Research Article

Asymptotic Behavior of Solutions of Even-Order Advanced Differential Equations

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Received 30 March 2020; Accepted 15 October 2020; Published 30 October 2020

Academic Editor: Praveen Agarwal

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In this paper, we establish the qualitative behavior of the even-order advanced differential equation $(a(v)(y^{(\kappa-1)}(v))^{\beta})' + \sum_{i=1}^j q_i(v)g(y(\eta_i(v))) = 0$, $v \geq v_0$. The results obtained are based on the Riccati transformation and the theory of comparison with first- and second-order equations. This new theorem complements and improves a number of results reported in the literature. Two examples are presented to demonstrate the main results.

1. Introduction

Advanced differential equations are of practical importance, which model a phenomenon in which the rate of change of a quantity depends on present and future values of the quantity. Myschkis was the first, who discussed such equations in 1955 [1] and after him Cooke and Bellman worked further on it in 1963 [2]. These types of equations have been used in modeling of various physical and engineering phenomena. For example, population genetics [3], the study of wavelets [4], population growth [5], the field of time symmetric electrodynamics [6], neural networks [7], optimal control problems with delay [8], economics [8], dynamical systems, mathematics of networks, optimization, electrical power systems, materials, energy $j \geq 1$, etc. [9] have been studied using advanced differential equations and many approaches discussed in [10–22] can be presented for solution of such equations.

In 1980, Shah et al. [23] discussed the uniqueness and existence of the solution to nonlinear and linear such types equations, while the oscillation properties of the solution were investigated by Ladas and Stavroulakis [24], and after that, particularly in the last decade, Further refinements and improvements in the theory of advanced differential

equations have been made by different researchers and it is still an active of research in engineering and applied sciences. The present paper deals with the investigation of the qualitative behavior of even-order advanced differential equation:

$$\begin{aligned} & \left(a(v)(y^{(\kappa-1)}(v))^{\beta} \right)' \\ & + \sum_{i=1}^j q_i(v)g(y(\eta_i(v))) = 0, \quad v \geq v_0, \end{aligned} \quad (1)$$

where $j \geq 1$ and β are a quotient of odd positive integers. Throughout this work, we suppose that

C1: $a \in C^1([v_0, \infty), \mathbb{R})$, $a(v) > 0$, $a'(v) \geq 0$

C2: $q_i, \eta_i \in C([v_0, \infty), \mathbb{R})$, $q_i(v) \geq 0$, $\eta_i(v) \geq v$, $\lim_{v \rightarrow \infty} \eta_i(v) = \infty$, $i = 1, 2, \dots, j$

C3: $g \in C(\mathbb{R}, \mathbb{R})$ such that $g(x)/x^{\beta} \geq k > 0$, for $x \neq 0$ and under the condition

$$\int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} ds = \infty. \quad (2)$$

By a solution of (1), we mean a function $y \in C^{\kappa-1}[v_y, \infty)$, $v_y \geq v_0$, which has the property $a(v)(y^{(\kappa-1)}(v))^\beta \in C^1[v_y, \infty)$, and satisfies (1) on $[v_y, \infty)$. We consider only those solutions y of (1) which satisfy $\sup\{|y(v)|: v \geq v_y\} > 0$. A solution of (1) is called oscillatory if it has arbitrarily large zeros on $[v_y, \infty)$; otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

2. The Motivation of Studying this Paper

During this decade, several works have been accomplished in the development of the oscillation theory of higher-order advanced equations by using the Riccati transformation and the theory of comparison between first- and second-order delay equations [25–39]. Further, the oscillation theory of fourth- and second-order equations has been studied and developed by using integral averaging technique and the Riccati transformation [40–45].

The study of oscillation has been carried to fractional equations in the setting of fractional operators with singular and nonsingular kernels as well (see [46, 47] and the references therein).

The main aim of this paper is to complement and improve the results of [48–49]. For this purpose we discuss these results.

Moaaz et al. [26] considered the fourth-order differential equation:

$$\left(a(v)(y'''(v))^\beta\right)' + q(v)y^\alpha(\eta(v)) = 0, \quad (3)$$

where β, α are quotients of odd positive integers.

Grace et al. [27] considered the equation

$$\left(a(v)(y''(v))^\beta\right)'' + q(v)g(y(\eta(v))) = 0, \quad (4)$$

where $\eta(v) \leq v$, β is a quotient of odd positive integers.

In particular, by using the comparison technique, the equation

$$\left((y^{(\kappa-1)}(v))^\beta\right)' + q(v)y^\beta(\eta(v)) = 0, \quad (5)$$

has been studied by Agarwal and Grace [48], and they proved it oscillatory if

$$\liminf_{v \rightarrow \infty} \int_v^{\eta(v)} (\eta(s) - s)^{\kappa-2} \left(\int_s^\infty q(v) dv \right)^{1/\beta} ds > \frac{(\kappa-2)!}{e}. \quad (6)$$

Agarwal and Grace [48] extended the Riccati transformation to obtain new oscillatory criteria for (5) as condition

$$\limsup_{v \rightarrow \infty} v^{\beta(\kappa-1)} \int_v^\infty q(s) ds > ((\kappa-1)!)^\beta. \quad (7)$$

Authors in [50] studied oscillatory behavior of (5) where $\beta = 1$ and if there exists a function $\tau \in C^1([v_0, \infty), (0, \infty))$, also, they proved it oscillatory by using the Riccati transformation if

$$\int_{v_0}^\infty \left(\tau(s)q(s) - \frac{(\kappa-2)! (\tau'(s))^2}{2^{3-2\kappa} s^{\kappa-2} \tau(s)} \right) ds = \infty. \quad (8)$$

To prove this, we apply the previous results to the equation

$$y^{(4)}(v) + \frac{q_0}{v^4} y(2v) = 0, \quad v \geq 1. \quad (9)$$

(1) By applying condition (6) in [48], we get

$$q_0 > 25.5. \quad (10)$$

(2) By applying condition (7) in [49], we get

$$q_0 > 18. \quad (11)$$

(3) By applying condition (8) in [50], we get

$$q_0 > 1728. \quad (12)$$

From the above, we find the results in [49] improve results [50]. Moreover, the results in [48] improve results [49, 50].

Thus, the motivation in studying this paper is complement and improve results [48–50].

We shall employ the following lemmas.

Lemma 1 (see [44]). *If $y^{(i)}(v) > 0$, $i = 0, 1, \dots, \kappa$, and $y^{(\kappa+1)}(v) < 0$, then*

$$\frac{y(v)}{v^\kappa/\kappa!} \geq \frac{y'(v)}{v^{\kappa-1}/(\kappa-1)!}. \quad (13)$$

Lemma 2 (see [44]). *Suppose that $y \in C^\kappa([v_0, \infty), (0, \infty))$, $y^{(\kappa)}$ is of a fixed sign on $[v_0, \infty)$, $y^{(\kappa)}$ not identically zero and there exists a $v_1 \geq v_0$ such that*

$$y^{(\kappa-1)}(v)y^{(\kappa)}(v) \leq 0, \quad (14)$$

for all $v \geq v_1$. If we have $\lim_{v \rightarrow \infty} y(v) \neq 0$, then there exists $v_\theta \geq v_1$ such that

$$y(v) \geq \frac{\theta}{(\kappa-1)!} v^{\kappa-1} |y^{(\kappa-1)}(v)|, \quad (15)$$

for every $\theta \in (0, 1)$ and $v \geq v_\theta$.

Lemma 3 (see [34]). *Let β be a ratio of two odd numbers, $V > 0$ and U are constants. Then*

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^\beta}, \quad V > 0. \quad (16)$$

Lemma 4 (see [29]). *Suppose that y is an eventually positive solution of (1). Then, there exist two possible cases:*

$$\begin{aligned} (\mathbf{S}_1) \quad & y(v) > 0, y'(v) > 0, \quad y''(v) > 0, y^{(\kappa-1)}(v) > 0, \\ & y^{(\kappa)}(v) < 0 \\ (\mathbf{S}_2) \quad & y(v) > 0, \quad y^{(r)}(v) > 0, \quad y^{(r+1)}(v) < 0 \text{ for all odd} \\ & \text{integer } r \in \{1, 3, \dots, \kappa-3\}, \quad y^{(\kappa-1)}(v) > 0, \quad y^{(\kappa)}(v) < 0 \end{aligned}$$

For $v \geq v_1$, where $v_1 \geq v_0$ is sufficiently large.

3. Comparison Theorems with Second/First-Order Equations

Theorem 1. Assume that (2) holds. If the differential equations

$$\left(\frac{(\kappa-2)! a^{1/\beta}(v)}{(\theta v^{\kappa-2})^\beta} (y'(v))^\beta \right)' + k \sum_{i=1}^j q_i(v) y^\beta(v) = 0, \quad (17)$$

$$\begin{aligned} & y''(v) + y(v) \frac{1}{(\kappa-4)!} \int_v^\infty (\zeta-v)^{\kappa-4} \\ & \cdot \left(\frac{1}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta = 0. \end{aligned} \quad (18)$$

are oscillatory. Then every solution of (1) is oscillatory.

Proof. Assume the contrary that y is a positive solution of (1). Then, we can suppose that $y(v)$ and $y(\eta_i(v))$ are positive

for all $v \geq v_1$ sufficiently large. From Lemma 4, we have two possible cases ($\mathbf{S}_1$) and ($\mathbf{S}_2$).

Let case ($\mathbf{S}_1$) holds. Using Lemma 2, we find

$$y'(v) \geq \frac{\theta}{2} v^{\kappa-2} y^{(\kappa-1)}(v), \quad (19)$$

for every $\theta \in (0, 1)$ and for all large v .

Define

$$\varphi(v) := \tau(v) \left(\frac{a(v) (y^{(\kappa-1)}(v))^\beta}{y^\beta(v)} \right), \quad (20)$$

we see that $\varphi(v) > 0$ for $v \geq v_1$, where $\tau \in C^1([v_0, \infty), (0, \infty))$ and

$$\begin{aligned} \varphi'(v) &= \tau'(v) \frac{a(v) (y^{(\kappa-1)}(v))^\beta}{y^\beta(v)} \\ &+ \tau(v) \frac{(a(y^{(\kappa-1)}(v))^\beta)'(v)}{y^\beta(v)} \\ &- \beta \tau(v) \frac{y^{\beta-1}(v) y'(v) a(v) (y^{(\kappa-1)}(v))^\beta}{y^{2\beta}(v)}. \end{aligned} \quad (21)$$

Using (19) and (20), we obtain

$$\begin{aligned} \varphi'(v) &\leq \frac{\tau'(v)}{\tau(v)} \varphi(v) + \tau(v) \frac{(a(v) (y^{(\kappa-1)}(v))^\beta)'(v)}{y^\beta(v)} - \beta \tau(v) \frac{\theta}{(\kappa-2)!} v^{\kappa-2} \frac{a(v) (y^{(\kappa-1)}(v))^\beta}{y^{\beta+1}(v)} \\ &\leq \frac{\tau'(v)}{\tau(v)} \varphi(v) + \tau(v) \frac{(a(v) (y^{(\kappa-1)}(v))^\beta)'(v)}{y^\beta(v)} - \frac{\beta \theta v^{\kappa-2}}{(\kappa-2)! (\tau(v) a(v))^{1/\beta}} \varphi(v)^{\beta+1/\beta}. \end{aligned} \quad (22)$$

From (1) and (22), we obtain

$$\begin{aligned} \varphi'(v) &\leq \frac{\tau'(v)}{\tau(v)} \varphi(v) - k \tau(v) \frac{\sum_{i=1}^j q_i(v) y^\beta(\eta_i(v))}{y^\beta(v)} \\ &- \frac{\beta \theta v^{\kappa-2}}{(\kappa-2)! (\tau(v) a(v))^{1/\beta}} \varphi(v)^{\beta+1/\beta}. \end{aligned} \quad (23)$$

Note that $y'(v) > 0$ and $\eta_i(v) \geq v$; thus, we find

$$\begin{aligned} \varphi'(v) &\leq \frac{\tau'(v)}{\tau(v)} \varphi(v) - k \tau(v) \sum_{i=1}^j q_i(v) \\ &- \frac{\beta \theta v^{\kappa-2}}{(\kappa-2)! (\tau(v) a(v))^{1/\beta}} \varphi(v)^{\beta+1/\beta}. \end{aligned} \quad (24)$$

If we set $\tau(v) = k = 1$ in (24), then we find

$$\varphi'(v) + \frac{\beta \theta v^{\kappa-2}}{(\kappa-2)! a^{1/\beta}(v)} \varphi(v)^{\beta+1/\beta} + \sum_{i=1}^j q_i(v) \leq 0. \quad (25)$$

From [25], we can see that equation (17) is non-oscillatory, which is a contradiction.

Let case ($\mathbf{S}_2$) holds. Define

$$\psi(v) := \vartheta(v) \frac{y'(v)}{y(v)}, \quad (26)$$

we see that $\psi(v) > 0$ for $v \geq v_1$, where $\vartheta \in C^1([v_0, \infty), (0, \infty))$. By differentiating $\psi(v)$, we find

$$\psi'(v) = \frac{\vartheta'(v)}{\vartheta(v)} \psi(v) + \vartheta(v) \frac{y''(v)}{y(v)} - \frac{1}{\vartheta(v)} \psi(v)^2. \quad (27)$$

Now, integrating (1) from v to m and using $y'(v) > 0$, we find

$$a(m)(y^{(\kappa-1)}(m))^\beta - a(v)(y^{(\kappa-1)}(v))^\beta = - \int_v^m \sum_{i=1}^j q_i(s) g(y(\eta_i(s))) ds. \quad (28)$$

By virtue of $y'(v) > 0$ and $\eta_i(v) \geq v$, we get

$$\begin{aligned} a(m)(y^{(\kappa-1)}(m))^\beta - a(v)(y^{(\kappa-1)}(v))^\beta \\ \leq -k y^\beta(v) \int_v^u \sum_{i=1}^j q_i(s) ds. \end{aligned} \quad (29)$$

Letting $m \rightarrow \infty$, we see that

$$a(v)(y^{(\kappa-1)}(v))^\beta \geq k y^\beta(v) \int_v^\infty \sum_{i=1}^j q_i(s) ds, \quad (30)$$

and so

$$y^{(\kappa-1)}(v) \geq y(v) \left(\frac{k}{a(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta}. \quad (31)$$

Integrating again from v to ∞ for a total of $(\kappa - 4)$ times, we get

$$\begin{aligned} y''(v) + \frac{y(v)}{(\kappa - 4)!} \int_v^\infty (\zeta - v)^{\kappa-4} \\ \cdot \left(\frac{k}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta \leq 0. \end{aligned} \quad (32)$$

From (27) and (32), we obtain

$$\begin{aligned} \psi'(v) \leq \frac{\vartheta'(v)}{\vartheta(v)} \psi(v) - \frac{\vartheta(v)}{(\kappa - 4)!} \int_v^\infty (\zeta - v)^{\kappa-4} \\ \cdot \left(\frac{k}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta - \frac{1}{\vartheta(v)} \psi(v)^2. \end{aligned} \quad (33)$$

If we now set $\vartheta(v) = k = 1$ in (33), then we obtain

$$\begin{aligned} \psi'(v) + \psi^2(v) + \frac{1}{(\kappa - 4)!} \int_v^\infty (\zeta - v)^{\kappa-4} \\ \cdot \left(\frac{1}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta \leq 0. \end{aligned} \quad (34)$$

From [25], we see equation (18) is nonoscillatory, which is a contradiction. Theorem 1 is proved. $\square$

Remark 1. It is well known (see [42]) that if

$$\int_{v_0}^\infty \frac{1}{a(v)} dv = \infty, \quad (35)$$

$$\liminf_{v \rightarrow \infty} \left(\int_{v_0}^v \frac{1}{a(s)} ds \right) \int_v^\infty q(s) ds > \frac{1}{4},$$

then equation

$$\left[a(v) (y'(v))^\beta \right]' + q(v) y^\beta(g(v)) = 0, \quad v \geq v_0, \quad (36)$$

where $\beta = 1$ is oscillatory.

Based on the above results and Theorem 1, we can easily obtain the following Hille and Nehari type oscillation criteria for (1) with $\beta = 1$.

Theorem 2. Let $\beta = k = 1$. Assume that (2) holds. If

$$\int_{v_0}^\infty \frac{\theta v^{\kappa-2}}{(\kappa - 2)! a(v)} dv = \infty, \quad (37)$$

$$\liminf_{v \rightarrow \infty} \left(\int_{v_0}^v \frac{\theta s^{\kappa-2}}{(\kappa - 2)! a(s)} ds \right) \int_v^\infty \sum_{i=1}^j q_i(s) ds > \frac{1}{4}, \quad (38)$$

also, if

$$\begin{aligned} \liminf_{v \rightarrow \infty} v \int_{v_0}^v \frac{1}{(\kappa - 4)!} \int_v^\infty (\zeta - v)^{\kappa-4} \\ \cdot \left(\frac{1}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta dv > \frac{1}{4}, \end{aligned} \quad (39)$$

for some constant $\theta \in (0, 1)$. Then all solution of (1) is oscillatory.

In the theorem, we compare the oscillatory behavior of (1) with the first-order differential equations:

Theorem 3. Assume that (2) holds. If the differential equations

$$x'(v) + k \sum_{i=1}^j q_i(v) \left(\frac{\theta v^{\kappa-2}}{(\kappa - 2)! a^{1/\beta}(v)} \right)^\beta x(\eta(v)) = 0, \quad (40)$$

$$\begin{aligned} z'(v) + z(v) \frac{v}{(\kappa - 4)!} \int_v^\infty (\zeta - v)^{\kappa-4} \\ \cdot \left(\frac{k}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta = 0, \end{aligned} \quad (41)$$

are oscillatory, then every solution of (1) is oscillatory.

Proof. Assume the contrary that y is a positive solution of (1). Then, we can suppose that $y(v)$ and $y(\eta_i(v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 4, we have two possible cases (S_1) and (S_2) .

In the case where (S_1) holds, from Lemma 2, we see

$$y(v) \geq \frac{\theta v^{\kappa-2}}{(\kappa-2)! a^{1/\beta}(v)} (a^{1/\beta}(v) y^{(\kappa-1)}(v)), \quad (42)$$

for every $\theta \in (0, 1)$ and for all large v . Thus, if we set

$$x(v) = a(v) (y^{(\kappa-1)}(v))^\beta > 0, \quad (43)$$

then we see that ψ is a positive solution of the inequality.

$$x'(v) + k \sum_{i=1}^j q_i(v) \left(\frac{\theta v^{\kappa-2}}{(\kappa-2)! a^{1/\beta}(v)} \right)^\beta x(\eta(v)) \leq 0. \quad (44)$$

From [?, Theorem 1], we see that the equation (40) also has a positive solution, which is a contradiction.

In the case where (S_2) holds, from Lemma 1, we get

$$y(v) \geq v y'(v). \quad (45)$$

From (32) and (45), we get

$$y''(v) + y'(v) \frac{v}{(\kappa-4)!} \int_v^\infty (\zeta-v)^{\kappa-4} \cdot \left(\frac{k}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta \leq 0. \quad (46)$$

Now, we set

$$z(v) = y'(v). \quad (47)$$

Thus, we find ψ is a positive solution of the inequality

$$z'(v) + z(v) \frac{v}{(\kappa-4)!} \int_v^\infty (\zeta-v)^{\kappa-4} \cdot \left(\frac{k}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta \leq 0. \quad (48)$$

It is well known (see [?, Theorem 1]) that the equation (41) also has a positive solution, which is a contradiction. The proof is complete. $\square$

Corollary 1. Let (2) holds. If

$$\liminf_{v \rightarrow \infty} \int_{\eta_i(v)}^v \sum_{i=1}^j q_i(s) \left(\frac{\theta v^{\kappa-2}}{(\kappa-2)! a^{1/\beta}(v)} \right)^\beta ds > \frac{((\kappa-1)!)^\beta}{e} \quad (49)$$

$$\liminf_{v \rightarrow \infty} \int_{\eta_i(v)}^v \frac{s}{(\kappa-4)!} \int_v^\infty (\zeta-v)^{\kappa-4} \left(\frac{k}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta ds > \frac{1}{e},$$

then every solution of (1) is oscillatory.

Example 1. Let the equation

$$y^{(4)}(v) + \frac{q_0}{v^4} y(3v) = 0, \quad v \geq 1, \quad (50)$$

where $q_0 > 0$ is a constant. Note that $\beta = 1, \kappa = 4, a(v) = 1, q(v) = q_0/v^4$, and $\eta(v) = 3v$. If we set $k = 1$, then condition (38) becomes

$$\liminf_{v \rightarrow \infty} \left(\int_{v_0}^v \frac{\theta s^{\kappa-2}}{(\kappa-2)! a(s)} ds \right) \int_v^\infty \sum_{i=1}^j q_i(s) ds \quad (51)$$

$$= \liminf_{v \rightarrow \infty} \left(\frac{v^3}{3} \right) \int_v^\infty \frac{q_0}{s^4} ds = \frac{q_0}{9} > \frac{1}{4}$$

and condition (39) becomes

$$\liminf_{v \rightarrow \infty} v \int_{v_0}^v \frac{1}{(\kappa-4)!} \int_v^\infty (\zeta-v)^{\kappa-4} \cdot \left(\frac{1}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta dv \quad (52)$$

$$= \liminf_{v \rightarrow \infty} v \left(\frac{q_0}{6v} \right) = \frac{q_0}{6} > \frac{1}{4}.$$

Therefore, from Theorem 2, all solution equation (51) is oscillatory if $q_0 > 2.25$.

Remark 2. We compare our result with the known related criteria for oscillation of this equation as follows (Table 1).

Therefore, our result improves results [48–50].

Example 2. Consider a differential equation (9) where $q_0 > 0$ is a constant. Note that $\beta = 1, \kappa = 4, a(v) = 1, q(v) = q_0/v^4$, and $\eta(v) = 2v$. If we set $k = 1$, then condition (38) becomes

TABLE 1: Comparison of results.

The condition	(6)	(7)	(8)
The criterion	$q_0 > 13.6$	$q_0 > 18$	$q_0 > 576$

$$\frac{q_0}{9} > \frac{1}{4}. \quad (53)$$

Therefore, from Theorem 2, all solution equation (9) is oscillatory if $q_0 > 2.25$.

Remark 3. Our result improves results [48–50].

4. Conclusion

In this article, we study the oscillatory behavior of a class of nonlinear even-order differential equations and establish sufficient conditions for oscillation of an even-order differential equation by using the theory of comparison with first- and second-order delay equations and Riccati substitution technique.

For researchers interested in this field, and as part of our future research, there is a nice open problem which is finding new results in the following case:

$$\int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} ds < \infty. \quad (54)$$

For all this, there is some research in progress.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest associated with this publication.

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