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Research Article

Probabilistic Analysis for Structures with Hybrid Uncertain Parameters

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In practical engineering problems, the distribution parameters of random variables cannot be determined precisely due to limited experimental data. The hybrid uncertain model of interval and probability can deal with the problem, but it will produce extensive computation and it is difficult to meet the requirement of the complex engineering problem analysis. In this scenario, this paper presents a vertex method for the uncertainty analysis of the hybrid model. By combining the traditional finite element method, it can be applied to the structural uncertainty analysis. The key of this method is to demonstrate the monotonicity between expectation and variance of the function and distribution parameters of random variables. Based on the monotonicity analysis, interval bounds of the expectation and variance are directly calculated by means of vertex of distribution parameter intervals. Two numerical examples are used to evaluate the effectiveness and accuracy of the proposed method. The results show the vertex method is computationally more efficient than the common interval Monte Carlo method under the same accuracy. Two practical engineering examples are to show that the vertex method makes the engineering application of the hybrid uncertain model easy.

1. Introduction

Uncertainties are widely encountered in practical engineering problems, and traditional analysis techniques for uncertainties are based on probability theory. In probability theory, uncertain parameters are represented by random variables, which are quantified by distribution function and correlation coefficient. Many probabilistic analysis methods that are based on probability models have been well developed, such as the Monte Carlo method [1, 2], full factorial numerical integration [3], univariate dimension reduction [4], first-order reliability method [5, 6], and Bayesian approach [7, 8]. For the uncertainty analysis of structures, the probabilistic theory and the finite element method are combined to generate stochastic finite element analysis (SFEM) [9]. SFEM techniques include Monte Carlo simulation, perturbation method [10, 11], and spectral stochastic finite element method [12, 13]. However, constructing the precise probability model requires a large amount of experimental data which is impractical, especially for the complex engineering problem.

To overcome the disadvantages of the probabilistic method, hybrid uncertain models are developed. The hybrid uncertain models integrate the classical probabilistic method and interval analysis, which are commonly divided into two types [14]. For the models of the first type, uncertain model parameters with sufficient experimental data are represented by the probability model, and uncertain parameters with limited information are quantified by intervals. Several analysis techniques for this model have been developed [15–19]. Hybrid models of the second type are also called probability-box or interval random variable models in the literature [20–23]. All of the uncertain parameters are quantified by the probability model, including some interval distribution parameters, due to the limited information. In the traditional probability analysis, this model can be directly constructed by interval estimation with a confidence level; hence, it is considered as an extension of the probability model. Since this model was proposed by Elishakoff and Colombi [24], it has been applied in the multifield of structural analysis, and several analysis methods for this model have been developed. Based on the monotonicity analysis for probability transformations of random variables, Jiang et al. [25, 26] proposed two efficient algorithms for solving the second hybrid reliability model. By combining Monte Carlo simulation and finite element method, Zhang et al. [27, 28] proposed an interval Monte Carlo method for computing the failure probabilistic intervals and the response probability bounds of structures with interval probability variables. Xia and Yu [13] developed an interval random perturbation method for obtaining the interval expectations and variances of the responses of acoustic fields and structural-acoustic systems with interval probability variables. Xiao and Lu [29] presented a sampling method for the structural response analysis with dependent interval random variables. Xiao et al. [30] improved the double-loop sampling method by using monotonicity analysis for the interval random variables. Liu et al. [21] used the optimized univariate dimension reduction method and the Johnson distributions fitting method to calculate the probability bounds of the response function. Liu et al. [31] proposed a new uncertainty propagation method that is based on the sparse grid technique and saddle point approximation.

From the overall perspective, significant successes have also been realized with the hybrid interval probability model. However, studies on interval random variables are still in their preliminary stage. Especially for the second type of hybrid uncertainty model, some important issues that are associated with its response analysis remain unsolved. One of the very challenging tasks is to improve the computational efficiency. The second hybrid model can be regarded as a set of many probability models that have the same distribution function and different distribution parameters; hence, uncertainty analysis of the hybrid model typically is of products with higher cost than that of the corresponding probability model. The Monte Carlo method is the simplest and the most versatile method for uncertain problems with random variables and can be combined with the finite element method to solve practical engineering problems. However, two-layer nesting sampling is commonly involved in the hybrid model analysis [32], which will lead to extremely low computational efficiency especially for complex problems. Various modified methods that are based on the perturbation technique are provided for improving the computational efficiency [33]. These methods use Taylor series expansion to calculate the relative matrix; thus, the available deterministic finite element analysis (FEA) cannot be directly utilized. To apply these methods to the practical problems, new and more complex FEA codes must be developed. To use the hybrid uncertain model in practical applications, the efficient and practical computational techniques must be developed.

In this paper, a new methodology is presented for the static structural response analysis with uncertain parameters, which is represented by the second type of hybrid model. This hybrid model is named the interval random variable model in this paper. For obtaining the intervals of expectation and standard variance of the response vector of the static structural system with interval random variables, a vertex method is proposed. A monotonicity analysis is conducted for the expectation and variance of the response vector with respect to interval variables. On the basis of the monotonicity analysis, the intervals of expectation and variance of the response vector are calculated by the classical probabilistic method. The response vector of the complex structural system is calculated by the determined FEA. The remainder of this paper is organized as follows. The mathematical background regarding the interval random arithmetic is discussed in Section 2. The method for the static structural system with interval random variables is proposed in Section 3. In Section 4, the effectiveness and efficiency of the proposed method are evaluated by four numerical examples. In Section 5, the conclusions of this study are presented.

2. Mathematical Background

2.1. Expectation and Variance of Interval Variables. Let $\mathbf{X}$ denote a random vector with n independent random variables:

$$\mathbf{X} = (X_1, \ldots, X_i, \ldots, X_n), \quad i = 1, 2, \ldots, n. \tag{1}$$

Assuming a distribution, the parameters of X_i are interval variables. The interval variables that are associated with the random parameter X_i can be represented as an interval vector $\mathbf{x}_i^I$

$$\mathbf{x}_i^I = \left[\mathbf{x}_i^L, \mathbf{x}_i^R\right], \tag{2}$$

or in component forms

$$x_{i,j}^I = \left[x_{i,j}^L, x_{i,j}^R\right], \quad j = 1, 2, \ldots, m, \tag{3}$$

where $\mathbf{x}_i^L$ and $\mathbf{x}_i^R$ are the lower and upper bounds, respectively, of the interval variable vector $\mathbf{x}_i^I$; j is the number of interval variables in the interval vector $\mathbf{x}_i^I$; and $x_{i,j}^L$ and $x_{i,j}^R$ are the lower and upper bounds, respectively, of the interval variable $x_{i,j}^I$. Thus, the interval random vector with interval parameters $\mathbf{x}_i^I$ can be denoted as

$$\mathbf{X}\left(\mathbf{x}^I\right) = \left(X_1\left(\mathbf{x}_1^I\right), \ldots, X_i\left(\mathbf{x}_1^I\right), \ldots, X_n\left(\mathbf{x}_n^I\right)\right). \tag{4}$$

The expectation $\mathrm{E}(\mathbf{X}(\mathbf{x}^I))$ and variance $\mathrm{var}(\mathbf{X}(\mathbf{x}^I))$ of the independent interval random vector $\mathbf{X}(\mathbf{x}^I)$ can be expressed as

$$\begin{aligned} E\left(\mathbf{X}\left(\mathbf{x}^I\right)\right) &= E\left(X_1\left(\mathbf{x}_1^I\right), \ldots, X_i\left(\mathbf{x}_i^I\right), \ldots, X_n\left(\mathbf{x}_n^I\right)\right) \\ &= \left(E\left(X_1\left(\mathbf{x}_1^I\right)\right), \ldots, \mathrm{E}\left(X_i\left(\mathbf{x}_i^I\right)\right), \ldots, E\left(X_n\left(\mathbf{x}_n^I\right)\right)\right), \\ \mathrm{var}\left(\mathbf{X}\left(\mathbf{x}^I\right)\right) &= \mathrm{var}\left(X_1\left(\mathbf{x}_1^I\right), \ldots, X_i\left(\mathbf{x}_i^I\right), \ldots, X_n\left(\mathbf{x}_n^I\right)\right) \\ &= \left(\mathrm{var}\left(X_1\left(\mathbf{x}_1^I\right)\right), \ldots, \mathrm{var}\left(X_i\left(\mathbf{x}_i^I\right)\right), \ldots, \mathrm{var}\left(X_n\left(\mathbf{x}_n^I\right)\right)\right). \end{aligned} \tag{5}$$

2.2. Calculation of the Expectation and Variance Based on Monotonicity Analysis. Consider the following linear function of n independent interval random variables:

$$f\left(\mathbf{X}\left(\mathbf{x}^I\right)\right) = a_0 + a_1 X_1\left(\mathbf{x}_1^I\right) + a_2 X_2\left(\mathbf{x}_2^I\right) + \cdots + a_n X_n\left(\mathbf{x}_n^I\right), \tag{6}$$

where $X_i(\mathbf{x}_i^I)$ is an interval random variable and a_i is a coefficient of the linear function. Based on classical statistical theory, the expectation and variance of the response distribution of $f(\mathbf{X}(\mathbf{x}^I))$ are

$$\begin{aligned} E\left(f\left(\mathbf{X}\left(\mathbf{x}^I\right)\right)\right) &= a_0 + \sum_{i=1}^{n} a_i E\left(X_i\left(\mathbf{x}_i^I\right)\right), \\ \operatorname{var}\left(f\left(\mathbf{X}\left(\mathbf{x}^I\right)\right)\right) &= \sum_{i=1}^{n} a_i^2 \operatorname{var}\left(f\left(X_i\left(\mathbf{x}_i^I\right)\right)\right). \end{aligned} \tag{7}$$

To obtain the lower and upper bounds of expectation and variance of the interval random function $f(\mathbf{X}(\mathbf{x}^I))$, a new arithmetic will be proposed in this section.

The spatial derivative of $E(f(\mathbf{X}(\mathbf{x}^I)))$ with respect to $x_{i,j}$ can be expressed as

$$\frac{\partial E(f(\mathbf{X}(\mathbf{x}^I)))}{\partial x_{i,j}} = a_i \frac{\partial E(X_i(\mathbf{x}_i^I))}{\partial x_{i,j}}. \tag{8}$$

A monotonicity analysis of the expectation for commonly used PDFs is conducted, and the results are presented in Table 1. According to Table 1, $(\partial E(X_i(\mathbf{x}_i^I))/\partial x_{i,j})$ remains negative or positive. Hence, $(\partial E(f(\mathbf{X}(\mathbf{x}^I)))/\partial x_{i,j})$ is monotonic, and extreme points of $E(f(\mathbf{X}(\mathbf{x}^I)))$ exist on the boundaries of $\mathbf{x}^I$. In this case, the vertex method can be used to calculate the extreme points. If $a_i(\partial E(X_i(\mathbf{x}_i^I))/\partial x_{i,j}) > 0$, $x_{i,j}^L$ minimizes $E(f(\mathbf{X}(\mathbf{x}^I)))$; if $a_i(\partial E(X_i(\mathbf{x}_i^I))/\partial x_{i,j}) < 0$, $x_{i,j}^R$ minimizes $E(f(\mathbf{X}(\mathbf{x}^I)))$. In other words, the optima of $E(f(\mathbf{X}(\mathbf{x}^I)))$ are the vertex of the interval $\mathbf{x}^I$.

According to the monotonicity analysis, the values that minimize and maximize $E(f(\mathbf{X}(\mathbf{x}^I)))$ can be determined. Subscripts L and R represent the minimum and maximum values, respectively, of $E(f(\mathbf{X}(\mathbf{x}^I)))$. For example, $(x_{i,j}^L)_L$ is the value at which the minimum is attained and $(x_{i,j}^L)_R$ is the value at which the maximum is attained. The extreme minimum value vector and the extreme maximum value vector are denoted as $\mathbf{x}_L$ and $\mathbf{x}_R$. Thus, extreme values of the expectations can be expressed as

$$\begin{aligned} \min\left\{E\left(f\left(\mathbf{X}\left(\mathbf{x}^I\right)\right)\right)\right\} &= E(f(\mathbf{X}(\mathbf{x}_L))), \\ \max\left\{E\left(f\left(\mathbf{X}\left(\mathbf{x}^I\right)\right)\right)\right\} &= E(f(\mathbf{X}(\mathbf{x}_R))). \end{aligned} \tag{9}$$

Hence,

$$E^I\left(f\left(\mathbf{X}\left(\mathbf{x}^I\right)\right)\right) = [E(f(\mathbf{X}(\mathbf{x}_L))), E(f(\mathbf{X}(\mathbf{x}_R)))], \tag{10}$$

where $E^I(f(\mathbf{X}(\mathbf{x}^I)))$ is the interval of expectation of $f(\mathbf{X}(\mathbf{x}^I))$.

The partial derivative of $\operatorname{var}(f(\mathbf{X}(\mathbf{x}^I)))$ with respect to $x_{i,j}$ is

$$\frac{\partial \operatorname{var}(f(\mathbf{X}(\mathbf{x}^I)))}{\partial x_{i,j}} = \frac{a_i^2 \partial \operatorname{var}(f(X_i(\mathbf{x}_i^I)))}{\partial x_{i,j}}, \tag{11}$$

which is always negative or positive for the commonly used PDFs in Table 1. Similar to the analysis above, the interval of variance of $f(\mathbf{X}(\mathbf{x}^I))$ can be expressed as

$$\operatorname{var}^I\left(f\left(\mathbf{X}\left(\mathbf{x}^I\right)\right)\right) = [\operatorname{var}(f(\mathbf{X}(\mathbf{x}_L))), \operatorname{var}(f(\mathbf{X}(\mathbf{x}_R)))], \tag{12}$$

where $\operatorname{var}^I(f(\mathbf{X}(\mathbf{x}^I)))$ is the interval of variance of $f(\mathbf{X}(\mathbf{x}^I))$.

According to the above analysis, the expectation and variance of equation (6) with the uncertain distribution parameters are monotonous. Based on the monotonicity analysis, the vertex method is presented to quickly calculate the interval bounds of expectation and variance by equations (11) and (12). This vertex method can avoid the optimal iterative search process.

3. Structural Uncertainty Analysis Static System with Interval Random Variables

3.1. Calculation of the Expectation and Variance of a Structural Static System. In the previous section, an interval random vector is defined. An interval random matrix is constructed in this section, which is a matrix whose elements are the interval random variables. Consider the real linear equations that are solved by a displacement-based linear static FEA.

In this paper, it focuses on the large finite element (FE) model with a load vector that includes interval random variables. Hence, the real linear equations can be expressed as

$$\mathbf{KU}(\mathbf{X}) = \mathbf{F}\left(\mathbf{X}\left(\mathbf{x}^I\right)\right), \tag{13}$$

where $\mathbf{K}$ is the real stiffness matrix, $\mathbf{F}(\mathbf{X}(\mathbf{x}^I))$ is the interval random structure load vector, and $\mathbf{U}$ is the interval random displacement response vector of the system. The solution of (13) that is considered in this paper is expressed in terms of the intervals of expectation and standard variance of $\mathbf{U}$.

Solving the matrix equation (13), the interval random displacement response vector can be expressed as

$$\mathbf{U}(\mathbf{X}) = \mathbf{K}^{-1}\mathbf{F}\left(\mathbf{X}\left(\mathbf{x}^I\right)\right), \tag{14}$$

where

$$\begin{aligned} \mathbf{U}(\mathbf{X}) &= \begin{bmatrix} u_1(\mathbf{X}(\mathbf{x}^I)) \\ u_2(\mathbf{X}(\mathbf{x}^I)) \\ \vdots \\ u_n(\mathbf{X}(\mathbf{x}^I)) \end{bmatrix}, \\ \mathbf{K}_1^{-1} &= \begin{bmatrix} k_{1,1}^{-1} & k_{1,2}^{-1} & \cdots & k_{1,n}^{-1} \\ k_{2,1}^{-1} & k_{2,2}^{-1} & \dots & k_{2,n}^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ k_{n,1}^{-1} & k_{n,2}^{-1} & \dots & k_{n,n}^{-1} \end{bmatrix}, \\ \mathbf{F} &= \begin{bmatrix} F_1(\mathbf{X}(\mathbf{x}^I)) \\ F_2(\mathbf{X}(\mathbf{x}^I)) \\ \vdots \\ F_n(\mathbf{X}(\mathbf{x}^I)) \end{bmatrix}. \end{aligned} \tag{15}$$

Then, the nodal displacement $u_i(\mathbf{x}^I)$ can be obtained as

Table 1: Monotonicity analysis of expectation and variance for commonly used PDFs.

Distribution type	Probability density function (PDF)	Distribution parameter	Expectation	Partial derivative of the expectation	Variance	Partial derivative of the variance
Weibull	$(k/\lambda)(x/\lambda)^{k-1}e^{-(x/\lambda)^k}$	$x \geq 0$, $\lambda > 0$, $k > 0$	$\lambda\Gamma(1+1/k)$	$\partial E(x)/\partial\lambda > 0$ $\partial E(x)/\partial k < 0$	$\lambda^2[\Gamma(1+(2/k)) - (\Gamma(1+(1/k)))^2]$	$\partial\mathrm{var}(x)/\partial\lambda > 0$ $\partial\mathrm{var}(x)/\partial k > 0$
Lognormal	$(1/x\sqrt{2\pi}\sigma)e^{-((\ln x-\mu)^2/2\sigma^2)}$	$x > 0$, $\mu \in R$, $\sigma^2 \geq 0$	$e^{\mu+\sigma^2/2}$	$\partial E(x)/\partial\mu > 0$ $\partial E(x)/\partial\sigma > 0$	$(e^{\sigma^2}-1)e^{2\mu+\sigma^2}$	$\partial\mathrm{var}(x)/\partial\mu > 0$ $\partial\mathrm{var}(x)/\partial\sigma > 0$
Normal	$(1/\sqrt{2\pi}\sigma)e^{-((x-\mu)^2/2\sigma^2)}$	$x \in R$, $\mu \in R$, $\sigma^2 \geq 0$	μ	$\partial E(x)/\partial\mu > 0$ $\partial E(x)/\partial\sigma = 0$	$\sigma^2 \geq 0$	$\partial\mathrm{var}(x)/\partial\mu > 0$ $\partial\mathrm{var}(x)/\partial\sigma > 0$
Type I extreme value (Gumbel)	$(1/\beta)e^{-((x-\mu/\beta)+e^{-(x-\mu/\beta)})}$	$x \in R$, $\beta > 0$, $\mu \in R$	$\mu+\beta\gamma(\gamma = 0.5772)$	$\partial E(x)/\partial\mu > 0$ $\partial E(x)/\partial\beta > 0$	$(\pi^2/6)\beta^2$	$\partial\mathrm{var}(x)/\partial\mu(x) = 0$ $\partial\mathrm{var}(x)/\partial\beta > 0$
Type II extreme value (Frechet)	$\alpha\exp(-x^{-\alpha})x^{-\alpha-1}$	$x > 0$, $\alpha > 0$	$\begin{cases}\Gamma(1-(1/\alpha))\text{for} & \alpha > 1\\ \infty & \text{otherwise}\end{cases}$	$\partial E(x)/\partial\alpha < 0$	$\begin{cases}\Gamma(1-(2/\alpha)) - (\Gamma(1-(1/\alpha)))^2\text{for} & \alpha > 2\\ \infty & \text{otherwise}\end{cases}$	$\partial\mathrm{var}(x)/\partial\alpha < 0$
Uniform	$x-a/b-a$	$0 < a \leq x \leq b$	$1/2(b+a)$	$\partial E(x)/\partial a > 0$ $\partial E(x)/\partial b > 0$	$(1/12)(b-a)^2$	$\partial\mathrm{var}(x)/\partial b > 0$ $\partial\mathrm{var}(x)/\partial a < 0$
Exponential	$\lambda e^{-\lambda x}$	$x > 0$, $\lambda > 0$	λ^{-1}	$\partial E(x)/\partial\lambda > 0$	λ^{-2}	$\partial\mathrm{var}(x)/\partial\lambda < 0$

$$u_i(\mathbf{X}(\mathbf{x}^I)) = k_{i,1}^{-1}F_1(\mathbf{X}(\mathbf{x}^I)) + k_{i,2}^{-1}F_2(\mathbf{X}(\mathbf{x}^I)) + \cdots + k_{i,n}^{-1}F_n(\mathbf{X}(\mathbf{x}^I)). \tag{16}$$

Merging the same interval random $X_i(\mathbf{x}_i^I)$, equation (16) can be expressed as

$$u_i(\mathbf{X}(\mathbf{x}^I)) = a_1X_1(\mathbf{x}_1^I) + a_2X_2(\mathbf{x}_2^I) + \cdots + a_nX_n(\mathbf{x}_n^I), \tag{17}$$

where $(a_1, a_2, \ldots, a_n)$ is the coefficient of the linear function. According to equations (10) and (12), the interval expectation and variance of $u_i(\mathbf{X}(\mathbf{x}^I))$ are

$$\begin{aligned} E^I(u_i(\mathbf{X}(\mathbf{x}^I))) &= [E(u_i(\mathbf{X}(\mathbf{x}_L))), E(u_i(\mathbf{X}(\mathbf{x}_R)))], \\ \operatorname{var}^I(u_i(\mathbf{X}(\mathbf{x}^I))) &= [\operatorname{var}(u_i(\mathbf{X}(\mathbf{x}_L))), \operatorname{var}(u_i(\mathbf{X}(\mathbf{x}_R)))]. \end{aligned} \tag{18}$$

3.2. Solving the Coefficient of Uncertainty Function. To obtain equation (17), it must solve equation (14) to obtain the inverse of the stiffness matrix, namely, $\mathbf{K}^{-1}$. However, the computation of $\mathbf{K}^{-1}$ is time-consuming, and $\mathbf{K}^{-1}$, which is typically nonsingular, requires a large amount of memory for storage, especially for large-scale engineering problems. In this section, a new method is proposed for computing the coefficient in equation (16), which avoids the computation of $\mathbf{K}^{-1}$.

Consider n independent interval random variables $X_i(\mathbf{x}_i^I)\,(i = 1, 2, \ldots, n)$. Sampling n sets from $X_i(\mathbf{x}_i^I)$ and substituting them into equation (13), the node displacements are calculated by the following equation:

$$\begin{aligned} u_{i,1} &= a_1X_{1,1} + a_2X_{1,2} + \cdots + a_nX_{1,n}, \\ u_{i,2} &= a_1X_{2,1} + a_2X_{2,2} + \cdots + a_nX_{2,n}, \\ &\cdots \\ u_{i,n} &= a_1X_{n,1} + a_2X_{n,2} + \cdots + a_nX_{n,n}, \end{aligned} \tag{19}$$

where $(X_{i,1}, X_{i,2}, \ldots, X_{i,n})\,(i = 1, 2, \ldots, n)$ is the i-th sampling data from $(X_1(\mathbf{x}_1^I), X_2(\mathbf{x}_2^I), \ldots, X_n(\mathbf{x}_n^I)), (a_1, a_2, \ldots, a_n)$ are coefficient that are associated with inverse of the stiffness matrix, and $(u_{i,1}, u_{i,2}, \ldots, u_{i,n})$ is the i-th node displacement under n sampling data. By solving the linear equation (19), the coefficient in (17) can be obtained. In engineering application, the node displacements can be calculated by the commercial FEA software. Therefore, the present algorithm avoids to solve the inverse matrix of the stiffness matrix and improves the computation efficiency of the vertex method.

4. Illustrative Examples

4.1. Static Response of a Truss Structure. To evaluate the efficiency of the approach that is presented in this paper, a linear elastic plane truss is considered, as illustrated in Figure 1. The design values of the structural parameters for all members are the cross-sectional areas for elements $A_1 \sim A_6 = 10.32\ \mathrm{cm}^2$, the cross-sectional areas for elements $A_7 \sim A_{15} = 6.45\ \mathrm{cm}^2$, and Young's modulus $E = 2 \times 10^{11}$ Pa.

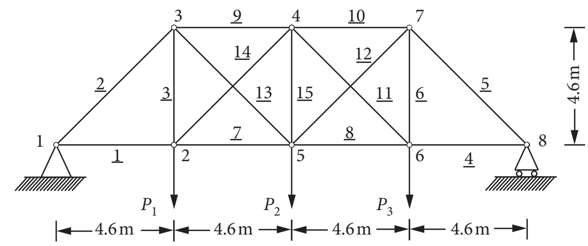


Figure 1: Truss structure.

Suppose that loads P_1, $P_2 = 0.3$, and $P_3 = 0.1$ m are interval random variables; their parameters are listed in Table 2.

In the following, the interval Monte Carlo method (IMC) [34] and the vertex method (VM) are used to calculate the lower and upper bounds of the expectation and variance of the structural displacement response at node 5. In every IMC method, the first step is to arbitrarily generate intervals via the inverse transform method, and the second step is to use interval FEA to compute the interval structural system. Then, the lower and upper bounds of the expectation and variance can be determined. This model has been investigated with the IMC method in [34]; hence, the results with 5000, 50000, and 500000 samples are presented in Table 3. The intervals of the expectation and variance for the proposed method are also specified in Table 3. To compare the two methods in terms of efficiency and accuracy, relative errors are calculated and are presented in Table 4.

According to Table 4, the accuracy of the results is influenced by the simulation times for IMC. To improve the accuracy, more computational time is required. Thus, the results of IMC 500,000 are closer to the exact values. The intervals of expectation and variance that are calculated by the proposed method and IMC 500,000 are close to each other. The IMC is time-consuming as large simulations and interval arithmetic are required. For realizing the same level of accuracy with the proposed method, the total number of simulations for IMC 500,000 is 5×10^5. Because of the monotonicity analysis, the proposed method saves much time.

4.2. Static Response of a Cantilever Plate. Consider the analysis of a cantilever plate, as illustrated in Figure 2. Suppose that there is no preload in any element, and only a force is applied at the top-right corner of the plate. The material of this plate is steel. The deterministic values of the plate are a Poisson ratio of $m = 0.3$ and a thickness of $H = 0.1$ m. Suppose that Young's modulus E, the concentrated loads P, and the density ρ are interval random variables. as presented in Table 5. The finite element method is used to compute the displacements in the plate A. To demonstrate the analysis technique, the 3-node triangular element idealization in Figure 2(b) is considered.

In this example, the finite element model includes 8 nodes and 6 elements. The self-weights of the triangular elements are considered and are equivalent to the element nodal load matrix. As Young's modulus E is an interval

Table 2: Statistics of random loadings that act on the truss.

Statistical parameters	90% confidence interval
Mean ln P_1	[4.4465, 4.5199]
Mean ln P_2 = 0.3	[5.5452, 5.6186]
Mean ln P_3 = 0.1 m	[4.4465, 4.5199]
ln standard dev. P_1, P_2, P_3	0.09975

Table 3: Expectation and variance of the displacement of node 5 with IMC and the vertex method.

Method	Expectation of the displacement of node 5	Variance × E of the displacement of node 5
IMC 5,000	[−0.0619094, −0.057528]	[1.7665, 2.0459]
IMC 50,000	[−0.0619236, −0.0575412]	[1.7833, 2.06539]
IMC 500,000	[−0.0619299, −0.057547]	[1.7724, 2.053]
The vertex method	[−0.0619326, −0.057549]	[1.7758, 2.0566]

Table 4: Relative error of expectation and variance.

Method	Relative error of expectation |(IMC − VM)/VM| (%)		Relative error of variance |(IMC − VM)/VM| (%)	
	Lower bound	Upper bound	Lower bound	Upper bound
IMC 5,000	0.0375	0.0365	0.5237	0.5203
IMC 50,000	0.0145	0.0136	0.4223	0.4274
IMC 500,000	0.0044	0.0035	0.1915	0.1750

random variable, the inverse of the stiffness matrix $\mathbf{K}$ cannot be calculated directly. Instead, it can obtain $\mathbf{K} = \mathrm{E}\mathbf{K}_1$ and $\mathbf{K}^{-1} = 1/\mathrm{E}\mathbf{K}_1^{-1}$, where $\mathbf{K}_1$ is a new matrix and $\mathbf{K}^{-1}$ and $\mathbf{K}_1^{-1}$ are the inverse matrices of $\mathbf{K}$ and $\mathbf{K}_1$, respectively. The node displacement vector is $\mathbf{U} = \mathbf{K}^{-1}\mathbf{F} = 1/\mathrm{E}\mathbf{K}_1^{-1}\mathbf{F}$. The displacement of node 4 can be expressed as $u_4 = (1/\mathrm{E}) f(P,\rho)$, where $f(P,\rho)$ is a linear function of P and ρ. The expectation and variance can be computed via the following equations:

$$E(u_4) = E\left(\frac{1}{\mathrm{E}}\right) E(F(P,\rho)),$$
$$\operatorname{var}(u_4) = \operatorname{var}\left(\frac{1}{\mathrm{E}}\right)\operatorname{var}(F(P,\rho)) + \operatorname{var}\left(\frac{1}{\mathrm{E}}\right) E(F(P,\rho))^2 + \operatorname{var}(F(P,\rho)) E\left(\frac{1}{\mathrm{E}}\right)^2, \tag{20}$$

where the interval bounds of $E(1/\mathrm{E})$ and $\operatorname{var}(1/\mathrm{E})$ can be calculated by classical numerical integration. As $F(P,\rho)$ is a linear function, the proposed method is used to be compute $E^I(F(P,\rho))$ and $\operatorname{var}^I(F(P,\rho))$. Synthesizing the above method, the interval expectation $E^I(u_4)$ and the interval variance $\operatorname{var}^I(u_4)$ are obtained:

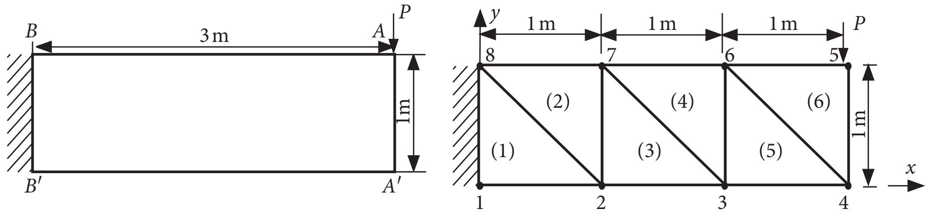
$$E^I(u_4) = [-1886\mathrm{e}-5, -1.687\mathrm{e}-5],$$
$$\operatorname{var}^I(u_4) = [0.0851\mathrm{e}-11, 0.1178\mathrm{e}-11]. \tag{21}$$

From this example, the proposed method can be applied to FE models in which the stiffness matrix includes common interval random variables.

4.3. Automobile Frame Structure. The proposed method is applied to the uncertainty analysis of a practical automobile frame, which is modified from the numerical example in [35]. As illustrated in Figure 3, the frame is composed of two side beams and eight cross beams, and four equivalent distributed forces, namely, Q_1, Q_2, Q_3, and Q_4, are applied to the frame, which result from the operator cabin, engine assembly, gasoline tank, and goods, respectively. Because the vertical stiffness affects the performance of the automobile, the real maximum displacement d of the frame in the Y-direction is analysed. In this application, Q_1, Q_2, Q_3, and Q_4 represent four uniformly distributed forces that act on the frame. Young's modulus E is 2.07×10^5 MPa. The density ρ and the four external forces $Q_1 \sim Q_4$ are treated as interval random variables, which are listed in Table 6.

In this example, FEA is used to compute the real displacement d of the frame in the Y-direction. FE model of the automobile frame structure includes 30854 nodes and 29038 shell elements, and it has 92562 degrees of freedom. So the inverse matrix of the stiffness matrix is a 92562×92562 nonsingular matrix. It is hard to solve and save the inverse matrix. However, it is very easy to solve the displacement vector of the automobile frame structure with commercial FE software. For this problem, the solve process is given as follows. By the method in Section 3.2, the function of uncertain parameter and output response can be calculated. Firstly, 5 sets of data sampling from the five interval variables are denoted as $(\rho_i, (Q_1)_i, (Q_2)_i, (Q_3)_i, (Q_4)_i)\,(i = 1, 2, \ldots, 5)$. Secondly, using the 5 sets of data solves the nodal displacement with the FEA and will get six nodal displacements of $d_i\,(i = 1, 2, \ldots, 5)$. Then, the linear coefficient of the interval random variables associated with displacement response d is solved by equation (19). Lastly, the interval expectation and variance of d can be computed by the present method. If it is to compute the other nodal interval expectation and variance, result from 5-time computations of FEA can be utilizable.

To evaluate the efficiency, multiple nodal interval expectations and variances and their computation times with the proposed method are listed in Table 7. Simulation of this FE model and solution of the expectation and variance are conducted using ANSYS 12.1 and MATLAB 7.1, respectively, on a 3.10 GHz Intel (R) Core (TM) $i3$-2100. The runtime for a single FE model with the ANSYS software is 14.1082 s, and 5 rounds of FEA require 70.5 s. The computation time of a nodal interval of expectation and variance with MATLAB code is 0.004 s. Hence, the total computing time for the expectation and variance of a node is 70.5 s. Point A in Figure 3 is the most likely point for generating the maximum displacement in the Y-direction. Via the



(a) (b)

Figure 2: Finite element idealization of the cantilever plate: (a) a cantilever plate; (b) 3-node triangular element model.

Table 5: Distributions of the three random variables for the cantilever beam.

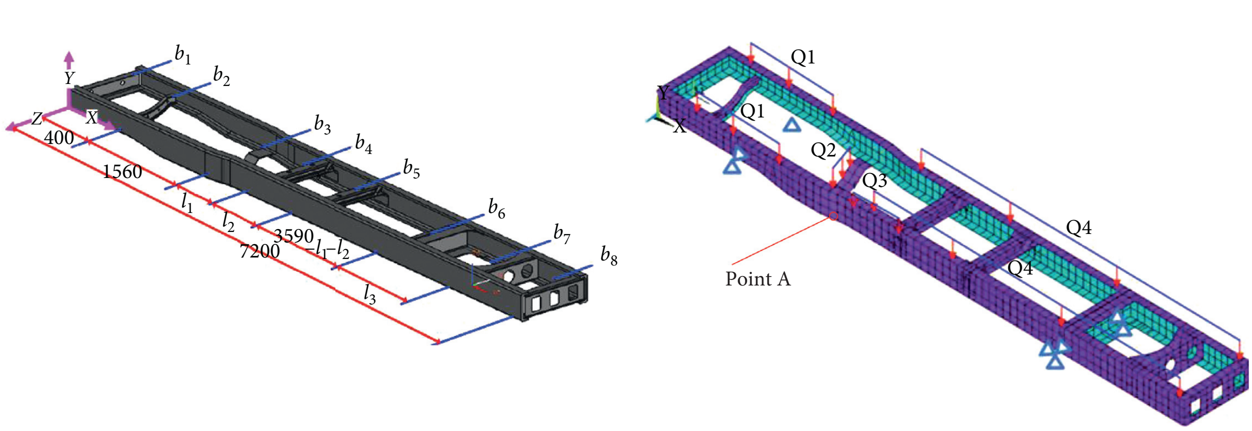
Random variable	Distribution parameter 1	Distribution parameter 2	Distribution type
E (Pa)	$u_E = 2.1 \times 10^{11}$	$\beta_E = [4.8, 5.15]$	Type I extreme value
$P(N)$	$u_p = [0.98 \times 10^4, 1.1 \times 10^4]$	$\beta_p = [0.85 \times 10^3, 1 \times 10^3]$	Type I extreme value
$\rho\,(\mathrm{kg/m^3})$	$\mu_p = [7600, 7920]$	$\sigma_p = 65$	Normal

Table 6: Distribution of the random variables for the automobile frame.

Uncertain variables	Parameter 1	Parameter 2	Distribution type
ρ (kg/mm^3)	$\mu_\rho \in [7.595 \times 10^{-6}, 7.905 \times 10^{-6}]$	$\sigma_\rho = 5.0 \times 10^{-8}$	Normal
$Q_1\,(N)$	$\mu_{Q_1} = 1.8 \times 10^4$	$\sigma_{Q_1} = [100, 120]$	Normal
$Q_2\,(N)$	$\mu_{Q_2} = [6.27 \times 10^3, 6.65 \times 10^3]$	$\sigma_{Q_2} = [50, 65]$	Normal
$Q_3\,(N)$	$a_{Q_3} = [6.27 \times 10^3, 6.65 \times 10^3]$	$b_{Q_3} = 6.76 \times 10^3$	Uniform
$Q_4\,(N)$	$a_{Q_4} = 1.2 \times 10^5$	$b_{Q_4} \in [1.55 \times 10^5, 1.65 \times 10^5]$	Uniform

Table 7: Intervals of expectation and variance of 8 nodes of the hexahedral element at Point A.

Node no.	Interval expectation	Interval variance	Computing time (s)
7535	[1.35818561, 1.40560837]	[0.05111004, 0.08449479]	0.004
7536	[1.37720014, 1.42532155]	[0.05238237, 0.08659866]	0.004
7537	[1.39622373, 1.44504495]	[0.05367141, 0.08873017]	0.004
7538	[1.41532345, 1.46484746]	[0.05498247, 0.09089812]	0.004
7539	[1.59172880, 1.64721704]	[0.06870772, 0.11359037]	0.004
7540	[1.59152348, 1.64702854]	[0.06874794, 0.11365677]	0.004
7541	[1.59078654, 1.64628807]	[0.06873925, 0.11364232]	0.004
7542	[1.59236121, 1.65139101]	[0.06875101, 0.11364232]	0.004



(a) (b)

Figure 3: An automobile frame structure [35].

Table 8: Comparison of the calculation time of all the nodal displacement expectation and variance.

	Double Monte Carlo	The vertex method
Total time(s)	4.3529×10^5	208.066

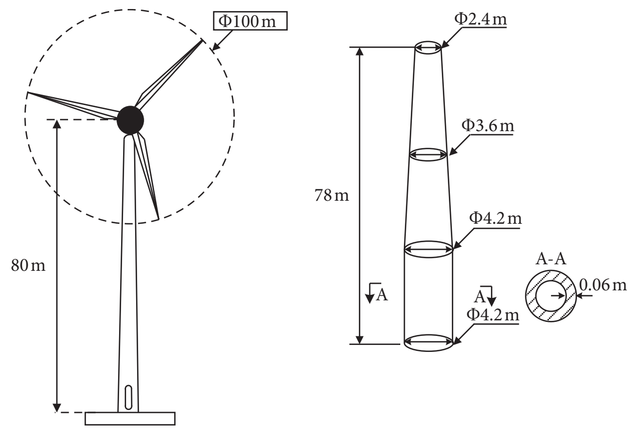


Figure 4: 2 MW horizontal-axis wind turbine and tower.

Table 9: Distribution of the random variables for the wind turbine tower.

Name	Uncertain variables	Expectation	Standard deviation	Distribution type
X-direction force (KN)	F_x	[44.6, 51.5]	1.78	Normal
Y-direction force (KN)	F_y	[880, 960]	35	Normal
Z-direction force (KN)	F_z	[1864, 1960]	75	Normal
X-direction moment (KN·m)	M_x	[9247, 9500]	240	Normal
Y-direction moment (KN·m)	M_y	[112, 120]	4.5	Normal
Z-direction moment (KN·m)	M_z	[4281, 4490]	171	Normal
Density (kg/mm^3)	ρ	7.75×10^{-6}	—	—
Elasticity modulus (Pa)	E	2.1×10^{11}	—	—
Poisson's rate	μ	0.2	—	—

proposed method, the intervals of expectation and variance of 8 nodes (nodes 7535 to 7542) of the hexahedral element at Point A are listed in Table 7. The acquisition of all nodal expectations and variances only requires 208.066 s, which is the sum of 6 times the computation time of FE model and the product of 30854 nodes and 0.004 s. The double Monte Carlo method with 10000 samples is used to calculate the displacements of 30854 nodes [30]. The calculation requires 4.3529×10^5 s, which is equivalent to 120.9151 h, which is more than 2000 times the computation time of the proposed method (Table 8). According to this numerical example, the proposed method is highly computationally efficient for this complex engineering problem.

4.4. A Tall Wind Turbine Tower. Tower is mainly bearing structure of a large wind turbine. It supports the gravitational and inertial loads of nacelle and wind rotor as well as aerodynamic loads, actuation loads, and so on [36]. Tower reliability and safety is key factor in ensuring the normal operation of the wind turbine. It is important to acquire the accurate loads under the limit state. However, tower loads are difficult to calculate. Load calculation involves multisource uncertainties that include variations in wind tunnels measurements, geometric distortions of the blade under loading, 3D rotational correction, and so on [37]. Therefore, tower loads under the extreme wind speed cannot quantify by the probability model. The interval random variables are an effective means of quantifying the loads. In this section, the loads are represented by interval random variables. The vertex method is used to calculate the displacement of tower.

In this example, tower is from a 2 MW horizontal-axis wind turbine. The diameter of the wind rotor is 100 m, and the high of hub center is 80 m. The geometry dimensions of tower are shown in Figure 4. ANSYS software is used to analyze the structural response. SOLD45 solid elements and hexahedron mesh are applied to establish the finite element model, which includes 153686 nodes and 190742 elements. Loads and the material parameters are given in Table 9. FE model is plotted in Figure 5. The displacement counter is also given in Figure 6, which is used to validate the FE model.

There are 6 interval random variables in the FE model of wind turbine tower. According to equation (22), 6 times of random sample is performed to acquire the

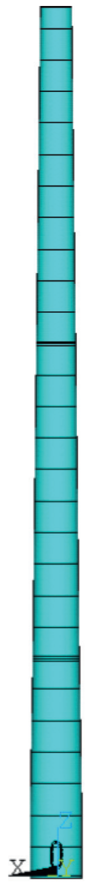


Figure 5: FE model of tower.

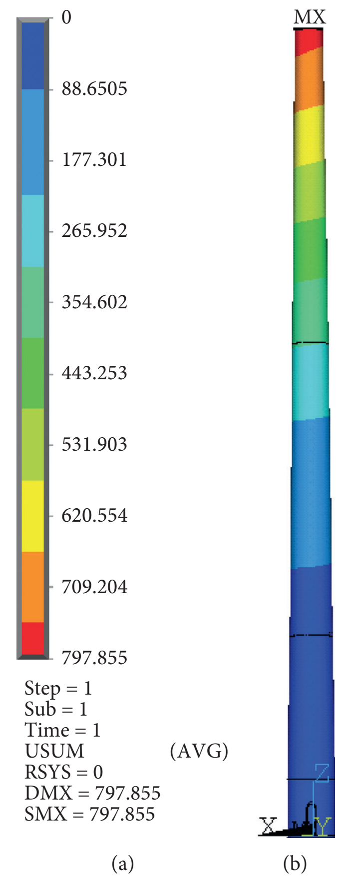


Figure 6: Displacement contour of vector sum (unit: mm).

$\{(F_x)_i, (F_y)_i, (F_z)_i, (M_x)_i, (M_y)_i\}\,(i = 1, 2, 3, \cdots, 6)$ from the interval random variables F_x, F_y, F_z, M_x, M_y, and M_z. Substituting them into the finite element model, 6 groups of nodal displacement are calculated by the AYSYS software. Solving equation (19), coefficient $a_i\,(i = 1, 2, 3, \ldots, 6)$ is computed. Based on theoretical analysis, the maximum displacement locates the top of the tower. So, nodes of top tower are selected and their coefficient $a_i\,(i = 1, 2, 3, \ldots, 6)$ is calculated. There is monotonicity between loads and displacement, so the vertex method can be used to calculate the interval expectation and variances. The nodal displacement of 121 nodes located in the top of tower is computed and shown in Figure 7. In Figure 7, red points and black points are the lower and upper bounds of expectation, respectively. Standard deviations of displacement are calculated and plotted in Figure 8. Because standard deviations of the interval random variables are constants, standard deviation of a nodal displacement is also a constant. In practical engineering design of wind turbines tower, the maximal displacements are concerned. So, the maximal upper bounds of expectation and the maximal standard deviations are selected as the design values, which are shown in Table 10. These parameters could be used for reliability analysis and structural design.

This example is to show the analysis and design produce using the interval random variables. The traditional uncertainty analysis methods for the interval random variables

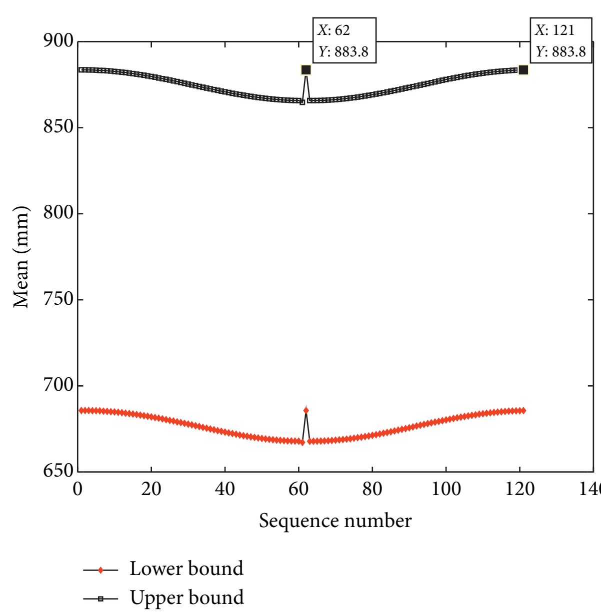


Figure 7: Interval bound curves of nodal expectations.

are time-consuming, which makes it difficult to solve the complex engineering problem. The proposed method takes full advantage of the monotonicity and makes the engineering application of the hybrid model possible.

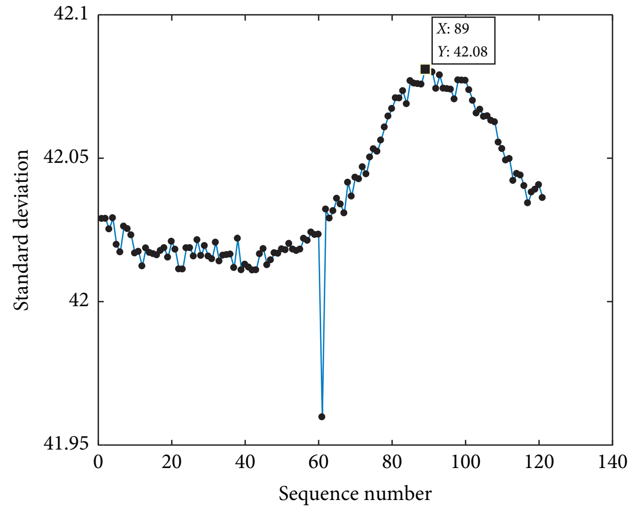


Figure 8: Nodal standard deviation curve.

Table 10: Maximal expectation and standard deviation of nodal displacement.

Maximal expectation (unit: mm)	Maximal standard deviation	Coefficient of variation
883.8	42.1	0.047

5. Conclusion

In this paper, a vertex method is proposed for the static analysis of structures that have interval random variables. Expressions for calculating the expectation and variance of interval random responses are developed. A monotonicity analysis of the response expectation and variance with respect to interval distribution parameters and variables is conducted. On the basis of the monotonicity, the intervals of the expectation and the standard variance of the response vector are obtained by the traditional probability method. The efficiency and accuracy of the proposed method are demonstrated based on four examples.

The proposed method is simple. The developed FEA code utilizes a highly popular method that has been widely used for analysis of practical engineering structures. Solving for the expectation and variance of a linear function with a random vector, traditional probability theory yields an accurate mathematical expression. Monotonicity is applied to compute the extreme values of the function. These traditional methods are integrated to form the vertex method. Therefore, the proposed method and the traditional method are compatible. Although the proposed method can be used only for a specified type of uncertain structures, it can address large-scale engineering problems with a traditional technique. In the future work, it will extend the proposed technique to the uncertain analysis of more general structural models.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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