

Research Article

Linear Feedback of Mean-Field Stochastic Linear Quadratic Optimal Control Problems on Time Scales

Yingjun Zhu and Guangyan Jia 

Zhongtai Securities Institute for Financial Studies, Shandong University, Jinan 250100, China

Correspondence should be addressed to Guangyan Jia; jiagy@sdu.edu.cn

Received 7 September 2019; Revised 17 April 2020; Accepted 13 May 2020; Published 30 May 2020

Academic Editor: Nicolae Herisanu

Copyright © 2020 Yingjun Zhu and Guangyan Jia. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper addresses a version of the linear quadratic control problem for mean-field stochastic differential equations with deterministic coefficients on time scales, which includes the discrete time and continuous time as special cases. Two coupled Riccati equations on time scales are given and the optimal control can be expressed as a linear state feedback. Furthermore, we give a numerical example.

1. Introduction

The linear quadratic control problem is one of the most important issues for optimal control problem. The study of the mean-field linear quadratic optimal control problem also has received much attention [1, 2], and it has a wide range of applications in engineering and finance [3, 4]. Until now, the mean-field linear quadratic control problem is well understood both from the continuous and discrete points of view. In this paper, the mean-field linear quadratic control problem is studied in the version of time scales.

Time scales were first introduced by Hilger [5] in 1988 in order to unite differential and difference equations into a general framework. Recently, time scales theory is extensively studied in many works [6–14]. It is well known that the optimal control problems on time scales are an important field for both theory and applications. Since the calculus of variations on time scales was studied by Bohner [15], results on related topics and their applications has become more and more. The existence of optimal control for the dynamic systems on time scales was discussed [16–18]. Subsequently, maximum principles were studied in several work [19, 20], which specify the necessary conditions for optimality. In addition, Bellman dynamic

programming on time scales for the deterministic optimal control problems was considered in [21]. At the same time, some results were obtained for the linear quadratic control problems for deterministic linear system on time scales in [22, 23]. In [24], the authors developed the linear quadratic control problems for stochastic linear system on time scales. To our best knowledge, the optimal control problems for the mean-field system on time scales have not been established.

We are interested in the mean-field stochastic linear quadratic control problem on time scales (MF-S Δ LQ for short). To deal with the well posedness of the state equation on time scales, we use the similar iteration method as [25]. Very similar to continuous and discrete cases, we can also get the associated Riccati equations (see [26, 27], for continuous and discrete cases) on time scales, and the optimal control can be expressed as a linear state feedback through the solutions of the coupled Riccati equations.

The organization of this paper is as follows. In Section 2, we show some preliminaries about time scales' theory and MF-S Δ LQ problem. We study the well posedness of the state equation on time scales and show the feedback representation of the optimal control by the associated Riccati equations on time scales in Section 3. Finally, an example is presented.

2. Preliminaries

A time scales \mathbb{T} is a nonempty closed subset of real numbers set \mathbb{R} and we denote $[0, T]_{\mathbb{T}} = [0, T] \cap \mathbb{T}$. In this paper, we always suppose \mathbb{T} is bounded. The forward jump operator σ and backward jump operator ρ are, respectively, defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad (1)$$

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad (2)$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, where \emptyset denotes the empty set). If $\sigma(t) = t$ ($\sigma(t) > t$, $\rho(t) = t$, and $\rho(t) = t$), the point t is called right-dense (right-scattered, left-dense, and left-scattered). Moreover, a point is called isolated if it is both left-scattered and right-scattered. For a function f , we denote $f^\sigma = f \cdot \sigma$ and $f^\rho = f \cdot \rho$. The definition of the graininess function μ is as follows:

$$\mu(t) = \sigma(t) - t. \quad (3)$$

We now present some basic concepts and properties about time scales (see [10, 11]).

Definition 1. Let f be a function on time scales \mathbb{T} , and f is called right-dense continuous function if f is continuous at every right-dense point and has finite left-sided limits at every left-dense point. Similarly, f is called left-dense continuous function if f is continuous at every left-dense point and has finite right-sided limits at every right-dense point. If f is right-dense continuous and also is left-dense continuous, then f is called continuous function.

Remark 1. If a function f is right-dense continuous, then f has an antiderivative F .

Define the set \mathbb{T}^κ as follows:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}), & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases} \quad (4)$$

Definition 2. Let $f: \mathbb{T} \rightarrow \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$, and if for all $\varepsilon > 0$, there exist a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \quad \text{for all } s \in U. \quad (5)$$

We call $f^\Delta(t)$ the Δ derivative of f at t .

Remark 2. If the functions f and g are differentiable at t , then the product fg is also differentiable at t and the product rule is given by

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t). \quad (6)$$

In this paper, we adopt the stochastic integral defined by Bohner et al. [25]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]_{\mathbb{T}}}, P)$ be a complete probability space with an increasing and continuous filtration $\{\mathcal{F}_t\}_{t \in [0, T]_{\mathbb{T}}}$. We define that $L^2_{\mathcal{F}}([0, T]_{\mathbb{T}}; \mathbb{R})$ is the set of all \mathcal{F}_t -adapted, \mathbb{R} -valued measurable process $X(t)$ such that $E[\int_0^T |X(t)|^2 \Delta t] < \infty$.

A Brownian motion indexed by time scales \mathbb{T} is defined by Grow and Sanyal [13]. Although the Brownian motion on time scales is very similar to that on continuous time, but there are also some differences between them. For example, the quadratic variation of a Brownian motion on time scales (see [14]) is an increasing process yet, but it is not deterministic. In fact, $\langle W \rangle_t = \lambda([0, t]_{\mathbb{T}}) + \sum_{s \leq t} (W_s^\sigma - W_s)^2$, where λ is the Lebesgue measure.

Next, we give the definition of the stochastic Δ integral and its properties.

Definition 3 (see [25]). The random process $X(t)$ is stochastic Δ integrable on $[0, T]_{\mathbb{T}}$ if the corresponding $\tilde{X}(t)$ is integrable, and define the integral value of $X(t)$ as

$$\int_0^T X(t) \Delta W(t) := \int_0^T \tilde{X}(t) dW(t), \quad (7)$$

where

$$\tilde{X}(t) = X(\sup [0, t]_{\mathbb{T}}), \quad \text{for all } t \in [0, T], \quad (8)$$

and the Brownian motion on the right side of (7) is indexed by continuous time.

We also have the following properties.

Let $X(t), Y(t) \in L^2_{\mathcal{F}}([0, T]_{\mathbb{T}}; \mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$. Then,

$$\begin{aligned} \text{(i)} \quad & \int_0^T (\alpha X(t) + \beta Y(t)) \Delta W(t) = \alpha \int_0^T X(t) \Delta W(t) + \beta \int_0^T Y(t) \Delta W(t), \\ \text{(ii)} \quad & \mathbb{E} \left[\int_0^T X(t) \Delta W(t) \right] = 0, \\ \text{(iii)} \quad & \mathbb{E} \left[\left| \int_0^T X(t) \Delta W(t) \right|^2 \right] = \mathbb{E} \left[\int_0^T |X(t)|^2 \Delta \langle W \rangle_t \right] = \mathbb{E} \left[\int_0^T |X(t)|^2 \Delta t \right], \end{aligned} \quad (9)$$

where the integral X with respect to the quadratic variation of Brownian motion $\langle W \rangle_t$ is defined by Stieltjes integral as $\int_0^T X_t(\omega) \Delta \langle W \rangle_t(\omega)$.

Notation 1. The following notation will be used:

M' : the transpose of any vector or matrix M

$|M|$: $= \sqrt{\sum_{i,j} m_{ij}^2}$ for any matrix or vector $M = (m_{ij})$

$M \geq 0$: M is a positive semidefinite matrix
 S^n : the space of all $n \times n$ symmetric matrices
 $\langle X, Y \rangle_t$: the quadratic covariation process of X and Y
 $L^\infty([0, T]_{\mathbb{T}}; \mathbb{R})$: the space of bounded, Δ -Lebesgue integrable, and \mathbb{R} -valued functions on $[0, T]_{\mathbb{T}}$

$C^1([0, T]_{\mathbb{T}}; \mathbb{R})$: the family of all \mathbb{R} -valued continuous functions $f(t)$ defined on $[0, T]_{\mathbb{T}}$ such that they are Δ differentiable in t

Finally, we introduce our MF-S Δ LQ problem. Consider the following stochastic Δ -differential equation:

$$\begin{cases} \Delta X(t) = \{A(t)X(t) + \bar{A}(t)\mathbb{E}[X(t)] + B(t)u(t) + \bar{B}(t)\mathbb{E}[u(t)]\}\Delta t + \{D(t)u(t) + \bar{D}(t)\mathbb{E}[u(t)]\}\Delta W, \\ X(0) = x, \end{cases} \quad (10)$$

where the coefficients $A(\cdot), \bar{A}(\cdot), B(\cdot), \bar{B}(\cdot), D(\cdot)$, and $\bar{D}(\cdot)$ are all deterministic matrix-valued functions and $u(\cdot) \in \mathcal{U}[0, T]_{\mathbb{T}} = L^2_{\mathcal{F}}([0, T]_{\mathbb{T}}; \mathbb{R}^m)$. The cost functional is

$$\begin{aligned} J(x; u(\cdot)) = \mathbb{E} \left\{ \int_0^T [X'(t)Q(t)X(t) + \mathbb{E}[X(t)]'\bar{Q}(t)\mathbb{E}[X(t)] \right. \\ \left. + u'(t)R(t)u(t) + \mathbb{E}[u(t)]'\bar{R}(t)\mathbb{E}[u(t)]\right]\Delta t \\ \left. + X'(T)GX(T) + \mathbb{E}[X(T)]'\bar{G}\mathbb{E}[X(T)] \right\}, \end{aligned} \quad (11)$$

where G and \bar{G} are symmetric matrices and $Q(\cdot), \bar{Q}(\cdot), R(\cdot)$, and $\bar{R}(\cdot)$ are given deterministic matrix-valued functions.

Problem (MF-S Δ LQ). For any given initial state $x \in \mathbb{R}^n$, find a $u^*(\cdot) \in \mathcal{U}[0, T]_{\mathbb{T}}$ such that

$$J(x; u^*(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]_{\mathbb{T}}} J(x; u(\cdot)). \quad (12)$$

$u^*(\cdot)$ is called an optimal control of the MF-S Δ LQ problems and the corresponding $X(\cdot; x, u^*(\cdot))$ is called an optimal state process.

3. Main Results

First, we introduce the following assumptions which are necessary for the proofs of our main results.

(H1) Assume that

$$A(\cdot), \bar{A}(\cdot) \in L^\infty([0, T]_{\mathbb{T}}; \mathbb{R}^{n \times n}), B(\cdot), \bar{B}(\cdot), D(\cdot), \bar{D}(\cdot) \in L^\infty([0, T]_{\mathbb{T}}; \mathbb{R}^{n \times m}). \quad (13)$$

(H2) Assume that

$$Q(\cdot), \bar{Q}(\cdot) \in L^\infty([0, T]_{\mathbb{T}}; S^n), \quad R(\cdot), \bar{R}(\cdot) \in L^\infty([0, T]_{\mathbb{T}}; S^m), \quad G, \bar{G} \in S^n, \quad (14)$$

and for some $\delta > 0$,

$$\begin{cases} Q(s) \geq 0, Q(s) + \bar{Q}(s) \geq 0, \quad R(s) \geq \delta I, R(s) + \bar{R}(s) \geq \delta I, \quad s \in [0, T]_{\mathbb{T}}, \\ G \geq 0, G + \bar{G} \geq 0. \end{cases} \quad (15)$$

Remark 3. Assumption (H1) can guarantee the existence and uniqueness of the solution of the mean-field stochastic linear system (10). Under Assumptions (H1) and (H2), we can establish two coupled Riccati equations to show the feedback control.

Now, we show the well posedness of the state equation (10) by the iteration method, which is very similar to the way as in [25].

Theorem 1. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]_{\mathbb{T}}}, P)$ be given and W be a standard $\{\mathcal{F}_t\}_{t \in [0, T]_{\mathbb{T}}}$ Brownian motion. Suppose that (H1) holds, then system (10) has a unique solution $X \in L^2_{\mathcal{F}}([0, T]_{\mathbb{T}}; \mathbb{R}^n)$ for any $(x, u(\cdot)) \in \mathbb{R}^n \times \mathcal{U}[0, T]_{\mathbb{T}}$.

Proof. For the existence, we adopt the iteration method and define

$$X^0(t) = x, \quad (16)$$

$$\begin{aligned} X^n(t) &= x + \int_0^t \{A(s)X^{n-1}(s) + \bar{A}(s)\mathbb{E}[X^{n-1}(s)] \\ &\quad + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)]\} \Delta s \\ &\quad + \int_0^t \{D(s)u(s) + \bar{D}(s)\mathbb{E}[u(s)]\} \Delta W(s), \end{aligned} \quad (17)$$

$n \in \mathbb{N}, t \in [0, T]_{\mathbb{T}}$.

Let $\delta^n(t) = \mathbb{E}[|X^{n+1}(t) - X^n(t)|^2]$, and we claim that

$$\delta^n(t) \leq M^{n+1} h_{n+1}(t, 0), \quad n \in \mathbb{N}, t \in [0, T]_{\mathbb{T}}, \quad (18)$$

where M is a generic constant and h_n is the generalized monomials defined in [28]. When $n = 0$, we obtain

$$\begin{aligned} \delta^0(t) &= \mathbb{E}\left[|X^1(t) - X^0(t)|^2\right] \\ &= \mathbb{E}\left[\left|\int_0^t \{A(s)X^0(s) + \bar{A}(s)\mathbb{E}[X^0(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)]\} \Delta s + \int_0^t \{D(s)u(s) + \bar{D}(s)\mathbb{E}[u(s)]\} \Delta W(s)\right|^2\right] \\ &\leq 2\mathbb{E}\left[\left|\int_0^t \{A(s)X^0(s) + \bar{A}(s)\mathbb{E}[X^0(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)]\} \Delta s\right|^2\right] \\ &\quad + 2\mathbb{E}\left[\int_0^t |D(s)u(s) + \bar{D}(s)\mathbb{E}[u(s)]|^2 \Delta s\right] \\ &\leq tM = Mh_1(t, 0). \end{aligned} \quad (19)$$

Suppose inequality (14) holds for $n - 1$, then

$$\begin{aligned} \delta^n(t) &= \mathbb{E}\left[|X^{n+1}(t) - X^n(t)|^2\right] \\ &= \mathbb{E}\left[\left|\int_0^t \{A(s)[X^n(s) - X^{n-1}(s)] + \bar{A}(t)\mathbb{E}[X^n(s) - X^{n-1}(s)]\} \Delta s\right|^2\right] \\ &= 2\mathbb{E}\left[\left|\int_0^t A(s)[X^n(s) - X^{n-1}(s)] \Delta s\right|^2\right] + 2\mathbb{E}\left[\left|\int_0^t \bar{A}(t)\mathbb{E}[X^n(s) - X^{n-1}(s)] \Delta s\right|^2\right] \\ &\leq M \int_0^t \mathbb{E}\left[|X^n(s) - X^{n-1}(s)|^2\right] \Delta s = M \int_0^t \delta^{n-1}(s) \Delta s \\ &\leq M \int_0^t M^n h_n(s, 0) \Delta s = M^{n+1} h_{n+1}(t, 0), \quad n \in \mathbb{N}, t \in [0, T]_{\mathbb{T}}. \end{aligned} \quad (20)$$

This proves the claim.
Similarly, we have

$$\begin{aligned} \sup_{t \in [0, T]_{\mathbb{T}}} |X^{n+1}(t) - X^n(t)|^2 &\leq 2TM \int_0^T (|X^n(s) - X^{n-1}(s)|^2 \\ &\quad + \mathbb{E}[|X^n(s) - X^{n-1}(s)|^2]) \Delta s, \quad n \in \mathbb{N}. \end{aligned} \quad (21)$$

By a martingale inequality and by inequality (18) (see [25], for details), one has

$$\mathbb{E}\left[\sup_{t \in [0, T]_{\mathbb{T}}} |X^{n+1}(t) - X^n(t)|^2\right] \leq CM^n h_n(T, 0), \quad n \in \mathbb{N}, \quad (22)$$

where $C = 4TM$. Note that a simple probability inequality is obtained from Markov's inequality, $P(|Y| > a) \leq (1/a^p)\mathbb{E}[|Y|^p]$, where $a > 0, p > 0$, and Y is a random variable. Using the probability inequality, where

$Y = \sup_{t \in [0, T]_{\mathbb{T}}} |X^{n+1}(t) - X^n(t)|, a = 1/2^n$ and $p = 2$, we obtain

$$P\left(\sup_{t \in [0, T]_{\mathbb{T}}} |X^{n+1}(t) - X^n(t)| > \frac{1}{2^n}\right) \leq 4^n CM^n h_n(T, 0), \quad n \in \mathbb{N}. \quad (23)$$

According to Borel–Cantelli lemma, this implies that

$$P\left(\sup_{t \in [0, T]_{\mathbb{T}}} |X^{n+1}(t) - X^n(t)| > \frac{1}{2^n} \text{ i.o.}\right) = 0, \quad (24)$$

where *i.o.* is the abbreviation of “infinitely often”. Consequently, $X^0 + \sum_{i=0}^{n-1} (X^{i+1} - X^i)$ converges uniformly. Let $n \rightarrow \infty$, we have

$$\begin{aligned} X(t) &= x + \int_0^t \{A(s)X(s) + \bar{A}(s)\mathbb{E}[X(s)] + B(s)u(s) \\ &\quad + \bar{B}(s)\mathbb{E}[u(s)]\} \Delta s + \int_0^t \{D(s)u(s) + \bar{D}(s)\mathbb{E}[u(s)]\} \Delta W(s). \end{aligned} \quad (25)$$

For the uniqueness, we assume X_1 and X_2 are both solution. Then,

$$X_1(t) - X_2(t) = \int_0^t \{A(s)(X_1(s) - X_2(s)) + \bar{A}(s)\mathbb{E}[X_1(s) - X_2(s)]\} \Delta s. \quad (26)$$

It follows that

$$\mathbb{E}[|X_1(t) - X_2(t)|^2] \leq C \int_0^t \mathbb{E}[|X_1(s) - X_2(s)|^2] \Delta s. \quad (27)$$

By Gronwall's inequality [29], we obtain $\mathbb{E}[|X_1(t) - X_2(t)|^2] = 0$. Thus, $X_1 = X_2$.

We are in a position to give the main results of the MF-SALQ optimal control problem. For this, we need a useful lemma. By some simple calculations, it is not hard for us to get the following product rule for stochastic processes on time scales, which is very similar to Du and Dieu [12]. \square

Lemma 1. For any two n -dimensional stochastic processes X_1 and X_2 with

$$\begin{cases} \Delta X_i(t) = a_i(t, X_i(t))\Delta t + b_i(t, X_i(t))\Delta W(t), & t \in [0, T]_{\mathbb{T}}, \\ X_i(0) = \xi_i, \end{cases} \quad (28)$$

where $a_i, b_i: \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, we have

$$\Delta X_1'(t)X_2(t) = X_1'(t)\Delta X_2(t) + (\Delta X_1'(t))X_2(t) + \Delta \langle X_1, X_2 \rangle_t, \quad t \in [0, T]_{\mathbb{T}}. \quad (29)$$

In this case,

$$\Delta \langle X_1, X_2 \rangle_t = \mu(t)a_1'a_2\Delta t + b_1'b_2\Delta \langle W \rangle_t + \mu(t)(a_1'b_2 + a_2'b_1)\Delta W(t), \quad t \in [0, T]_{\mathbb{T}}, \quad (30)$$

where the function μ is the graininess function as defined in (3) on time scales.

Remark 4. Another form of the abovementioned product rule is as follows:

$$\Delta X_1'(t)X_2(t) = X_1'(t)\Delta X_2(t) + (\Delta X_1'(t))X_2(t) + \Delta X_1'(t)\Delta X_2(t), \quad (31)$$

where $\Delta t \Delta t = \mu(t)\Delta t$, $\Delta t \Delta W = \Delta W \Delta t = \mu(t)\Delta W$, and $\Delta W \Delta W = \Delta \langle W \rangle_t$. When $\mathbb{T} = \mathbb{R}$, it is consistent with Itô's formula.

Remark 5. As mentioned before, because the quadratic variation of a process depends on not only the process itself but also the structure of time, the quadratic variation of a process becomes a little more complicated than the classical one. For instance, the quadratic variation of a deterministic continuous process is no longer zero. Therefore, we can have different forms of the product rule on time scales. For example, the product rule (6) is equivalent to

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(t)g^\Delta(t) + \mu(t)f^\Delta(t)g^\Delta(t). \quad (32)$$

Now, we use the square completion technique to present a state feedback optimal control via two coupled Riccati equations on time scales.

Theorem 2. Let (H1) and (H2) hold; then, the following Riccati equations on time scales (RDEs) admit unique solution $P(\cdot), \hat{P}(\cdot) \in C^1([0, T]_{\mathbb{T}}; S^n)$:

$$\begin{cases} -P^\Delta(t) = Q(t) + A'(t)P^\sigma(t) + P^\sigma(t)A(t) + \mu(t)A'(t)P^\sigma(t)A(t) - (I + \mu(t)A'(t))P^\sigma(t)B(t)K^{-1}(t)B'(t)P^\sigma(t) \\ \quad (I + \mu(t)A(t)), & t \in [0, \rho(T)]_{\mathbb{T}}, \\ P(T) = G, \end{cases} \quad (33)$$

$$\begin{cases} -\hat{P}^\Delta(t) = Q(t) + \bar{Q}(t) + (A(t) + \bar{A}(t))'\hat{P}^\sigma(t) + \hat{P}^\sigma(t)(A(t) + \bar{A}(t)) + \mu(t)(A(t) + \bar{A}(t))'\hat{P}^\sigma(t)(A(t) + \bar{A}(t)) \\ \quad -(I + \mu(t)(A(t) + \bar{A}(t))')\hat{P}^\sigma(t)(B(t) + \bar{B}(t))\hat{K}^{-1}(t)(B(t) + \bar{B}(t))'\hat{P}^\sigma(t)(I + \mu(t)(A(t) + \bar{A}(t))), \\ \hat{P}(T) = G + \bar{G}, & t \in [0, \rho(T)]_{\mathbb{T}}, \end{cases} \quad (34)$$

where K and \hat{K} are given as

$$K(t) = R(t) + \mu(t)B'(t)P^\sigma(t)B(t) + D'(t)P^\sigma(t)D(t), \quad (35)$$

$$\hat{K}(t) = R(t) + \bar{R}(t) + \mu(t)(B(t) + \bar{B}(t))'P^\sigma(t)(B(t) + \bar{B}(t)) + (D(t) + \bar{D}(t))'P^\sigma(t)(D(t) + \bar{D}(t)). \quad (36)$$

Furthermore, the optimal control of the MF-SALQ problems can be presented as

$$u^*(t) = -K^{-1}(t)B'(t)P^\sigma(t)(I + \mu(t)A(t))(X(t) - \mathbb{E}[X(t)]) - \hat{K}^{-1}(t)(B(t) + \bar{B}(t))'\hat{P}^\sigma(t)(I + \mu(t)(A(t) + \bar{A}(t)))\mathbb{E}[X(t)], \quad t \in [0, \rho(T)]_{\mathbb{T}}. \quad (37)$$

In this case, the optimal cost functional is

$$J^* = x' \hat{P}(0)x. \quad (38)$$

Assume that $P^\Delta(t) = \Gamma(t)$ and $\hat{P}^\Delta(t) = \Lambda(t)$ for some deterministic and differentiable functions P and \hat{P} on time scales $[0, T]_{\mathbb{T}}$. Applying Lemma 1 to $(X - \mathbb{E}[X])' P(X - \mathbb{E}[X])$ and $\mathbb{E}[X'] \hat{P} \mathbb{E}[X]$, we can obtain

Proof. From the state equation, we have

$$\Delta \mathbb{E}[X(t)] = \{(A(t) + \bar{A}(t))\mathbb{E}[X(t)] + (B(t) + \bar{B}(t))\mathbb{E}[u(t)]\} \Delta t, \quad (39)$$

$$\begin{aligned} \Delta(X(t) - \mathbb{E}[X(t)]) &= \{A(t)(X(t) - \mathbb{E}[X(t)]) \\ &+ B(t)(u(t) - \mathbb{E}[u(t)])\} \Delta t + \{D(t)(u(t) - \mathbb{E}[u(t)]) \\ &+ (D + \bar{D})\mathbb{E}[u(t)]\} \Delta W(t). \end{aligned} \quad (40)$$

$$\begin{aligned} &\Delta(X(t) - \mathbb{E}[X(t)])' P(t)(X(t) - \mathbb{E}[X(t)]) \\ &= \{(X(t) - \mathbb{E}[X(t)])' \Gamma(t)(X(t) - \mathbb{E}[X(t)]) + \mu(t)(u(t) - \mathbb{E}[u(t)])' B'(t) P^\sigma(t) B(t)(u(t) - \mathbb{E}[u(t)]) \\ &+ 2(u(t) - \mathbb{E}[u(t)])' B'(t) P^\sigma(t)(I + \mu(t)A(t))(X(t) - \mathbb{E}[X(t)])\} \Delta t \\ &+ \{(u(t) - \mathbb{E}[u(t)])' D'(t) P^\sigma(t) D(t)(u(t) - \mathbb{E}[u(t)]) + \mathbb{E}[u(t)]' (D(t) + \bar{D}(t))' P^\sigma(t)(D(t) + \bar{D}(t))\mathbb{E}[u(t)]\} \Delta \langle W \rangle_t \\ &+ 2\{(X(t) - \mathbb{E}[X(t)] + \mu(t)(A(t)(X(t) - \mathbb{E}[X(t)]) + B(t)(u(t) - \mathbb{E}[u(t)]))\}' P^\sigma(t) D(t)(u(t) - \mathbb{E}[u(t)]) \\ &+ [X(t) - \mathbb{E}[X(t)] + \mu(t)(A(t)(X(t) - \mathbb{E}[X(t)]) + B(t)(u(t) - \mathbb{E}[u(t)]))\}' P^\sigma(t)(D(t) + \bar{D}(t))\mathbb{E}[u(t)]\} \Delta W(t), \end{aligned} \quad (41)$$

$$\begin{aligned} &\Delta \mathbb{E}[X'(t)] \hat{P}(t) \mathbb{E}[X(t)] \\ &= \{\mathbb{E}[X(t)]' \Lambda(t) \mathbb{E}[X(t)] + \mu(t) \mathbb{E}[u(t)]' (B(t) + \bar{B}(t))' \hat{P}^\sigma(t) (B(t) + \bar{B}(t)) \mathbb{E}[u(t)] \\ &+ 2 \mathbb{E}[u(t)]' (B(t) + \bar{B}(t))' \hat{P}^\sigma(t) (I + \mu(t)(A(t) + \bar{A}(t))) \mathbb{E}[X(t)]\} \Delta t. \end{aligned} \quad (42)$$

By property (9), integrating from 0 to T and taking expectation from the both sides of the above two equations (41) and (42), we see that

$$\begin{aligned} &(X(T) - \mathbb{E}[X(T)])' P(T)(X(T) - \mathbb{E}[X(T)]) \\ &= \int_0^T \{(X(t) - \mathbb{E}[X(t)])' \Gamma(t)(X(t) - \mathbb{E}[X(t)]) \\ &+ \mu(t)(u(t) - \mathbb{E}[u(t)])' B'(t) P^\sigma(t) B(t)(u(t) - \mathbb{E}[u(t)]) \\ &+ 2(u(t) - \mathbb{E}[u(t)])' B'(t) P^\sigma(t)(I + \mu(t)A(t))(X(t) - \mathbb{E}[X(t)]) \\ &+ (u(t) - \mathbb{E}[u(t)])' D'(t) P^\sigma(t) D(t)(u(t) - \mathbb{E}[u(t)]) \\ &+ \mathbb{E}[u(t)]' (D(t) + \bar{D}(t))' P^\sigma(t)(D(t) + \bar{D}(t))\mathbb{E}[u(t)]\} \Delta t, \end{aligned} \quad (43)$$

$$\begin{aligned} &\mathbb{E}[X'(T)] \hat{P}(T) \mathbb{E}[X(T)] - x' \hat{P}(0)x \\ &= \int_0^T \{\mathbb{E}[X(t)]' \Lambda(t) \mathbb{E}[X(t)] + \mu(t) \mathbb{E}[u(t)]' (B(t) + \bar{B}(t))' \hat{P}^\sigma(t) (B(t) + \bar{B}(t)) \mathbb{E}[u(t)] \\ &+ 2 \mathbb{E}[u(t)]' (B(t) + \bar{B}(t))' \hat{P}^\sigma(t) (I + \mu(t)(A(t) + \bar{A}(t))) \mathbb{E}[X(t)]\} \Delta t. \end{aligned} \quad (44)$$

Moreover, the cost functional can be rewritten as

$$\begin{aligned}
 J(x; u(\cdot)) = & \mathbb{E} \left\{ \int_0^T [(X(t) - \mathbb{E}[X(t)])' Q(t) (X(t) - \mathbb{E}[X(t)]) \right. \\
 & + \mathbb{E}[X(t)]' (Q(t) + \bar{Q}(t)) \mathbb{E}[X(t)] \\
 & + (u(t) - \mathbb{E}[u(t)])' R(t) (u(t) - \mathbb{E}[u(t)]) \\
 & + \mathbb{E}[u(t)]' (R(t) + \bar{R}(t)) \mathbb{E}[u(t)] \Delta t \\
 & + (X(T) - \mathbb{E}[X(T)])' G X(T) \\
 & \left. + \mathbb{E}[X(T)]' (G + \bar{G}) \mathbb{E}[X(T)] \right\}.
 \end{aligned} \tag{45}$$

Inserting (43) and (44) into the cost functional (45) gives

$$\begin{aligned}
 J(x; u(\cdot)) = & \mathbb{E} \left[\int_0^T \left\{ \left(\mathbb{E}[u(t)] + \hat{K}^{-1}(t) (B(t) + \bar{B}(t))' \hat{P}^\sigma(t) (I + \mu(t) (A(t) + \bar{A}(t))) \mathbb{E}[X(t)] \right)' \hat{K}(t) \right. \right. \\
 & \times \left(\mathbb{E}[u(t)] + \hat{K}^{-1}(t) (B(t) + \bar{B}(t))' \hat{P}^\sigma(t) (I + \mu(t) (A(t) + \bar{A}(t))) \mathbb{E}[X(t)] \right) \\
 & \left. \left(u(t) - \mathbb{E}[u(t)] + K^{-1}(t) B'(t) P^\sigma(t) (I + \mu(t) A(t)) (X(t) - \mathbb{E}[X(t)]) \right)' K(t) \right. \\
 & \times \left. \left(u(t) - \mathbb{E}[u(t)] + K^{-1} B'(t) P^\sigma(t) (I + \mu(t) A(t)) (X(t) - \mathbb{E}[X(t)]) \right) \right. \\
 & + (X(t) - \mathbb{E}[X(t)])' [\Gamma(t) + Q(t) + A'(t) P^\sigma(t) + P^\sigma(t) A(t) + \mu(t) A'(t) P^\sigma(t) A(t) \\
 & \left. - (I + \mu(t) A'(t)) P^\sigma(t) B(t) K^{-1}(t) B'(t) P^\sigma(t) (I + \mu(t) A(t))] (X(t) - \mathbb{E}[X(t)]) \right. \\
 & + \mathbb{E}[X'(t)] [\Lambda(t) + Q(t) + \bar{Q}(t) + (A(t) + \bar{A}(t))' \hat{P}^\sigma(t) + \hat{P}^\sigma(t) (A(t) + \bar{A}(t)) \\
 & + \mu(t) (A(t) + \bar{A}(t))' \hat{P}^\sigma(t) (A(t) + \bar{A}(t)) - (I + \mu(t) (A(t) + \bar{A}(t)))' \hat{P}^\sigma(t) (B(t) + \bar{B}(t)) \hat{K}^{-1}(t) \\
 & \left. \times (B(t) + \bar{B}(t))' \hat{P}^\sigma(t) (I + \mu(t) (A(t) + \bar{A}(t))) \right] \mathbb{E}[X(t)] \Delta t \Big] \\
 & + x' \hat{P}(0) x + (X(T) - \mathbb{E}[X(T)])' (G - P(T)) (X(T) - \mathbb{E}[X(T)]) + \mathbb{E}[X'(T)] (G + \bar{G} - \hat{P}(T)) \mathbb{E}[X(T)].
 \end{aligned} \tag{46}$$

If P and \hat{P} satisfy the Riccati equations (33) and (34), then

$$\begin{aligned}
 J(x; u(\cdot)) = & \mathbb{E} \left[\int_0^T \left(\mathbb{E}[u(t)] + \hat{K}^{-1}(t) (B(t) + \bar{B}(t))' \hat{P}^\sigma(t) (I + \mu(t) (A(t) + \bar{A}(t))) \mathbb{E}[X(t)] \right)' \hat{K}(t) \right. \\
 & \times \left. \left(\mathbb{E}[u(t)] + \hat{K}^{-1}(t) (B(t) + \bar{B}(t))' \hat{P}^\sigma(t) (I + \mu(t) (A(t) + \bar{A}(t))) \mathbb{E}[X(t)] \right) \Delta t \right] \\
 & + \mathbb{E} \left[\int_0^T \left(u(t) - \mathbb{E}[u(t)] + K^{-1}(t) B'(t) P^\sigma(t) (I + \mu(t) A(t)) (X(t) - \mathbb{E}[X(t)]) \right)' K(t) \right. \\
 & \left. \times \left(u(t) - \mathbb{E}[u(t)] + K^{-1} B'(t) P^\sigma(t) (I + \mu(t) A(t)) (X(t) - \mathbb{E}[X(t)]) \right) \Delta t \right] + x' \hat{P}(0) x.
 \end{aligned} \tag{47}$$

Since $K > 0$ and $\widehat{K} > 0$, the optimal control should satisfy

$$\mathbb{E}[u(t)] + \widehat{K}^{-1}(t)(B(t) + \overline{B}(t))' \widehat{P}^\sigma(t)(I + \mu(t)(A(t) + \overline{A}(t)))\mathbb{E}[X(t)] = 0, \quad t \in [0, \rho(T)]_{\mathbb{T}}, \quad (48)$$

$$u(t) - \mathbb{E}[u(t)] + K^{-1}B'(t)P^\sigma(t)(I + \mu(t)A(t))(X(t) - \mathbb{E}[X(t)]) = 0, \quad t \in [0, \rho(T)]_{\mathbb{T}}. \quad (49)$$

Making some calculations, we get the optimal control as (37). Substituting it into (47), we have the optimal cost functional can be expressed as (38). For the existence and uniqueness of the solutions to the RΔEs, it is assert [24] that Riccati equation (33) admits as a unique positive semi-definite solution P since (H2) holds. It follows that $\widehat{K} > 0$. Using the similar method, we can get the solvability of the RΔE (34). \square

Remark 6. When $\overline{A}, \overline{B}, \overline{D}, \overline{Q}, \overline{R}$, and \overline{G} are all equal to zero, then $P = \widehat{P}$. This recovers the result of the classical SΔLQ problem [24].

Remark 7. When $\mathbb{T} = \mathbb{R}^+$, the coupled RΔEs (33) and (34) reduce to the result in [26]. On the contrary, when $\mathbb{T} = \mathbb{Z}^+$, the coupled RΔEs are consistent with the case in [27].

Similarly, we have the following theorem which can be regarded as an equivalent form of Theorem 2.

Theorem 3. *Let (H1) and (H2), then RΔE (33) and the following RΔE (50) admit unique solution $P(\cdot), \overline{P}(\cdot) \in C^1([0, T]_{\mathbb{T}}; S^n)$:*

$$\left\{ \begin{array}{l} -\overline{P}^\Delta(t) = \overline{Q}(t) + \overline{A}'(t)P^\sigma(t)(I + \mu(t)A(t)) + (I + \mu(t)A'(t))P^\sigma(t)\overline{A}(t) + \mu(t)\overline{A}'(t)P^\sigma(t)\overline{A}(t) \\ + (A(t) + \overline{A}(t))'\overline{P}^\sigma(t) + \overline{P}^\sigma(t)(A(t) + \overline{A}(t)) + \mu(t)(A(t) + \overline{A}(t))'\overline{P}^\sigma(t)(A(t) + \overline{A}(t)) \\ - (I + \mu(t)(A(t) + \overline{A}(t)))'(P + \overline{P})^\sigma(t)(B(t) + \overline{B}(t))\widehat{K}^{-1}(t) \\ \times (B(t) + \overline{B}(t))'(P + \overline{P})^\sigma(t)(I + \mu(t)(A(t) + \overline{A}(t))) \\ + (I + \mu(t)A'(t))P^\sigma(t)B(t)K^{-1}(t)B'(t)P^\sigma(t)(I + \mu(t)A(t)), \quad t \in [0, \rho(T)]_{\mathbb{T}}, \\ \widehat{P}(T) = \overline{G}, \end{array} \right. \quad (50)$$

where K and \widehat{K} are given as before. For the MF-SΔLQ problem,

$$u^*(t) = - \left\{ \widehat{K}^{-1}(t)(B(t) + \overline{B}(t))'(P + \overline{P})^\sigma(t)(I + \mu(t)(A(t) + \overline{A}(t))) - K^{-1}(t)B'(t)P^\sigma(t)(I + \mu(t)A(t)) \right\} \mathbb{E}[X(t)] \\ - K^{-1}(t)B'(t)P^\sigma(t)(I + \mu(t)A(t))X(t), \quad t \in [0, \rho(T)]_{\mathbb{T}}, \quad (51)$$

is an optimal control. Moreover, the optimal cost functional with respect to u^* is

$$J^* = x'P(0)x + x'\overline{P}(0)x. \quad (52)$$

Proof. As the statement in previous Theorem 2, we can obtain the solvability of the RΔEs (33) and (50). We need only to prove (51) and (42). Taking integral of

$\Delta(X'(t)P(t)X(t) + \mathbb{E}[X(t)]'\bar{P}(t)\mathbb{E}[X(t)])$ from 0 to T and taking expectation, we obtain

$$\begin{aligned} & \mathbb{E}[X(T)'P(T)X(T)] + \mathbb{E}[X(T)']\bar{P}(T)\mathbb{E}[X(T)] - x'P(0)x - x'\bar{P}(0)x \\ &= -\mathbb{E}\left[\int_0^T (X'(t)Q(t)X(t) + \mathbb{E}[X(t)]'\bar{Q}(t)\mathbb{E}[X(t)] + u'(t)R(t)u(t) + \mathbb{E}[u(t)]'\bar{R}(t)\mathbb{E}[u(t)])\Delta t\right] \\ &+ \mathbb{E}\left[\int_0^T \left(\mathbb{E}[u(t)] + \hat{K}^{-1}(t)(B(t) + \bar{B}(t))'(P + \bar{P})^\sigma(t)(I + \mu(t)(A(t) + \bar{A}(t)))\mathbb{E}[X(t)]\right)'\hat{K}(t)\right. \\ &\times \left.\left(\mathbb{E}[u(t)] + \hat{K}^{-1}(t)(B(t) + \bar{B}(t))'(P + \bar{P})^\sigma(t)(I + \mu(t)(A(t) + \bar{A}(t)))\mathbb{E}[X(t)]\right)\Delta t\right] \\ &+ \mathbb{E}\left[\int_0^T (u(t) - \mathbb{E}[u(t)] + K^{-1}B'(t)P^\sigma(t)(I + \mu(t)A(t))(X(t) - \mathbb{E}[X(t)]))'K(t)\right. \\ &\times \left.(u(t) - \mathbb{E}[u(t)] + K^{-1}B'(t)P^\sigma(t)(I + \mu(t)A(t))(X(t) - \mathbb{E}[X(t)]))\Delta t\right]. \end{aligned} \quad (53)$$

Consequently, by completing the squares, one has

$$\begin{aligned} J(x; u(\cdot)) &= \mathbb{E}\left[\int_0^T \left(\mathbb{E}[u(t)] + \hat{K}^{-1}(t)(B(t) + \bar{B}(t))'(P + \bar{P})^\sigma(t)(I + \mu(t)(A(t) + \bar{A}(t)))\mathbb{E}[X(t)]\right)'\hat{K}(t)\right. \\ &\times \left.\left(\mathbb{E}[u(t)] + \hat{K}^{-1}(t)(B(t) + \bar{B}(t))'(P + \bar{P})^\sigma(t)(I + \mu(t)(A(t) + \bar{A}(t)))\mathbb{E}[X(t)]\right)\Delta t\right] \\ &+ \mathbb{E}\left[\int_0^T (u(t) - \mathbb{E}[u(t)] + K^{-1}B'(t)P^\sigma(t)(I + \mu(t)A(t))(X(t) - \mathbb{E}[X(t)]))'K(t)\right. \\ &\times \left.(u(t) - \mathbb{E}[u(t)] + K^{-1}B'(t)P^\sigma(t)(I + \mu(t)A(t))(X(t) - \mathbb{E}[X(t)]))\Delta t\right] + x'P(0)x + x'\bar{P}(0)x. \end{aligned} \quad (54)$$

Since $K > 0$ and $\hat{K} > 0$, we must select u such that

$$\begin{cases} \mathbb{E}[u(t)] = -\hat{K}^{-1}(t)(B(t) + \bar{B}(t))'(P + \bar{P})^\sigma(t)(I + \mu(t)(A(t) + \bar{A}(t)))\mathbb{E}[X(t)], \\ u(t) - \mathbb{E}[u(t)] = -K^{-1}B'(t)P^\sigma(t)(I + \mu(t)A(t))(X(t) - \mathbb{E}[X(t)]), \quad t \in [0, \rho(T)]_{\mathbb{T}}. \end{cases} \quad (55)$$

Therefore, the optimal control satisfies (51). In this case, the optimal cost functional is (52). The conclusions are proved. \square

Remark 8. The solution \hat{P} of the RAE (34) equals to the sum of P and \bar{P} , where P and \bar{P} are the solutions of the RAEs (33) and (50).

4. Example

The theorems in Section 3 tell us that we can solve the MF-SALQ problems if the corresponding Riccati equations can be solved. Now, we discuss a numerical example based on the method developed in the previous sections and compare the difference among the time scales, continuous time, and discrete time.

Consider the following example with one-dimensional state equation on time scales $\mathbb{T}_1 = [0, 1/2] \cup \{1\} \cup [3/2, 2]$:

$$\begin{cases} \Delta X(t) = \left(\mathbb{E}[X(t)] + \frac{1}{2}\mathbb{E}[u(t)]\right)\Delta t + u(t)\Delta W(t), \\ X(0) = 1, \end{cases} \quad (56)$$

and the cost functional

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^2 [(u(t))^2 + (\mathbb{E}[u(t)])^2]\Delta t + (\mathbb{E}[X(2)])^2\right]. \quad (57)$$

The corresponding coupled Riccati equations are

$$\begin{cases} P^\Delta(t) = 0, \\ P(2) = 0, \end{cases} \quad (58)$$

$$\begin{cases} -\hat{P}^\Delta(t) = (2 + \mu)\hat{P}^\sigma(t) - \frac{1}{8}(1 + \mu)^2(\hat{P}^\sigma(t))^2, \\ \hat{P}(2) = 1. \end{cases} \quad (59)$$

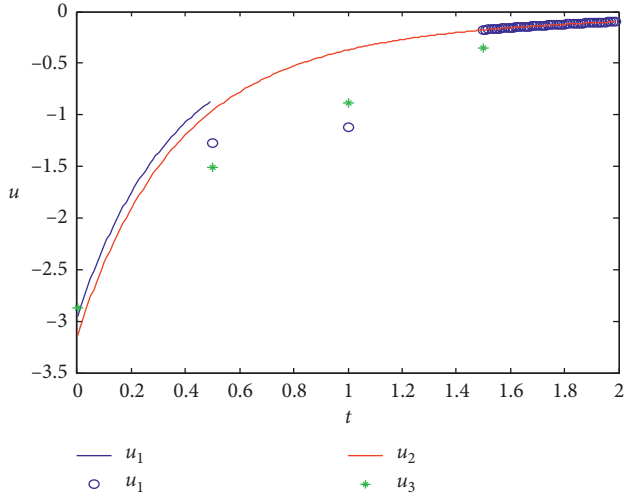


FIGURE 1: Optimal controls of the dynamic systems in time scales T_1 , continuous time T_2 , and discrete time T_3 , respectively.

By solving the coupled Riccati equations and using Theorem 2, the optimal control can be expressed as

$$u_1^*(t) = \begin{cases} -\frac{4}{1 + 0.97e^{2t-1}} \mathbb{E}[X(t)], & t \in \left[0, \frac{1}{2}\right), \\ -3.05 \mathbb{E}[X(t)], & t = \frac{1}{2}, \\ -3.99 \mathbb{E}[X(t)], & t = 1, \\ -\frac{4}{1 + 15e^{-2(2-t)}} \mathbb{E}[X(t)], & t \in \left[\frac{3}{2}, 2\right). \end{cases} \quad (60)$$

If we regard the time as the continuous time $T_2 = [0, 2]$, then the optimal control is

$$u_2^*(t) = -\frac{4}{1 + 15e^{-2(2-t)}} \mathbb{E}[X(t)], \quad t \in [0, 2]. \quad (61)$$

On the contrary, if we treat the time as discrete time $T_3 = \{0, 1, 2, 3, 4\}$, then the optimal control is

$$u_3^*(t) = \begin{cases} -2.87 \mathbb{E}[X(t)], & t = 0, \\ -3.61 \mathbb{E}[X(t)], & t = \frac{1}{2}, \\ -3.13 \mathbb{E}[X(t)], & t = 1, \\ -1.21 \mathbb{E}[X(t)], & t = \frac{3}{2}. \end{cases} \quad (62)$$

The comparison result of the optimal controls is shown in Figure 1.

The example implies that we should take an impulsive control when $t = 1/2$ and $t = 1$ in the time scales setting T_1 . Although the optimal control in the interval $[3/2, 2)$ on the

time scales T_1 is the same as in the continuous case, they are different in the interval $[0, 1/2)$. It is to say that the time gap μ influences not only the impulsive control but also the optimal control in the interval $[0, 1/2)$. We can see that the optimal control depends on the structure of time domain. This interesting result is hidden in the classical continuous and discrete formulation.

5. Conclusions

The linear quadratic optimal control problems for mean-field stochastic differential equations on time scales are studied. It unifies and extends the mean-field optimal control problems in continuous and discrete time formulations. Via two coupled RDEs on time scales, we get the corresponding optimal control with the state feedback representation. The optimal control problems established in this paper offer a more practical scheme in tackling directly the issue on the mixture of continuous time and discrete time.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the National Key R&D Program of China (Grant no. 2018YFA0703900), the Tian Yuan Projection of the National Natural Science Foundation of China (Grant no. 11626247) and the Major Project of National Social Science Foundation of China (Grant no. 19ZDA091).

References

- [1] Y.-H. Ni, J.-F. Zhang, and M. Krstic, "Time-inconsistent mean-field stochastic lq problem: open-loop time-consistent control," *IEEE Transactions on Automatic Control*, vol. 63, no. 9, pp. 2771–2786, 2017.
- [2] R. Buckdahn, B. Djehiche, and J. Li, "A general stochastic maximum principle for sdes of mean-field type," *Applied Mathematics & Optimization*, vol. 64, no. 2, pp. 197–216, 2011.
- [3] X. Cui, X. Li, and D. Li, "Unified framework of mean-field formulations for optimal multi-period mean-variance portfolio selection," *IEEE Transactions on Automatic Control*, vol. 59, no. 7, pp. 1833–1844, 2014.
- [4] M. Fischer and G. Livieri, "Continuous time mean-variance portfolio optimization through the mean field approach," *ESAIM: Probability and Statistics*, vol. 20, pp. 30–44, 2016.
- [5] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.
- [6] G. S. Guseinov, "Integration on time scales," *Journal of Mathematical Analysis and Applications*, vol. 285, no. 1, pp. 107–127, 2003.

- [7] M. Bohner and G. Guseinov, "Improper integrals on time scales," *Dynamic Systems and Applications*, vol. 12, no. 1/2, pp. 45–66, 2003.
- [8] J. M. Davis, J. Henderson, and K. R. Prasad, "Upper and lower bounds for the solution of the general matrix riccati differential equation on a time scale," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 133–145, 2002.
- [9] M. Bohner and G. S. Guseinov, "Partial differentiation on time scales," *Dynamic Systems and Applications*, vol. 13, no. 3-4, pp. 351–379, 2004.
- [10] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Springer Science & Business Media, Berlin, Germany, 2001.
- [11] M. Bohner and A. C. Peterson, *Advances in Dynamic Equations on Time Scales*, Springer Science & Business Media, Berlin, Germany, 2002.
- [12] N. H. Du and N. T. Dieu, "The first attempt on the stochastic calculus on time scale," *Stochastic Analysis and Applications*, vol. 29, no. 6, pp. 1057–1080, 2011.
- [13] D. Grow and S. Sanyal, "Brownian motion indexed by a time scale," *Stochastic Analysis and Applications*, vol. 29, no. 3, pp. 457–472, 2011.
- [14] D. Grow and S. Sanyal, "The quadratic variation of brownian motion on a time scale," *Statistics & Probability Letters*, vol. 82, no. 9, pp. 1677–1680, 2012.
- [15] M. Bohner, "Calculus of variations on time scales," *Dynamic Systems and Applications*, vol. 13, no. 3-4, pp. 339–349, 2004.
- [16] Z. Zhan and W. Wei, "On existence of optimal control governed by a class of the first-order linear dynamic systems on time scales," *Applied Mathematics and Computation*, vol. 215, no. 6, pp. 2070–2081, 2009.
- [17] Y. Gong, X. Xiang, and X. Xiang, "A class of optimal control problems of systems governed by the first order linear dynamic equations on time scales," *Journal of Industrial & Management Optimization*, vol. 5, no. 1, pp. 1–10, 2009.
- [18] Y. Peng, X. Xiang, and Y. Jiang, "Nonlinear dynamic systems and optimal control problems on time scales," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 17, no. 3, pp. 654–681, 2011.
- [19] Z. Zhan, S. Chen, and W. Wei, "A unified theory of maximum principle for continuous and discrete time optimal control problems," *Mathematical Control and Related Fields*, vol. 2, no. 2, pp. 195–215, 2012.
- [20] M. Bohner, K. Kenzhebaev, O. Lavrova, and O. Stanzhytskiy, "Pontryagin's maximum principle for dynamic systems on time scales," *Journal of Difference Equations and Applications*, vol. 23, no. 7, pp. 1161–1189, 2017.
- [21] J. Seiffert, S. Sanyal, and D. C. Wunsch, "Hamilton-Jacobi-Bellman equations and approximate dynamic programming on time scales," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 38, no. 4, pp. 918–923, 2008.
- [22] M. Bohner and N. Wintz, "The linear quadratic regulator on time scales," *International Journal of Difference Equations*, vol. 5, no. 2, pp. 149–174, 2010.
- [23] R. Š. Hilscher and V. Zeidan, "Hamilton-Jacobi theory over time scales and applications to linear-quadratic problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 2, pp. 932–950, 2012.
- [24] Y. Zhu and G. Jia, Stochastic Linear Quadratic Control Problem on Time Scales.
- [25] M. Bohner, O. M. Stanzhytskiy, and A. O. Bratochkina, "Stochastic dynamic equations on general time scales," *Electronic Journal of Differential Equations*, vol. 2013, no. 57, pp. 1–15, 2013.
- [26] J. Yong, "Linear-quadratic optimal control problems for mean-field stochastic differential equations," *SIAM Journal on Control and Optimization*, vol. 51, no. 4, pp. 2809–2838, 2013.
- [27] Y.-H. Ni, X. Li, and J.-F. Zhang, "Indefinite mean-field stochastic linear-quadratic optimal control: from finite horizon to infinite horizon," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3269–3284, 2015.
- [28] R. P. Agarwal and M. Bohner, "Basic calculus on time scales and some of its applications," *Results in Mathematics*, vol. 35, no. 1-2, pp. 3–22, 1999.
- [29] R. P. Agarwal, M. J. Bohner, and A. Peterson, "Inequalities on time scales: a survey," *Mathematical Inequalities & Applications*, vol. 4, no. 4, pp. 535–557, 2001.