Research Article

An Accelerated Conjugate Gradient Algorithm for Solving Nonlinear Monotone Equations and Image Restoration Problems

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Received 6 July 2020; Accepted 25 August 2020; Published 5 October 2020

Guest Editor: Wenjie Liu

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Combining the three-term conjugate gradient method of Yuan and Zhang and the acceleration step length of Andrei with the hyperplane projection method of Solodov and Svaiter, we propose an accelerated conjugate gradient algorithm for solving nonlinear monotone equations in this paper. The presented algorithm has the following properties: (i) All search directions generated by the algorithm satisfy the sufficient descent and trust region properties independent of the line search technique. (ii) A derivative-free search technique is proposed along the direction to obtain the step length α_k . (iii) If $\phi_k = -\alpha_k(h_k - h(w_k))^T d_k > 0$, then an acceleration scheme is used to modify the step length in a multiplicative manner and create a point. (iv) If the point satisfies the given condition, then it is the next point; otherwise, the hyperplane projection technique is used to obtain the next point. (v) The global convergence of the proposed algorithm is established under some suitable conditions. Numerical comparisons with other conjugate gradient algorithms show that the accelerated computing scheme is more competitive. In addition, the presented algorithm can also be applied to image restoration.

1. Introduction

In this paper, the following nonlinear equation is considered:

$$h(x) = 0, \text{ subject to } x \in \mathcal{R}^n, \quad (1)$$

where $h: \mathcal{R}^n \rightarrow \mathcal{R}^n$ is continuous and monotone, and $h(x)$ satisfies

$$(h(x) - h(y))^T (x - y) \geq 0, \forall x, y \in \mathcal{R}^n. \quad (2)$$

It is not difficult to show that the solution set of monotone equation (1), unless empty, is convex. This problem has many significant applications in applied mathematics, economics, and engineering. For example, the economic equilibrium problem [1] can be transformed into problem (1). Generally, an iteration formula generates the next iteration point by

$$x_{k+1} = x_k + \alpha_k d_k, \quad (3)$$

where α_k is the step length and d_k is a search direction, which are two important factors for solving nonlinear equations.

Some derivative-free line search techniques [2–5] were proposed to search for step length α_k . Li and Li [6] presented a derivative-free line search to find $\alpha_k = \max\{\rho s^i: i = 0, 1, \dots, \}$ such that

$$-h(x_k + \alpha_k d_k)^T d_k \geq \lambda \alpha_k \|h(x_k + \alpha_k d_k)\| \|d_k\|^2, \quad (4)$$

where $\rho, \lambda > 0$ and $s \in (0, 1)$. $\|\cdot\|$ represents the Euclidean norm. The line search technique (4) is different from other existing derivative-free line search techniques because it does not use a merit function. If d_k satisfies $h(x_k)^T d_k < 0$, the inequality (4) holds for all sufficiently small $\alpha_k > 0$. As a result, α_k can be obtained by some backtracking processes.

For d_k , it is well known that the Newton methods [7, 8], quasi-Newton methods [9–13], spectral gradient methods [14, 15], and conjugate gradient methods [16, 17] can deal with large-scale nonlinear equations. For solving large-scale optimization problems, the conjugate gradient methods are quite effective since they only calculate and store the gradient value of the objective function. Many scholars have applied conjugate gradient theory to solve nonlinear

monotone equations and have achieved good results [17–19]. Classical conjugate gradient methods include HS method [20], FR method [21], PRP method [22, 23], LS method [24], CD method [25], and DY method [26]. In particular, the PRP method, as one of the most effective methods, generates a small step near a minimum point; then, the subsequence generated for the search direction d_k will automatically approach the negative gradient direction, which avoids continuously generating a small step. However, the global convergence of the PRP method is not established under inexact line search techniques for general functions. Many scholars have performed continuous research and have reached satisfactory conclusions. Zhang [27] proposed the MPRP method, where d_k is designed as follows:

$$d_k = \begin{cases} -h_k + \beta_k^{\text{PRP}} d_{k-1} - \theta_k y_k, & \text{if } k \geq 1, \\ -h_k, & \text{if } k = 0, \end{cases} \quad (5)$$

where $h(x_k) = h_k$, $h(x_{k-1}) = h_{k-1}$, $y_k = h_k - h_{k-1}$, $\beta_k^{\text{PRP}} = h_k^T y_k / \|h_{k-1}\|^2$, and $\theta_k = h_k^T d_{k-1} / \|h_{k-1}\|^2$. It is easy to obtain from (5) that

$$d_k^T h_k = -\|h_k\|^2. \quad (6)$$

The above equation indicates that d_k is a descent direction of h at x_k . If the exact line search is used, then we have $\theta_k = 0$. Consequently, formula (5) is inferred to be the standard PRP method. Under some mild conditions, the MPRP method is globally convergent under the Armijo-type line search, but global convergence cannot be established under the weak Wolfe–Powell line search. The main reason is that the MPRP method does not satisfy the trust region property. Inspired by the above discussions, Yuan and Zhang [28] proposed a three-term PRP (TTPRP) method in which d_k is defined by

$$d_k = \begin{cases} -h_k + \frac{h_k^T y_k d_{k-1} - h_k^T d_{k-1} y_k}{\max\{\kappa \|d_{k-1}\| \|y_k\|, \|h_{k-1}\|^2\}}, & \text{if } k \geq 1, \\ -h_k, & \text{if } k = 0, \end{cases} \quad (7)$$

where $\kappa > 0$ is a constant. It is worth noting that the denominator $\|h_{k-1}\|^2$ in the formula of the MPRP method is adjusted to $\max\{\kappa \|d_{k-1}\| \|y_k\|, \|h_{k-1}\|^2\}$ in the formula of the TTPRP method. The TTPRP method automatically maintains the trust region property. Its global convergence is also established under certain conditions. Numerical results show that the TTPRP method is effective for large-scale nonlinear monotone equations.

In addition, the hyperplane projection method [17, 29–32] is the most effective method for solving large-scale nonlinear monotone equations. The concept of projection was first proposed by Goldstein [33] for convex programming in a Hilbert space. Furthermore, Solodov and Svaiter [34] proposed the hyperplane projection method for solving optimization problems. The specific process of the hyperplane projection method is as follows: let x_k be the current iteration point, and obtain a point $w_k = x_k + \alpha_k d_k$

along a certain line search direction d_k such that $h(w_k)^T (x_k - w_k) > 0$. Due to the monotonicity of $h(x)$, for a certain point $\bar{x}$ that satisfies $h(\bar{x}) = 0$, it can be deduced that

$$h(w_k)^T (\bar{x} - tw_k) \leq 0. \quad (8)$$

Obviously, the hyperplane $H_k = \{x \in \mathcal{R}^n \mid \langle h(w_k), x - w_k \rangle = 0\}$ strictly separates the current iteration point x_k from the solution set of the equation (1). The point x_k is projected onto the hyperplane H_k to obtain the next iteration point x_{k+1} , i.e.,

$$x_{k+1} = x_k - \frac{h(w_k)^T (x_k - w_k)}{\|h(w_k)\|^2} h(w_k). \quad (9)$$

The hyperplane projection method has been proved to possess good theoretical properties and numerical performance for nonlinear monotone equations [6, 34].

Furthermore, we know that the search directions tend to be poorly scaled in conjugate gradient methods. Consequently, in the line search, more function evaluations must be carried out to obtain an appropriate step length α_k . Andrei [35] presented an acceleration scheme that modifies the step length α_k in a multiplicative manner to improve the reduction of the function values along the iterations. The step length is defined as follows:

$$\tilde{\alpha}_k = \xi_k \alpha_k, \quad (10)$$

where $\xi_k = -\varphi_k / \phi_k$, $\varphi_k = \alpha_k h_k^T d_k$, $\phi_k = -\alpha_k (h_k - h(w_k))^T d_k$. If $\phi_k > 0$, let $\alpha_k = \tilde{\alpha}_k$. A numerical comparison with some conjugate gradient algorithms shows that the computational scheme is effective.

Inspired by the above discussions, we proposed an accelerated conjugate gradient algorithm that combines the TTPRP method, the acceleration step length, and the hyperplane projection method. The main contributions of the algorithm are as follows:

An accelerated conjugate gradient algorithm is introduced for solving nonlinear monotone equations

All search directions of the algorithm satisfy the sufficient descent condition

All search directions of the algorithm belong to a trust region

The global convergence of the presented algorithm is proved

The numerical results show that the proposed algorithm is more effective for nonlinear monotone equations

The algorithm can be applied to restore an original image from an image damaged by impulse noise

This paper is organized as follows: in the next section, we discuss the ATTTPRP algorithm and global convergence analysis. In Section 3, we report the preliminary numerical experiments to show that the algorithm is efficient for nonlinear monotone equations and applicable to image restoration problems. In Section 4, the conclusion regarding the proposed algorithm is given.

2. Accelerated Algorithm and Convergence Analysis

In this section, we will propose an accelerated algorithm and prove its global convergence. The steps of the given algorithm are as follows.

2.1. Accelerated Three-Term PRP Conjugate Gradient (ATTPRP) Algorithm

Step 0: choose any $x_0 \in \mathfrak{R}^n$ as the initial point and constants $s \in (0, 1)$, $\rho > 0$, $\lambda > 0$, $\kappa > 0$, and $\varepsilon \in (0, 1)$, let $k := 0$.

Step 1: stop if $\|h_k\| \leq \varepsilon$. Otherwise, compute d_k by using formula (5).

Step 2: choose α_k satisfying the inequality (4).

Step 3: if $\phi_k > 0$, then $\alpha_k = \tilde{\alpha}_k$.

Step 4: let the next iterative value be $w_k = x_k + \alpha_k d_k$.

Step 5: if $\|h(w_k)\| \leq \varepsilon$, stop and let $x_{k+1} = w_k$. Otherwise, determine x_{k+1} using formula (9).

Step 6: let $k := k + 1$. Go to Step 1.

The following lemma shows that the search direction d_k designed by using formula (7) has not only the sufficient descent property but also the trust region property independent of the line search.

Lemma 1. d_k is defined by using formula (7); then, we obtain

$$d_k^T h_k = -\|h_k\|^2, \quad (11)$$

$$\|h_k\| \leq \|d_k\| \leq \left(1 + \frac{2}{\kappa}\right) \|h_k\|. \quad (12)$$

Proof. If $k = 0$, formulas (7) and (12) are obviously true. If $k \geq 1$, we obtain from formula (7) that

$$\begin{aligned} h_k^T d_k &= h_k^T \left[-h_k + \frac{h_k^T y_k d_{k-1} - h_k^T d_{k-1} y_k}{\max\{\kappa \|d_{k-1}\| \|y_k\|, \|h_{k-1}\|^2\}} \right] \\ &= -\|h_k\|^2 + \frac{h_k^T y_k d_{k-1}^T h_k - h_k^T d_{k-1} y_k^T h_k}{\max\{\kappa \|d_{k-1}\| \|y_k\|, \|h_{k-1}\|^2\}} = -\|h_k\|^2. \end{aligned} \quad (13)$$

In addition, by formula (7), we get $\|d_k\| \geq \|h_k\|$ and

$$\begin{aligned} \|d_k\| &= \left\| -h_k + \frac{h_k^T y_k d_{k-1} - h_k^T d_{k-1} y_k}{\max\{\kappa \|d_{k-1}\| \|y_k\|, \|h_{k-1}\|^2\}} \right\| \\ &\leq \|h_k\| + \frac{\|h_k\| \|y_k\| \|d_{k-1}\| + \|h_k\| \|d_{k-1}\| \|y_k\|}{\max\{\kappa \|d_{k-1}\| \|y_k\|, \|h_{k-1}\|^2\}} \\ &\leq \|h_k\| + \frac{2\|h_k\| \|y_k\| \|d_k\|}{\kappa \|d_k\| \|y_k\|} = \left(1 + \frac{2}{\kappa}\right) \|h_k\|, \end{aligned} \quad (14)$$

where $\max\{\kappa \|d_{k-1}\| \|y_k\|, \|h_{k-1}\|^2\} \geq \kappa \|d_k\| \|y_k\|$. Then, the proof is completed.

The following assumption need to be established in order to study some properties of the ATTPRP algorithm: $\square$

Assumption 1.

- (i) The solution set of the problem (1) is nonempty.
- (ii) The function $h(x)$ is Lipschitz continuous on $\mathfrak{R}^n$; that is, there exists a positive constant K satisfying

$$\|h(x) - h(y)\| \leq K \|x - y\|, \quad \forall x, y \in \mathfrak{R}^n. \quad (15)$$

Remark 1. Assumption 1(ii) implies that $\{\|h_k\|\}$ is bounded; then, there exists a constant φ such that

$$\|h_k\| \leq \varphi. \quad (16)$$

In the following paper, if not specifically stated, we always assume that the conditions in Assumption 1 hold.

Lemma 2. Let $\{x_k\}$ and $\{w_k\}$ be generated by using the ATTPRP algorithm. The step length α_k generated by the ATTPRP algorithm satisfies

$$\alpha_k \geq \min \left\{ \rho, \frac{s}{K + \lambda \|h(w'_k)\|} \frac{\|h_k\|^2}{\|d_k\|^2} \right\}, \quad (17)$$

where $w'_k = x_k + \alpha'_k d_k$ and $\alpha'_k = \alpha_k s^{-1}$.

Proof. By line search (4), assuming $\alpha_k \neq \rho$, let $\alpha'_k \triangleq \alpha_k s^{-1}$, by the definition α'_k , does not satisfy the line search (4). That is,

$$-h(w'_k)^T d_k < \lambda \alpha'_k \|h(w'_k)\| \|d_k\|^2. \quad (18)$$

Since $h(x)$ is Lipschitz continuous and by using formula (11), we have

$$\begin{aligned} \|h_k\|^2 &= -h_k^T d_k = (h(w'_k) - h_k - h(w'_k))^T d_k \\ &= (h(w'_k) - h_k)^T d_k - h(w'_k)^T d_k \\ &\leq K \|w'_k - x_k\| \|d_k\| + \lambda \alpha'_k \|h(w'_k)\| \|d_k\|^2 \\ &= K \alpha'_k \|d_k\|^2 + \lambda \alpha'_k \|h(w'_k)\| \|d_k\|^2 \\ &= \alpha'_k (K + \lambda \|h(w'_k)\|) \|d_k\|^2 \\ &= \alpha_k s^{-1} (K + \lambda \|h(w'_k)\|) \|d_k\|^2, \end{aligned} \quad (19)$$

namely,

$$\alpha_k \geq \frac{s}{K + \lambda \|h(w'_k)\|} \frac{\|h_k\|^2}{\|d_k\|^2}. \quad (20)$$

This leads to the ideal inequality (17). The proof is completed.

The following lemma is similar to Lemma 1 in the study of Solodov and Svaiter [34], which also holds for the

ATTPRP algorithm. Therefore, we only state it as follows but omit its proof. $\square$

Lemma 3. *Let the sequence $\{x_k\}$ be generated by using the ATTPRP algorithm. Suppose that $\bar{x}$ is a solution of problem (1) with $h(\bar{x}) = 0$. We obtain*

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_k - \bar{x}\|^2 - \|x_{k+1} - x_k\|^2. \quad (21)$$

In particular, the sequence $\{x_k\}$ is bounded, and

$$\sum_{k=0}^{\infty} \|x_{k+1} - x_k\|^2 < \infty. \quad (22)$$

Remark 2. The above lemma reveals that the distance from the iterative points to the solution set of the problem (1) decreases along iterations. Otherwise, for any k , it is followed from formulas (9) and (4) that

$$\|x_{k+1} - x_k\| = \frac{|h(w_k)^T (x_k - w_k)|}{\|h(w_k)\|} = \frac{-\alpha_k h(w_k)^T d_k}{\|h(w_k)\|} \geq \lambda \alpha_k^2 \|d_k\|^2. \quad (23)$$

Particularly, we obtain

$$\lim_{k \rightarrow \infty} \alpha_k \|d_k\| = 0. \quad (24)$$

In the following part, the global convergence and the strong global convergence properties of the ATTPRP algorithm will be proven.

Theorem 1. *Let $\{x_k\}$ be generated by using the ATTPRP algorithm. Then, we have*

$$\lim_{k \rightarrow \infty} \inf \|h_k\| = 0. \quad (25)$$

Proof. We will prove this theorem by contradiction. Supposing that the equation (25) does not hold, there exists a constant $\eta > 0$ such that $\|h_k\| \geq \eta$ holds for all $k \geq 0$. From formula (12), we have

$$\|d_k\| \geq \|h_k\| \geq \eta, \quad \forall k \geq 0. \quad (26)$$

According to Lemma 3 and equation (24), the sequences $\{x_k\}$ and $\{w_k\}$ are bounded. By formulas (7) and (16), for all $k \geq 1$, we obtain

$$\begin{aligned} \|d_k\| &\leq \|h_k\| + \left\| \frac{h_k^T y_k d_{k-1} - h_k^T d_{k-1} y_k}{\max\{\kappa \|d_{k-1}\| \|y_k\|, \|h_{k-1}\|^2\}} \right\| \\ &\leq \|h_k\| + \frac{\|h_k\| \|y_k\| \|d_{k-1}\| + \|h_k\| \|d_{k-1}\| \|y_k\|}{\max\{\kappa \|d_{k-1}\| \|y_k\|, \|h_{k-1}\|^2\}} \\ &\leq \|h_k\| + \frac{\|h_k\| \|y_k\| \|d_{k-1}\| + \|h_k\| \|d_{k-1}\| \|y_k\|}{\|h_{k-1}\|^2} \\ &\leq \varphi + 2\varphi\eta^{-2}K \|x_k - x_{k-1}\| \|d_{k-1}\|, \end{aligned} \quad (27)$$

where $\max\{\kappa \|d_{k-1}\| \|y_k\|, \|h_{k-1}\|^2\} \geq \|h_{k-1}\|^2$. Since $\{\|x_{k+1} - x_k\|\}$ converges to zero, the last inequality shows that $\{\|d_k\|\}$ is bounded. By formulas (12) and (16), we obtain

$$\|d_k\| \leq M, \quad (28)$$

where $M = (1 + 2/\kappa)\varphi$. Thus, from the formulas (16) and (17), we obtain that

$$\begin{aligned} \alpha_k \|d_k\| &\geq \min \left\{ \rho, \frac{s}{K + \lambda \|h(w_k')\|} \frac{\|h_k\|^2}{\|d_k\|^2} \right\} \|d_k\| \\ &\geq \min \left\{ \rho\eta, \frac{s\eta^2}{(K + \lambda\varphi)M} \right\} > 0. \end{aligned} \quad (29)$$

This contradicts with formula (24). Consequently, the proof is completed.

The following theorem indicates the strong global convergence of the ATTPRP algorithm, which is similar to Theorem 1 in [6]. We also give a specific proof for convenience of understanding. $\square$

Theorem 2. *Let $\{x_k\}$ be generated by using the ATTPRP algorithm. Then, the whole sequence $\{x_k\}$ converges to a solution of the problem (1).*

Proof. Theorem 1 shows that there exists a subsequence of $\{x_k\}$ converging to a solution $\bar{x}$ of the problem (1). On the other side, it follows from Lemma 3 that the sequence $\{\|x_{k+1} - \bar{x}\|\}$ converges. Therefore, the whole sequence $\{x_k\}$ converges to $\bar{x}$. $\square$

3. Numerical Experiments

In this section, the numerical experiments will be divided into two parts for illustration. The first subsection involves normal nonlinear equations, and the second subsection describes image restoration problems. All tests in this section are coded in MATLAB R2017a, run on a PC with Intel (R) Core (TM) i5-4460 3.20 GHz, 8.00 GB of SDRAM memory, and Windows 7 operating system.

3.1. Normal Nonlinear Equations. In this subsection, we perform some numerical experiments to show the effectiveness of the ATTPRP algorithm. Some test problems and their relevant initial points are listed as follows:

$$h(x) = (f_1(x), f_2(x), \dots, f_n(x))^T. \quad (30)$$

Function 1. Exponential Function 1:

$$\begin{aligned} f_1(x) &= e^{x_1-1} - 1, \\ f_i(x) &= i(e^{x_i-1} - x_i), \quad i = 2, 3, \dots, n. \end{aligned} \quad (31)$$

Initial guess: $x_0 = (n/n - 1, n/n - 1, \dots, n/n - 1)^T$.

Function 2. Exponential Function 2:

$$\begin{aligned} f_1(x) &= e^{x_1} - 1, \\ f_i(x) &= \frac{i(e^{x_i} + x_{i-1} - 1)}{10}, \quad i = 2, 3, \dots, n. \end{aligned} \quad (32)$$

Initial guess: $x_0 = (1/n^2, 1/n^2, \dots, 1/n^2)^t$.

Function 3. Singular function:

$$\begin{aligned} f_1(x) &= \frac{x_1^3}{3} + \frac{x_2^2}{2}, \\ f_i(x) &= -\frac{x_i^2}{2} + \frac{i}{3}x_i^3 + \frac{x_{i+1}^2}{2}, \quad i = 2, 3, \dots, n-1, \\ f_n(x) &= -\frac{x_n^2}{2} + \frac{n}{3}x_n^3. \end{aligned} \quad (33)$$

Initial guess: $x_0 = (1, 1, \dots, 1)^t$.

Function 4. Logarithmic function:

$$f_i(x) = \ln(x_i + 1) - \frac{x_i}{n}, \quad i = 1, 2, \dots, n. \quad (34)$$

Initial guess: $x_0 = (1, 1, \dots, 1)^t$.

Function 5. Broyden tridiagonal function:

$$\begin{aligned} f_1(x) &= (3 - 0.5x_1)x_1 - 2x_2 + 1, \\ f_i(x) &= (3 - 0.5x_i)x_i - x_{i-1} - 2x_{i+1} + 1, \quad i = 2, 3, \dots, n-1, \\ f_n(x) &= (3 - 0.5x_n)x_n - x_{n-1} + 1. \end{aligned} \quad (35)$$

Initial guess: $x_0 = (-1, -1, \dots, -1)^t$.

Function 6. Trigexp function:

$$\begin{aligned} f_1(x) &= 3x_1^3 + 2x_2 - 5 + \sin(x_1 - x_2)\sin(x_1 + x_2), \\ f_i(x) &= -x_{i-1}e^{(x_{i-1}-x_i)} + x_i(4 + 3x_i^2) + 2x_{i+1} \\ &\quad + \sin(x_i - x_{i+1})\sin(x_i + x_{i+1}) - 8, \quad i = 2, 3, \dots, n-1, \\ f_n(x) &= -x_{n-1}e^{(x_{n-1}-x_n)} + 4x_n - 3. \end{aligned} \quad (36)$$

Initial guess: $x_0 = (0, 0, \dots, 0)^t$.

Function 7. Strictly convex Function 1: $h(x)$ is the gradient of $f(x) = \sum i = 1n(e^{x_i} - x_i)$.

$$f_i(x) = e^{x_i} - 1, \quad i = 1, 2, \dots, n. \quad (37)$$

Initial guess: $x_0 = (1/n, 2/n, \dots, 1)^t$.

Function 8. Variable dimensioned function:

$$f_i(x) = x_i - 1, \quad i = 1, 2, \dots, n-2,$$

$$f_{n-1}(x) = \sum_{j=1}^{n-2} j(x_j - 1), \quad (38)$$

$$f_n(x) = \left(\sum_{j=1}^{n-2} j(x_j - 1) \right)^2.$$

Initial guess: $x_0 = (1 - 1/n, 1 - 2/n, \dots, 0)^t$.

Function 9. Tridiagonal system:

$$\begin{aligned} f_1(x) &= 4(x_1 - x_2^2), \\ f_i(x) &= 8x_i(x_i^2 - x_{i-1}) - 2(1 - x_i) \\ &\quad + 4(x_i - x_{i+1}^2), \quad i = 2, \dots, n-1, \\ f_n(x) &= 8x_n(x_n^2 - x_{n-1}) - 2(1 - x_n). \end{aligned} \quad (39)$$

Initial guess: $x_0 = (12, \dots, 12)^t$.

Function 10. Five-diagonal system:

$$\begin{aligned} f_1(x) &= 4(x_1 - x_2^2) + x_2 - x_3^2, \\ f_2(x) &= 8x_2(x_2^2 - x_1) - 2(1 - x_2) + 4(x_2 - x_3^2) + x_3 - x_4^2, \\ f_i(x) &= 8x_i(x_i^2 - x_{i-1}) - 2(1 - x_i) + 4(x_i - x_{i+1}^2) \\ &\quad + x_{i-1}^2 - x_{i-2} + x_{i+1} - x_{i+2}^2, \quad i = 3, \dots, n-2, \\ f_{n-1}(x) &= 8x_{n-1}(x_{n-1}^2 - x_{n-2}) - 2(1 - x_{n-1}) \\ &\quad + 4(x_{n-1} - x_n^2) + x_{n-2}^2 - x_{n-3}, \\ f_n(x) &= 8x_n(x_n^2 - x_{n-1}) - 2(1 - x_n) + x_{n-1}^2 - x_{n-2}. \end{aligned} \quad (40)$$

Initial guess: $x_0 = (-2, \dots, -2)^t$.

Function 11. Extended Freudenstein and Roth function (n is even):

For $i = 1, 2, \dots, n/2$,

$$\begin{aligned} f_{2i-1}(x) &= x_{2i-1} + ((5 - x_{2i})x_{2i} - 2)x_{2i} - 13, \\ f_{2i}(x) &= x_{2i-1} + ((x_{2i} + 1)x_{2i} - 14)x_{2i} - 29. \end{aligned} \quad (41)$$

Initial guess: $x_0 = (6, 3, 6, 3, \dots, 6, 3)^t$.

Function 12. Brent problem:

$$\begin{aligned} f_1(x) &= 3x_1(x_2 - 2x_1) + x_2^2/4, \\ f_i(x) &= 3x_i(x_{i+1} - 2x_i + x_{i-1}) \\ &\quad + (x_{i+1} - x_{i-1})^2/4, \quad \text{for } i = 2, \dots, n-1, \\ f_n(x) &= 3x_n(20 - 2x_n + x_{n-1}) + (20 - x_{n-1})^2/4. \end{aligned} \quad (42)$$

Initial guess: $x_0 = (0, \dots, 0, 20, 20)^t$.

To test the numerical performances of the ATTPRP algorithm, we also perform the experiments with the LS algorithm and the TTPRP algorithm. The columns of Tables 1–3 have the following meanings:

TABLE 1: Test results of the ATTPRP algorithm.

No.	Dim	NI/NF	ATTPRP algorithm CPU	GN
1	3000	123/124	0.608404	$9.97E-06$
	9000	88/89	0.733205	$9.86E-06$
	30000	57/58	1.092007	$9.96E-06$
	90000	38/39	1.310408	$9.85E-06$
2	3000	28/514	0.686404	$9.50E-06$
	9000	18/386	1.372809	$9.22E-06$
	30000	15/371	3.432022	$9.83E-06$
	90000	10/276	3.946825	$9.95E-06$
3	3000	14016/20709	107.172687	$1.00E-05$
	9000	15769/23986	331.158923	$9.99E-06$
	30000	15609/27514	574.09928	$1.00E-05$
	90000	17136/35208	1503.022835	$1.00E-05$
4	3000	63/558	1.029607	$5.24E-07$
	9000	106/1096	5.241634	$8.39E-07$
	30000	195/2337	24.258155	$1.20E-07$
	90000	336/4521	88.031364	$2.21E-08$
5	3000	90/615	0.780005	$8.52E-06$
	9000	96/759	2.464816	$7.76E-06$
	30000	130/1194	11.154072	$8.62E-06$
	90000	176/1984	29.250188	$8.82E-06$
6	3000	91/1223	1.918812	$8.09E-06$
	9000	136/2093	8.845257	$9.86E-06$
	30000	221/3905	38.282645	$9.97E-06$
	90000	379/7177	125.814807	$9.33E-06$
7	3000	48/335	0.468003	$9.66E-06$
	9000	71/636	2.480416	$8.57E-06$
	30000	122/1339	13.135284	$9.31E-06$
	90000	203/2562	43.63348	$8.53E-06$
8	3000	1/2	0.000001	$0.00E+00$
	9000	1/2	0.0468	$0.00E+00$
	30000	1/2	0.000001	$0.00E+00$
	90000	1/2	0.0156	$0.00E+00$
9	3000	6014/88059	100.885847	$9.93E-06$
	9000	6437/103883	312.345202	$9.91E-06$
	30000	7345/137142	1209.803355	$9.96E-06$
	90000	8950/196012	2763.386114	$9.99E-06$
10	3000	1811/19670	25.693365	$9.99E-06$
	9000	1947/22554	76.36249	$9.82E-06$
	30000	2202/28399	261.644877	$9.95E-06$
	90000	2638/38968	577.983705	$1.00E-05$
11	3000	351/4205	5.382034	$9.78E-06$
	9000	409/5194	16.645307	$9.44E-06$
	30000	510/7139	64.475213	$9.64E-06$
	90000	683/10789	161.351834	$9.52E-06$
12	3000	184/188	0.156001	$9.98E-06$
	9000	184/188	0.577204	$9.98E-06$
	30000	184/188	1.107607	$9.98E-06$
	90000	184/188	3.07322	$9.98E-06$

NO: the serial number of the problem

Dim: the variable x dimensions

NI: the number of iterations

NF: the number of iterations of the function value

CPU: the calculation time in seconds

GN: the final function norm evaluations when the program is stopped

TABLE 2: Test results of the TTPRP algorithm.

No.	Dim	NI/NF	TTPRP algorithm CPU	GN
1	3000	129/130	0.436803	$9.97E-06$
	9000	89/90	0.873606	$9.96E-06$
	30000	59/60	1.232408	$9.96E-06$
	90000	41/42	1.435209	$9.73E-06$
2	3000	46/1055	1.435209	$9.99E-06$
	9000	10/267	0.951606	$6.76E-06$
	30000	9/278	2.667617	$9.83E-06$
	90000	8/279	4.040426	$9.00E-06$
3	3000	17401/18908	109.481502	$9.98E-06$
	9000	19999/23227	355.089476	$1.37E-05$
	30000	19431/26598	612.241525	$9.99E-06$
	90000	19999/34409	1542.678289	$2.56E-05$
4	3000	70/662	1.123207	$3.49E-06$
	9000	113/1326	6.099639	$2.14E-06$
	30000	196/2809	28.766584	$6.47E-07$
	90000	334/5473	101.884253	$3.11E-06$
5	3000	54/464	0.514803	$3.38E-06$
	9000	72/711	2.152814	$3.75E-06$
	30000	100/1225	10.99807	$7.38E-06$
	90000	160/2256	31.403001	$5.61E-06$
6	3000	84/1417	2.246414	$6.94E-06$
	9000	127/2473	10.311666	$9.41E-06$
	30000	211/4669	45.349491	$8.60E-06$
	90000	346/8524	141.773709	$8.53E-06$
7	3000	45/366	0.546003	$2.67E-06$
	9000	68/729	2.808018	$1.23E-06$
	30000	117/1558	14.617294	$8.34E-07$
	90000	195/3039	48.219909	$7.08E-07$
8	3000	1/2	0.0624	$0.00E+00$
	9000	1/2	0.000001	$0.00E+00$
	30000	1/2	0.0624	$0.00E+00$
	90000	1/2	0.0624	$0.00E+00$
9	3000	6521/123863	139.277693	$9.81E-06$
	9000	6827/141140	425.679929	$9.97E-06$
	30000	7711/182289	1559.916399	$9.99E-06$
	90000	9107/251132	3460.398582	$9.85E-06$
10	3000	4904/65319	77.750898	$1.00E-05$
	9000	5271/71854	229.820673	$9.73E-06$
	30000	5280/75948	689.150018	$9.97E-06$
	90000	5655/88105	1269.302137	$9.99E-06$
12	3000	300/4529	4.898431	$9.88E-06$
	9000	387/6163	17.846514	$9.61E-06$
	30000	472/8397	72.836867	$9.83E-06$
	90000	620/12490	168.652681	$9.27E-06$
13	3000	193/198	0.124801	$9.99E-06$
	9000	193/198	0.405603	$9.99E-06$
	30000	193/198	0.826805	$9.99E-06$
	90000	193/198	2.589617	$9.99E-06$

Initialization: the parameters are chosen as $\rho = 1$, $s = 0.69$, $\lambda = 0.925$, $\kappa = 0.0001$, and $\epsilon = 10^{-5}$

Stop rule: when the condition $\|h(x)\| \leq 10^{-5}$ or NI ≥ 20000 is satisfied, we stop the process

From Tables 1–3, it is obvious that the three methods can successfully solve most of the test problems with NI < 20000. However, for Function 3 with 9000 and 90000 variables, the

TABLE 3: Test results of the LS algorithm.

No.	Dim	NI/NF	LS algorithm CPU	GN
1	3000	174/175	0.982806	9.96E-06
	9000	94/95	0.670804	9.88E-06
	30000	60/61	1.107607	9.93E-06
	90000	41/42	1.248008	9.97E-06
2	3000	61/1405	1.762811	9.85E-06
	9000	34/916	3.05762	9.98E-06
	30000	15/470	3.946825	9.46E-06
	90000	13/456	6.864044	9.12E-06
3	3000	19999/21506	125.721206	1.43E-05
	9000	19999/23222	351.267452	1.44E-05
	30000	19999/27153	634.003664	2.24E-05
	90000	19999/34389	1582.942147	1.76E-05
4	3000	75/666	1.185608	7.17E-06
	9000	118/1331	6.24004	6.87E-06
	30000	202/2813	30.186194	7.82E-06
	90000	338/5467	96.159016	7.65E-06
5	3000	136/1145	1.404009	8.63E-06
	9000	150/1358	4.305628	9.81E-06
	30000	171/1806	16.801308	5.69E-06
	90000	234/2865	42.08907	9.58E-06
6	3000	109/1695	2.698817	7.38E-06
	9000	147/2694	11.263272	8.98E-06
	30000	235/4934	47.502305	8.67E-06
	90000	364/8724	147.062143	7.57E-06
7	3000	69/400	0.530403	6.22E-06
	9000	94/765	2.948419	3.29E-06
	30000	142/1590	15.085297	1.79E-06
	90000	217/3070	50.450723	4.55E-07
8	3000	1/2	0.0624	0.00E+00
	9000	1/2	0.000001	0.00E+00
	30000	1/2	0.0468	0.00E+00
	90000	1/2	0.0156	0.00E+00
9	3000	5643/107964	117.484353	9.84E-06
	9000	6131/128776	373.934397	9.86E-06
	30000	6997/169540	1461.323767	9.99E-06
	90000	8449/239554	3276.130201	9.88E-06
10	3000	4171/55825	65.47362	9.93E-06
	9000	4213/58137	183.113974	1.00E-05
	30000	4616/67344	608.747102	9.88E-06
	90000	4965/79166	1150.008172	9.92E-06
11	3000	3484/49132	52.073134	9.92E-06
	9000	371/5921	16.504906	9.85E-06
	30000	3625/52579	454.727315	9.93E-06
	90000	3802/57077	801.049535	9.87E-06
12	3000	194/199	0.109201	9.96E-06
	9000	194/199	0.405603	9.96E-06
	30000	194/199	0.998406	9.96E-06
	90000	194/199	2.776818	9.96E-06

TTPRP and LS algorithms cannot handle the function, but the proposed algorithm can do with NI <20000. To more directly show the methods' performance, Dolan and Moré [36] proposed a drawing tool that can obtain the performance profiles of methods. Therefore, using the drawing tool, we obtain Figures 1–3, which are related to the NI, NF,

FIGURE 1: Performance profiles of the methods (NI).

FIGURE 2: Performance profiles of the methods (NF).

and CPU in Tables 1–3. In Figure 1, the ATTPRP algorithm solves all test problems at approximately $\tau = 1.8$, while the LS algorithm solves 80% of the test problems at approximately $\tau = 1.88$, and the TTPRP algorithm solves 92% at approximately $\tau = 1.9$. Thus, we can obtain the result that the ATTPRP algorithm performs slightly better than the other two algorithms. When $\tau = 1.45$ in Figure 2, the presented algorithm solves all test problems, the TTPRP algorithm solves 87% of all test problems, and the LS algorithm only

FIGURE 3: Performance profiles of the methods (CPU).

FIGURE 4: Restoration of the Cameraman, Barbara, and Man images by using the ATTPRP algorithm and TTPRP algorithm. From left to right: a noisy image with 30% salt-and-pepper noise and the restorations obtained with the ATTPRP algorithm and the TTPRP algorithm by minimizing z .

FIGURE 5: Restoration of the Cameraman, Barbara, and Man images by using the ATTPRP algorithm and TTPRP algorithm. From left to right: a noisy image with 50% salt-and-pepper noise and the restorations obtained with the ATTPRP algorithm and TTPRP algorithm by minimizing z .

solves approximately 70% of the test problems. Thus, it is not difficult to see that the ATTPRP algorithm is more competitive than the other two methods. In Figure 3, the curve of the ATTPRP algorithm is above those of the TTPRP and LS algorithms, which indicates that the proposed algorithm is more robust than the other two algorithms in terms of the CPU. In summary, the enhancement of the presented method is noticeable.

3.2. Image Restoration Problems. The purpose of this subsection is to recover the original image from an image damaged by impulse noise. It has important practical significance in optimization fields. The selection of parameters is similar to that in the above subsection. The stop condition is $\|h_{k+1}\| - \|h_k\| / \|h_k\| < 10^{-3}$ or $\|x_{k+1} - x_k\| / \|x_k\| < 10^{-3}$. For the experiments, Cameraman (256×256), Barbara (512×512), and Man (1024×1024) are chosen as the test

images. We also perform experiments to compare the ATTPRP algorithm with the TTPRP algorithm, where the step length α_k is generated by Step 2 and Step 3 in the ATTPRP algorithm. More detailed performance results are shown in Figures 4–6. It is not difficult to see that both the ATTPRP and TTPRP algorithms are successful in the image restoration of the three images. The expenditure of the CPU time is listed in Table 4 to compare the ATTPRP algorithm with the TTPRP algorithm.

From Figures 1–4, we can obviously note that both algorithms can perfectly restore a noisy image with 30%, 50%, and 70% salt-and-pepper noise. In addition, the results in Table 4 show that the ATTPRP algorithm and the TTPRP algorithm are both successful in restoring these images with an approximate CPU time. The presented algorithm is slightly competitive with the TTPRP algorithm for 30% noise problems, 50% noise problems, and 70% noise problems.

FIGURE 6: Restoration of the Cameraman, Barbara, and Man images by using the ATTPRP algorithm and TTPRP algorithm. From left to right: a noisy image with 70% salt-and-pepper noise and the restorations obtained with the ATTPRP algorithm and TTPRP algorithm by minimizing z .

TABLE 4: CPU times of the ATTPRP algorithm and TTPRP algorithm in seconds.

30% noise	Cameraman	Barbara	Man	Total
ATTPRP algorithm	2.184	4.789	20.514	27.487
TTPRP algorithm	2.23	5.179	20.748	28.157
50% noise	Cameraman	Barbara	Man	Total
ATTPRP algorithm	3.276	9.142	35.475	47.893
TTPRP algorithm	3.307	9.204	35.677	48.188
70% noise	Cameraman	Barbara	Man	Total
ATTPRP algorithm	3.619	13.073	58.812	75.504
TTPRP algorithm	3.978	13.4	59.249	76.627

4. Conclusions

In this paper, an accelerated conjugate gradient algorithm that combines the TTPRP method, the acceleration step length, and the hyperplane projection technique is proposed. All search directions d_k generated by using the algorithm automatically have sufficient descent and trust region properties. The global convergence property of the proposed

algorithm is established under suitable conditions. The numerical results show that the proposed algorithm is effective. The image restoration problems also demonstrate that the proposed algorithm is successful.

For future research, we have some ideas as follows: (i) If the acceleration system is introduced into the quasi-Newton method, does it have some good properties? (ii) Can the acceleration system be introduced into the trust region method to solve unconstrained optimization problems and nonlinear equations? (iii) Can the proposed algorithm be applied to machine learning?

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 11661009), the High Level Innovation Teams and Excellent Scholars Program in Guangxi Institutions of Higher Education (Grant No. (2019) 52), and the Guangxi Natural Science Key Fund (No. 2017GXNSFDA198046).

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