb{U}) \end{bmatrix},$$

$$\begin{aligned} \Gamma_1 &= aP_{11} + bP_{12} + aP_{21} - bP_{22}, \\ \Gamma_2 &= -bP_{11} + aP_{12} + bP_{21} + aP_{22}, \\ a &= \sin\left(\frac{\pi\alpha}{2}\right), \\ b &= \cos\left(\frac{\pi\alpha}{2}\right), \\ Q_{ij} &= \mathrm{diag}\left\{G_{1i}C_j^T, G_{2i}C_j^T, G_{3i}C_j^T, G_{4i}C_j^T\right\}, \\ \mathbb{U} &= \mathrm{diag}\{U, U, U, U\}, \\ W_{ij} &= \mathrm{diag}\left\{\left[0, K_j^T B_i^T; 0, K_j^T B_i^T\right], \left[0, K_j^T B_i^T; 0, K_j^T B_i^T\right]\right\}. \end{aligned} \tag{17}$$

Proof. Let

$$\Omega = \begin{bmatrix} -\mathrm{sym}(\overline{G}_{1i}) & \Gamma_1 + \overline{G}_{1i}\overline{A}_i^T - \overline{G}_{2i}^T & 0 & \Gamma_2 & \\ * & \mathrm{sym}\left(\overline{G}_{2i}\overline{A}_i^T\right) & -\Gamma_2^T & 0 & \xi \overline{Q}_i + \overline{W}_{iu}^T \mathbb{U} \\ * & * & -\mathrm{sym}(\overline{G}_{3i}) & \Gamma_1 + \overline{G}_{3i}\overline{A}_i^T - \overline{G}_{4i}^T & \\ * & * & * & \mathrm{sym}\left(\overline{G}_{4i}\overline{A}_i^T\right) & \\ * & * & * & * & -\mathrm{sym}(\xi \mathbb{U}) \end{bmatrix}, \tag{18}$$

where $\overline{Q}_i = \mathrm{diag}\left\{\overline{G}_{1i}\overline{C}_i^T, \overline{G}_{2i}\overline{C}_i^T, \overline{G}_{3i}\overline{C}_i^T, \overline{G}_{4i}\overline{C}_i^T\right\}$, $\overline{W}_{iu} = \mathrm{diag}\{[0, U^{-T}\ \overline{K}_i^T \overline{B}_i^T; 0, U^{-T}\overline{K}_i^T \overline{B}_i^T], [0, U^{-T}\overline{K}_i^T \overline{B}_i^T; 0,\ U^{-T}\overline{K}_i^T \overline{B}_i^T]\}$, and $[\overline{A}_i, \overline{B}_i, \overline{C}_i, \overline{K}_i, \overline{G}_{1i}, \overline{G}_{2i}, \overline{G}_{3i}, \overline{G}_{4i}] = \sum_{i=1}^r h_i(\theta(t))[A_i, B_i, C_i, K_i, G_{1i}, G_{2i}, G_{3i}, G_{4i}]$.

According to inequalities (14) and (15) and the properties of the membership function, we can get

$$\begin{aligned}\Omega &= \sum_{i=1}^{r}\sum_{i=1}^{r} h_i(\theta(t))h_j(\theta(t))\Omega_{ij}\\ &= \sum_{i=1}^{r} h_i^2(\theta(t))\Omega_{ii} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} h_i(\theta(t))h_j(\theta(t))\left(\Omega_{ij}+\Omega_{ji}\right) < 0.\end{aligned} \tag{19}$$

Replacing $\overline{K}_i U^{-1}$ with $\overline{F}_i$ in Ω, we obtain

$$\begin{bmatrix} -\operatorname{sym}(\overline{G}_{1i}) & \Gamma_1 + \overline{G}_{1i}\overline{A}_i^T - \overline{G}_{2i}^T & 0 & \Gamma_2 & \\ * & \operatorname{sym}\left(\overline{G}_{2i}\overline{A}_i^T\right) & -\Gamma_2^T & 0 & \xi\overline{Q}_i + \overline{W}_{if}^T\mathbb{U} \\ * & * & -\operatorname{sym}(\overline{G}_{3i}) & \Gamma_1 + \overline{G}_{3i}\overline{A}_i^T - \overline{G}_{4i}^T & \\ * & * & * & \operatorname{sym}\left(\overline{G}_{4i}\overline{A}_i^T\right) & \\ * & * & * & * & -\operatorname{sym}(\xi\mathbb{U}) \end{bmatrix} < 0, \tag{20}$$

where $\overline{W}_{if} = \operatorname{diag}\left\{[0, \overline{F}_i^T\overline{B}_i^T; 0, \overline{F}_i^T\overline{B}_i^T], [0, \overline{F}_i^T\overline{B}_i^T; 0, \overline{F}_i^T\overline{B}_i^T]\right\}$.

According to Lemma 3, if inequality (20) holds, then

$$\begin{bmatrix} -\operatorname{sym}(\overline{G}_{1i}) & \Gamma_1 + \overline{G}_{1i}\overline{A}_i^T - \overline{G}_{2i}^T & 0 & \Gamma_2 \\ * & \operatorname{sym}\left(\overline{G}_{2i}\overline{A}_i^T\right) & -\Gamma_2^T & 0 \\ * & * & -\operatorname{sym}(\overline{G}_{3i}) & \Gamma_1 + \overline{G}_{3i}\overline{A}_i^T - \overline{G}_{4i}^T \\ * & * & * & \operatorname{sym}\left(\overline{G}_{4i}\overline{A}_i^T\right) \end{bmatrix} + \operatorname{sym}\left(\overline{Q}_i\overline{W}_{if}\right) < 0. \tag{21}$$

Let

$$\widetilde{A} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(\theta(t))h_j(\theta(t))h_k(\theta(t))\left(A_i + B_iF_jC_k\right). \tag{22}$$

Then, $\widetilde{A} = \overline{A}_i + \overline{B}_i\overline{F}_i\overline{C}_i$.

Therefore, inequality (21) can be rewritten as

$$\begin{bmatrix} 0 & \Gamma_1 & 0 & \Gamma_2 \\ * & 0 & -\Gamma_2^T & 0 \\ * & * & 0 & \Gamma_1 \\ * & * & * & 0 \end{bmatrix} + \operatorname{sym}\left(\begin{bmatrix} \overline{G}_{1i} & 0 \\ \overline{G}_{2i} & 0 \\ 0 & \overline{G}_{3i} \\ 0 & \overline{G}_{4i} \end{bmatrix}\begin{bmatrix} -I & \widetilde{A}^T & 0 & 0 \\ 0 & 0 & -I & \widetilde{A}^T \end{bmatrix}\right) < 0. \tag{23}$$

Taking $\Psi = \begin{bmatrix} \widetilde{A} & I & 0 & 0 \\ 0 & 0 & \widetilde{A} & I \end{bmatrix}$ and using Lemma 4, we have

$$\Psi\begin{bmatrix} 0 & \Gamma_1 & 0 & \Gamma_2 \\ * & 0 & -\Gamma_2^T & 0 \\ * & * & 0 & \Gamma_1 \\ * & * & * & 0 \end{bmatrix}\Psi^T + \operatorname{sym}\left(\Psi\begin{bmatrix} \overline{G}_{1i} & 0 \\ \overline{G}_{2i} & 0 \\ 0 & \overline{G}_{3i} \\ 0 & \overline{G}_{4i} \end{bmatrix}\begin{bmatrix} -I & \widetilde{A}^T & 0 & 0 \\ 0 & 0 & -I & \widetilde{A}^T \end{bmatrix}\Psi^T\right) < 0,$$

which is equivalent to

$$\begin{bmatrix} \widetilde{A}\Gamma_1 + \Gamma_1^T\widetilde{A}^T & \widetilde{A}\Gamma_2 - \Gamma_2^T\widetilde{A}^T \\ * & \widetilde{A}\Gamma_1 + \Gamma_1^T\widetilde{A}^T \end{bmatrix} < 0. \tag{24}$$

The above inequality can be rewritten as

$$\operatorname{sym}\left\{\begin{bmatrix} \widetilde{A}\Gamma_1 & \widetilde{A}\Gamma_2 \\ -\widetilde{A}\Gamma_2 & \widetilde{A}\Gamma_1 \end{bmatrix}\right\} < 0. \tag{25}$$

According to the definition of the Kronecker product, one has from (25)

$$\begin{aligned}\operatorname{sym}\Bigg\{&\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\otimes\left(\widetilde{A}P_{11}\right) + \begin{bmatrix} b & a \\ -a & b \end{bmatrix}\otimes\left(\widetilde{A}P_{12}\right)\\ &+ \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\otimes\left(\widetilde{A}P_{21}\right) + \begin{bmatrix} -b & a \\ -a & -b \end{bmatrix}\otimes\left(\widetilde{A}P_{22}\right)\Bigg\} < 0.\end{aligned} \tag{26}$$

According to Lemma 1, if inequalities (13) and (26) hold, closed-loop system (12) with $0 < \alpha < 1$ is asymptotically stable. This completes the proof. □

Remark 1. Compared with [19–26], this paper studies the design of the fuzzy static output feedback controller for fractional-order fuzzy systems. In general, the fuzzy static output feedback controller will generate the term $B_iF_jC_k$ $(i, j, k = 1, 2, \ldots, r)$, which increases the fuzzy r-times relationship. At the same time, the controller gain matrix is located between the two matrices, which makes the design of the control more difficult. In this paper, Lemma 3 is used to ingeniously separate this item and eliminate the above-mentioned difficulties. Furthermore, the relationship between fuzzy systems is considered.

Considering the relationship between fuzzy subsystems, we can prove the following conclusions.

Theorem 2. *Fractional-order closed-loop system (12) with $0<\alpha<1$ is asymptotically stable if there is scalar $\xi>0$, proper dimensional matrices $U, G_{1i}, G_{2i}, G_{3i}, G_{4i}, K_i, Z_{ij}$ $(i=1,2,\ldots,r)$, symmetric matrices P_{11}, P_{21}, Z_{ii}, and skew-symmetric matrices P_{21}, P_{22} such that (13) and the following LMIs hold:*

$$\Omega_{ii}<Z_{ii},\quad i=1,2,\ldots,r, \tag{27}$$

$$\begin{gathered}\Omega_{ij}+\Omega_{ji}<Z_{ij}+Z_{ij}^T,\quad i=1,2,\ldots,r-1,\ j=i+1,\ldots,r,\\ Z=\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1r}\\ Z_{12}^T & Z_{22} & \cdots & Z_{2r}\\ \vdots & \vdots & \ddots & \vdots\\ Z_{1r}^T & Z_{2r}^T & \cdots & Z_{rr}\end{bmatrix}<0.\end{gathered} \tag{28}$$

If the above matrix inequalities hold, the controller gain matrices can be designed as

$$F_i=K_iU^{-1},\quad i=1,2,\ldots,r. \tag{29}$$

Proof. According to inequalities (27) and (28), we can easily get

$$\begin{aligned}&\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\theta(t))h_j(\theta(t))\Omega_{ij}\\ &\quad=\sum_{i=1}^{r}h_i^2(\theta(t))\Omega_{ii}+\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}h_i(\theta(t))h_j(\theta(t))\left(\Omega_{ij}+\Omega_{ji}\right)\\ &\quad<\sum_{i=1}^{r}h_i^2(\theta(t))Z_{ii}+\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}h_i(\theta(t))h_j(\theta(t))\left(Z_{ij}+Z_{ij}^T\right)\\ &\quad=\left[h_1(\theta(t)),h_2(\theta(t)),\ldots,h_r(\theta(t))\right]\\ &\qquad\cdot Z\left[h_1(\theta(t)),h_2(\theta(t)),\ldots,h_r(\theta(t))\right]^T.\end{aligned} \tag{30}$$

If $Z<0$ holds, there is $\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\theta(t))h_j(\theta(t))\Omega_{ij}<0$, i.e., inequality (19) holds. According to the proof process of Theorem 1, closed-loop system (12) with 0 asymptotically stable. This completes the proof. □

For fractional-order system (12) with order greater than zero and less than one, two new controller design methods are proposed. Using the similar method, when the order is greater than or equal to 1 but less than 2, we can prove the following results. □

Theorem 3. *Fractional-order closed-loop system (12) with $1\le\alpha<2$ is asymptotically stable if there is scalar $\xi>0$, proper dimensional matrices $U, G_{1i}, G_{2i}, G_{3i}, G_{4i}, K_i$ $(i=1,2,\ldots,r)$, symmetric positive definite matrices P_i $(i=1,2,\ldots,r)$, and the following LMIs are satisfied:*

$$\Xi_{ii}<0,\quad i=1,2,\ldots,r, \tag{31}$$

$$\begin{gathered}\Xi_{ij}+\Xi_{ji}<0,\quad i=1,2,\ldots,r-1, j=i+1,\ldots,r,\\ F_i=K_iU^{-1},\quad i=1,2,\ldots,r,\end{gathered} \tag{32}$$

where

$$\Omega_{ij}=\begin{bmatrix}-\mathrm{sym}(G_{1i}) & cP_i+G_{1i}A_j^T-G_{2i}^T & 0 & dP_i & \\ * & \mathrm{sym}\left(G_{2i}A_j^T\right) & -dP_i & 0 & \xi Q_{ij}+W_{ij}^T\\ * & * & -\mathrm{sym}(G_{3i}) & cP_i+G_{3i}A_j^T-G_{4i}^T & \\ * & * & * & \mathrm{sym}\left(G_{4i}A_j^T\right) & \\ * & * & * & * & -\mathrm{sym}(\xi\mathbb{U})\end{bmatrix}, \tag{33}$$

$c=\sin(\pi-\pi\alpha/2), d=\cos(\pi-\pi\alpha/2)$, and the definitions of other symbols are the same as those in Theorem 1.

Proof. Let $\overline{P}_i=\sum_{i=1}^{r}h_i(\theta(t))P_i$. Because every P_i is a symmetric positive definite matrix and the membership function is greater than 0, $\overline{P}_i$ is also a symmetric positive definite matrix.

In the process of proving Theorem 2, let $\Gamma_1=c\overline{P}_i$ and $\Gamma_2=d\overline{P}_i$; starting from inequalities (31) and (32), we can get the following results from (24):

$$\begin{bmatrix}c\left(\widetilde{A}_{ijk}\overline{P}_i+\overline{P}_i\widetilde{A}_{ijk}^T\right) & d\left(\widetilde{A}_{ijk}\overline{P}_i-\overline{P}_i\widetilde{A}_{ijk}^T\right)\\ * & c\left(\widetilde{A}_{ijk}\overline{P}_i+\overline{P}_i\widetilde{A}_{ijk}^T\right)\end{bmatrix}<0. \tag{34}$$

According to Lemma 2, Theorem 3 ensures that closed-loop system (12) with $1\le\alpha<2$ is asymptotically stable. This completes the proof. □

Similar to Theorem 1, we can prove the following result. □

Theorem 4. *Fractional-order closed-loop system (12) is asymptotically stable if there is scalar $\xi > 0$, proper dimensional matrices $U, G_{1i}, G_{2i}, G_{3i}, G_{4i}, K_i$, $Z_{ij}\,(i = 1, 2, \ldots, r)$, and symmetric positive definite matrices $P_i\,(i = 1, 2, \ldots, r)$ such that the following linear matrix inequalities hold:*

$$\Xi_{ii} < Z_{ii}, \quad i = 1, 2, \ldots, r, \tag{35}$$

$$\Xi_{ij} + \Omega_{ji} < Z_{ij} + Z_{ij}^T, \quad i = 1, 2, \ldots, r-1,\ j = i+1, \ldots, r. \tag{36}$$

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1r} \\ Z_{12}^T & Z_{22} & \cdots & Z_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{1r}^T & Z_{2r}^T & \cdots & Z_{rr} \end{bmatrix} < 0. \tag{37}$$

The corresponding controller can be selected as

$$F_i = K_i U^{-1}, \quad i = 1, 2, \ldots, r. \tag{38}$$

Proof. The proof process of Theorem 4 is exactly the same as that of Theorem 2, so it is omitted here. □

5. Numerical Example

In order to illustrate the effectiveness of the proposed methods, a numerical simulation example is given in this part. Since the simulation methods are similar, we only verify Theorem 1.

Example 1. For system (9), the corresponding simulation parameters are selected as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & 1 \\ -2 & -1 & 2 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} -6 & -3 & 4 \\ 1 & 0 & 0 \\ -2 & -2 & 2 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 0.8 \\ -3 \\ 3.8 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} -2 & 1 & 0 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} 4 & -3 & 2 \end{bmatrix}. \end{aligned} \tag{39}$$

Membership functions are taken as $h_1(\theta(t)) = 0.5(1-\sin(x_1(t)))$, $h_2(\theta(t)) = 1 - h_1(\theta(t))$.

Take fractional order $\alpha = 0.75$ and initial condition of the system $x_0 = [1, -3, 1.8]^T$. When the controller $u(t) = 0$, the state trajectory of the system is shown in Figure 1. Obviously, the system is unstable.

According to Theorem 1, the following feasible solutions can be obtained by using the linear matrix inequality toolbox and making $\xi = 1$:

$$\begin{aligned} P_{11} = P_{21} &= \begin{bmatrix} 47.9949 & -4.9576 & 10.1684 \\ -4.9576 & 9.6889 & 7.9151 \\ 10.1684 & 7.9151 & 25.3468 \end{bmatrix}, \\ P_{12} = P_{22} &= \begin{bmatrix} 0 & -1.0326 & 5.4116 \\ 1.0326 & 0 & -2.1499 \\ -5.4116 & 2.1499 & 0 \end{bmatrix}, \\ G_{11} &= \begin{bmatrix} 17.3758 & -2.2678 & -11.9404 \\ -0.8326 & 27.6204 & -8.6416 \\ 15.0427 & 23.0184 & 17.4726 \end{bmatrix}, \\ G_{12} &= \begin{bmatrix} 6.3362 & -3.5964 & -15.0566 \\ 0.6207 & 32.8234 & 14.5413 \\ 9.2638 & 33.9112 & 34.1965 \end{bmatrix}, \\ G_{21} &= \begin{bmatrix} 9.6221 & -16.8697 & -25.2363 \\ -20.4052 & 4.4510 & -25.1853 \\ 21.6206 & 42.4479 & 27.6159 \end{bmatrix}, \\ G_{22} &= \begin{bmatrix} 11.1564 & -5.6354 & -5.2083 \\ -8.3296 & 7.0806 & -4.8982 \\ 10.2057 & 25.5957 & 24.3865 \end{bmatrix}, \\ G_{31} &= \begin{bmatrix} 25.1420 & 8.5343 & -88.6732 \\ 13.2014 & 34.9178 & -175.9623 \\ 79.6915 & 184.1028 & 38.7372 \end{bmatrix}, \\ G_{32} &= \begin{bmatrix} 23.7128 & 192.8158 & 254.9889 \\ -163.7837 & 31.1085 & 364.0394 \\ -275.7049 & -347.9581 & 36.6636 \end{bmatrix}, \\ G_{41} &= \begin{bmatrix} -19.3450 & -67.0576 & -2.1058 \\ -32.3201 & -11.7544 & -7.8586 \\ 1.8301 & -29.0457 & -38.1671 \end{bmatrix}, \\ G_{42} &= \begin{bmatrix} -14.0099 & -123.5301 & -127.9458 \\ -13.0018 & -23.2141 & -30.4210 \\ 6.7847 & 9.8925 & -0.7920 \end{bmatrix}, \\ K_1 &= 6.1395, \\ K_2 &= -21.5572, \\ U &= 101.1738. \end{aligned} \tag{40}$$

Gain matrices of the controller can be obtained by calculation:

$$\begin{aligned} F_1 &= K_1^{-1} U^{-1} = 0.0607, \\ F_2 &= K_2^{-1} U^{-1} = -0.2131. \end{aligned} \tag{41}$$

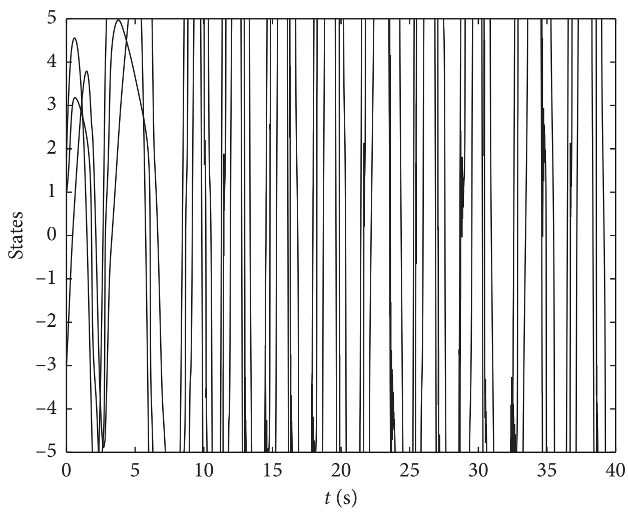

Figure 1: The state trajectories of the system when the controller $u(t) = 0$.

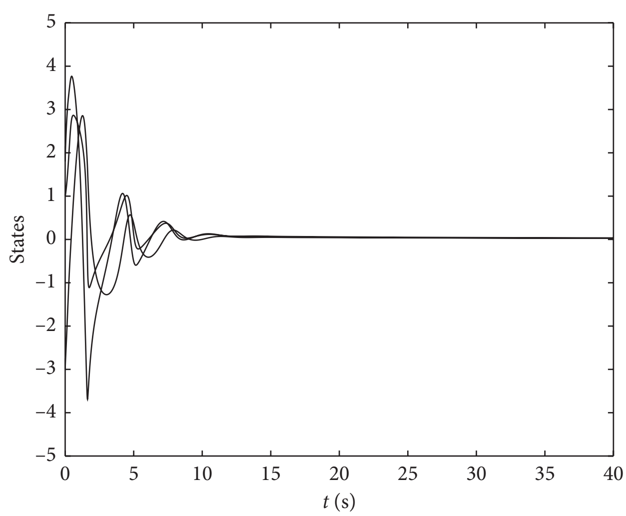

Figure 2: State trajectories of the system after adding the controller.

Using fuzzy controller (11), the state trajectories of closed-loop system (12) can be obtained as shown in Figure 2. As can be seen from Figure 2, the controller designed is effective.

6. Conclusion

For the fractional-order T-S fuzzy system, the controller design methods with order in two different intervals are studied. Four theorems are given to ensure that the closed-loop system is asymptotically stable. The result is expressed by the linear matrix inequality, which fully considers the feasibility and conservatism. In order to consider the feasibility, the condition of each theorem is the strict linear matrix inequality. This can be solved directly by the linear matrix inequality toolbox of Matlab. In order to reduce the conservatism, we try to make the matrix correspond to the fuzzy rule. Theorems 2 and 4 further consider the relationship between fuzzy subsystems.

Data Availability

The simulation results of this paper can be obtained by MATLAB software.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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