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*Research Article*

# High Accuracy Analysis of Nonconforming Mixed Finite Element Method for the Nonlinear Sivashinsky Equation

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The fourth-order nonlinear Sivashinsky equation is often used to simulate a planar solid-liquid interface for a binary alloy. In this paper, we study the high accuracy analysis of the nonconforming mixed finite element method (MFEM for short) for this equation. Firstly, by use of the special property of the nonconforming $EQ_1^{\text{rot}}$ element (see Lemma 1), the superclose estimates of order $O(h^2 + \Delta t)$ in the broken $H^1$-norm for the original variable $u$ and intermediate variable $p$ are deduced for the back-Euler (B-E for short) fully-discrete scheme. Secondly, the global superconvergence results of order $O(h^2 + \Delta t)$ for the two variables are derived through interpolation postprocessing technique. Finally, a numerical example is provided to illustrate validity and efficiency of our theoretical analysis and method.

## 1. Introduction

Under certain conditions, the dilute binary alloy will solidify, at which point the solid-liquid interface is unstable and has a cellular structure. When the solute rejection coefficient is close to unity, near the stability threshold, the characteristic cell size may significantly be beyond the diffusion width of the solidification zone. The Sivashinsky equation describes the dynamic of the onset and stabilization of the cellular structure, which is considered as the following fourth-order nonlinear equation [1, 2]:

$$\begin{cases} u_t + \Delta^2 u + \alpha u = \Delta f(u), & (X,t) \in \Omega \times (0,T], \\ u = \Delta u = 0, & (X,t) \in \partial\Omega \times (0,T], \\ u(X,0) = u^0(X), & X \in \Omega, \end{cases} \tag{1}$$

where $\Omega$ is the interior of the rectangle $[0,a] \times [0,b]$, $X = (x,y)$, $T > 0$, $a > 0$, $b > 0$, $\alpha > 0$ are fixed constants, $u^0(X)$ is a given smooth function, and $f(u) = (1/2)u^2 - 2u$. Due to the nonlinearity of this equation, it is very difficult to find out the true solution. Thus, a lot of numerical simulation methods have been considered for (1), such as the finite difference method, finite element method (FEM for short), and region decomposition method. For one-dimensional case, Benammou and Omrani [3] studied the FEM and obtained the convergence analysis of the original variable $u$ in $L^2$-norm; Momani [4] presented a numerical scheme based on the region decomposition method; and Omrani and Reza and Kenan [5, 6] provided two kinds of finite difference schemes and proved the uniqueness and convergence, respectively. For two-dimensional case, Denet [7] gave the stability of the solution under the rectangular region; Rouis and Omrani [8] proposed a linearized three-level difference scheme; and Ilati and Dehghan [9] derived an error analysis by a meshless method based on radial point interpolation technique.

As it is known to all that in regard to the fourth-order problem, the conforming Galerkin finite element (FE for short) approximation space belongs to $H^2(\Omega)$, and FE solution in turn shall be $C^1$-continuous. This leads to the higher degree of piecewise polynomials, and the related computation is complicated and difficult (both triangular Bell element and rectangular Bogner–Fox–Schmit element [10] are typical examples). The MFEM is an optimal choice to overcome the above deficiencies, which transforms a fourth-order problem into 2 coupled second-order problems by introducing an intermediate variable; thus, the low-order elements can be used to solve. The nonconforming MFEM brings down the smoothness requirement on FE solution compared to the conforming case. Readers with more interests may refer [11–15] and the references listed. For problem (1), Omrani [1] developed the convergence analysis of the corresponding variables in the semidiscrete and fully-discrete schemes by using conforming MFEM; however, situation involving nonconforming MFEM was not available till now.

It is also well known that the superconvergence analysis is an important approach to improve the precision of FE solution. More precisely, based on the so-called integral identity technique, the order of error in $H^1$-norm between FE approximation $u_h$ and the interpolation of the exact solution $I_h u$ is much better than that of $u$ and $I_h u$; this fascinating characteristic is called superclose. The global superconvergence will then be investigated by adding a simple postprocessing without changing the existing FE program. Meanwhile, superconvergence is critical in practical engineering numerical calculation and has always been a research hotspot. To find out more applications, readers may refer [12, 15–23]. As far as our knowledge is concerned, research on superconvergence for Sivashinsky equation is yet to be found.

The main purpose of this article is to develop a nonconforming MFE scheme for problem (1), and the superclose and superconvergence results of the original variable $u$ and auxiliary variable $p$ in the broken $H^1$-norm are obtained for the B-E fully-discrete scheme. The outline is organized as follows: in Section 2, the MFE spaces and variational formulation are introduced. In Section 3, based on the special property of the nonconforming $EQ_1^{\mathrm{rot}}$ element (when $u \in H^3(\Omega)$, the consistency error is of order $O(h^2)$ which is one order higher than the interpolation error), the superclose results for the above two variables are deduced. In Section 4, the global superconvergence properties are derived with the help of interpolation postprocessing technique. In Section 5, a numerical example is given to verify the theoretical analysis. In the last section, a brief conclusion is drawn.

Throughout this article, $C$ denotes a positive constant that may take different values at different places but remains independent of the subdivision parameter $h$ and time step $\Delta t$. Meanwhile, we use the notations as in [10] for the Sobolev spaces $W^{m,p}(\Omega)$ with norm $\|\cdot\|_{m,p}$ and seminorm $|\cdot|_{m,p}$, where $m$ and $p$ are nonnegative integer numbers. Especially, for $p = 2$, $p$ will be omitted in the above norms and seminorms. Furthermore, we define the space $L^p(a,b;Y)$ with the norm $\|\Phi\|_{L^p(0,T;Y)} = \int_0^T \|\Phi(\cdot,t)\|_Y^p \mathrm{d}t \quad (1 \le p < \infty)$ and $\|\Phi\|_{L^\infty(0,T;Y)} = ess \sup_{0<t<T} \|\Phi(\cdot,t)\|_Y \quad (p = \infty)$.

## 2. The MFE Spaces and Variational Formulation

Let $\Omega$ be a rectangular domain with edges parallel to the coordinate axes, $T_h$ be a rectangular subdivision of $\Omega$ which need not satisfy the regular condition [10]. For all $K \in T_h$, $K = [x_K - h_{x_K}, x_K + h_{x_K}] \times [y_K - h_{y_K}, y_K + h_{y_K}]$, assume that the barycenter of $K$ by $(x_K, y_K)$, and the four vertices and four sides are $z_i$, $l_i = \overline{z_i z_{i+1}}$ (mod4) $(i = 1,2,3,4)$, respectively. $h_K = \max\{h_{x_K}, h_{y_K}\}$, $h = \max_{K\in T_h} h_K$.

The nonconforming $EQ_1^{\mathrm{rot}}$ element space [17–21, 24, 25] is defined by

$$V_h = \Big\{v_h\colon v_h|_K \in span\{1, x, y, x^2, y^2\},\quad \forall K \in T_h, \int_F [v_h]\mathrm{d}s = 0, F \subset \partial K\Big\}, \tag{2}$$

where $[v_h]$ stands for the jump of $v_h$ across the boundary $F$ and $[v_h] = v_h$ if $F \subset \partial\Omega$.

Then, we denote the norm on $V_h$ as $\|\cdot\|_h = (\sum_{K\in T_h} |\cdot|_{1,K}^2)^{1/2}$.

The corresponding interpolation operator is defined as $I_h\colon v \in V = H_0^1(\Omega) \longrightarrow I_h v \in V_h$, $I_h \mid_K = I_K$, satisfying

$$\begin{aligned} &\int_{l_i} (v - I_K v)\mathrm{d}s = 0,\\ &\int_K (v - I_K v)\mathrm{d}x\,\mathrm{d}y = 0,\\ &\qquad i = 1,2,3,4, \forall K \in T_h. \end{aligned} \tag{3}$$

Let $p = f(u) - \Delta u$; then, the mixed variational formulation for (1) is find $(u,p) \in V \times V$ such that

$$\begin{cases} (u_t, v) + (\nabla p, \nabla v) + \alpha(u,v) = 0, & \forall v \in V,\\ (p,q) - (\nabla u, \nabla q) = (f(u), q), & \forall q \in V,\\ u(X,0) = u^0(X), & p(X,0) = p^0(X) = f(u^0(X)) - \Delta u^0(X), \end{cases} \tag{4}$$

where $(u,v) = \int_\Omega uv\mathrm{d}x\,\mathrm{d}y$.

## 3. Superclose Analysis for the Fully-Discrete Approximation Scheme

In this section, the superclose analysis for the B-E fully-discrete scheme will be studied.

Let $\{t_n \mid t_n = n\Delta t; n = 0,1,2,\ldots,N\}$ be a uniform partition of $[0,T]$ with the time step $\Delta t = (T/N)$. For a given continuous function $u$ on $[0,T]$, we define that $u^n = u(X,t_n), \overline{\partial}_t u^n = u^n - u^{n-1}/\Delta t$.

The following lemma is introduced first which is important in the superclose analysis.

**Lemma 1** (see [18]). *For all $v_h \in V_h$, we get*

$$\|u - I_h u\|_0 + h\|u - I_h u\|_h \leq Ch^2\|u\|_2, \quad u \in H^2(\Omega), \tag{5}$$

$$(\nabla(u - I_h u), \nabla v_h)_h = 0, \tag{6}$$

$$\|v_h\|_0 \leq C\|v_h\|_h, \tag{7}$$

$$\sum_K \int_{\partial K} \frac{\partial u}{\partial n} v_h \mathrm{d}s \leq Ch^2\|u\|_3\|v_h\|_h, \quad u \in H^3(\Omega), \tag{8}$$

*where* $(u, v)_h = \sum_K \int_K uv dx\, dy$.

Then, the B-E fully-discrete approximation scheme for (4) is find $(U_h^n, P_h^n) \in V_h \times V_h$ such that

$$\begin{cases} \left(\overline{\partial}_t U_h^n, v_h\right) + (\nabla P_h^n, \nabla v_h)_h + \alpha(U_h^n, v_h) = 0, & \forall v_h \in V_h, \\ (P_h^n, q_h) - (\nabla U_h^n, \nabla q_h)_h = (f(U_h^n), q_h), & \forall q_h \in V_h, \\ U_h^0 = I_h u^0, & P_h^0 = I_h p^0. \end{cases} \tag{9}$$

The existence and uniqueness of the solution for problem (9) can be found in [1].

Next focus will be placed on the superclose of $\|U_h^n - I_h u^n\|_h$ and $\|P_h^n - I_h p^n\|_h$.

**Theorem 1.** *Let $(u^n, p^n)$ and $(U_h^n, P_h^n)$ be the solutions of (4) and (9), respectively, $u, p \in L^\infty(0,T; H^3(\Omega))$, $u_t, p_t \in L^2(0,T; H^3(\Omega))$, and $u_{tt} \in L^2(0,T; H^2(\Omega))$; then, for $n = 0, 1, 2, \ldots, N$, we have*

$$\|U_h^n - I_h u^n\|_h + \|P_h^n - I_h p^n\|_h \leq C\left(h^2 + \Delta t\right). \tag{10}$$

*Proof.* Let $u^n - U_h^n = (u^n - I_h u^n) + (I_h u^n - U_h^n) := \theta^n + \rho^n$, $p^n - P_h^n = (p^n - I_h p^n) + (I_h p^n - P_h^n) := \xi^n + \eta^n$.

The error equations can be derived from (1), (6), and (9):

$$\left(\overline{\partial}_t \rho^n, v_h\right) + (\nabla \eta^n, \nabla v_h)_h + \alpha(\rho^n, v_h) = -\left(\overline{\partial}_t \theta^n, v_h\right) - \alpha(\theta^n, v_h) + \sum_K \int_{\partial K} \frac{\partial p^n}{\partial n} v_h \mathrm{d}s - (R_1^n, v_h), \tag{11a}$$

$$(\eta^n, q_h) - (\nabla \rho^n, \nabla q_h)_h = -(\xi^n, q_h) + (f(u^n) - f(U_h^n), q_h) - \sum_K \int_{\partial K} \frac{\partial u^n}{\partial n} q_h \mathrm{d}s, \tag{11b}$$

where $R_1^n = u_t^n - \overline{\partial}_t u^n = (1/\Delta t)\int_{t_{n-1}}^{t_n} (\tau - t_{n-1}) u_{tt}(\tau)\mathrm{d}\tau$.

Firstly, taking $v_h = \eta^n$ in (11a) and $q_h = \overline{\partial}_t \rho^n$ in (11b) and then subtracting them, there holds

$$\begin{aligned} \left(\nabla \rho^n, \nabla \overline{\partial}_t \rho^n\right)_h + \|\eta^n\|_h^2 &= -\left(\overline{\partial}_t \theta^n, \eta^n\right) - \alpha(\rho^n, \eta^n) - \alpha(\theta^n, \eta^n) + \sum_K \int_{\partial K} \frac{\partial p^n}{\partial n} \eta^n \mathrm{d}s - (R_1^n, \eta^n) \\ &\quad + \left(\xi^n, \overline{\partial}_t \rho^n\right) - \left(f(u^n) - f(U_h^n), \overline{\partial}_t \rho^n\right) + \sum_K \int_{\partial K} \frac{\partial u^n}{\partial n} \rho_t \mathrm{d}s \\ &= \sum_{i=1}^{8} A_i. \end{aligned} \tag{12}$$

It is easy to verify that

$$\left(\nabla \overline{\partial}_t \rho^n, \nabla \rho^n\right)_h \geq \frac{1}{2\Delta t}\left(\|\rho^n\|_h^2 - \|\rho^{n-1}\|_h^2\right), |A_5| = |(R_1^n, \eta^n)| \leq C\Delta t \int_{t_{n-1}}^{t_n} \|u_{tt}\|_0^2 \mathrm{d}\tau + \frac{1}{6}\|\eta^n\|_h^2. \tag{13}$$

By virtue of Lemma 1, we arrive at

$$\begin{aligned}|A_1+A_2+A_3| &= \left|\left(\overline{\partial}_t\theta^n,\eta^n\right)+\alpha\left(\rho^n,\eta^n\right)+\alpha\left(\theta^n,\eta^n\right)\right| \\ &\leq Ch^4\left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\|u_t\|_2^2 d\tau+\|u^n\|_2^2\right)+C\|\rho^n\|_h^2 \\ &\quad+\frac{1}{6}\|\eta^n\|_h^2,\end{aligned} \tag{14}$$

$$|A_4| = \left|\sum_K\int_{\partial K}\frac{\partial p^n}{\partial n}\eta^n ds\right| \leq Ch^4\|p^n\|_3^2+\frac{1}{6}\|\eta^n\|_h^2. \tag{15}$$

By using the derivative transfer technique and (5), there holds

$$\begin{aligned}|A_6+A_8| &= \left|\left(\xi^n,\overline{\partial}_t\rho^n\right)+\sum_K\int_{\partial K}\frac{\partial u^n}{\partial n}\overline{\partial}_t\rho^n ds\right| \\ &= \left|\overline{\partial}_t\left(\sum_K\int_{\partial K}\frac{\partial u^n}{\partial n}\rho^n ds\right)+\left(\xi^n,\overline{\partial}_t\rho^n\right)\right. \\ &\quad\left.-\sum_K\int_{\partial K}\frac{\partial\left(\overline{\partial}_t u^n\right)}{\partial n}\rho^{n-1}ds\right| \\ &\leq Ch^4\left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\|u_t\|_3^2 d\tau+\|p^n\|_2^2\right)+C\|\rho^{n-1}\|_h^2 \\ &\quad+\varepsilon\|\overline{\partial}_t\rho^n\|_0^2+\left|\overline{\partial}_t\left(\sum_K\int_{\partial K}\frac{\partial u^n}{\partial n}\rho^n ds\right)\right|.\end{aligned} \tag{16}$$

In order to estimate $A_7$, the following assumption is given which will be proved later:

$$\|U_h^n\|_{0,\infty}<M,\quad n=0,1,2,\ldots,N, \tag{17}$$

where $M=1+\|u\|_{L^\infty(0,T;L^\infty(\Omega))}$.

Then, we have

$$\begin{aligned}|A_7| &= \left(f(u^n)-f(U_h^n),\overline{\partial}_t\rho^n\right) \\ &= -\left(\frac{1}{2}\left((u^n)^2-(U_h^n)^2\right),\overline{\partial}_t\rho^n\right)+2\left(u^n-U_h^n,\overline{\partial}_t\rho^n\right) \\ &= -\left(\frac{1}{2}(\theta^n+\rho^n)(u^n+U_h^n),\overline{\partial}_t\rho^n\right)+2\left(\theta^n+\rho^n,\overline{\partial}_t\rho^n\right) \\ &\leq C\|\theta^n+\rho^n\|_0\|u^n+U_h^n\|_{0,\infty}\|\overline{\partial}_t\rho^n\|_0+C\|\theta^n+\rho^n\|_0\|\overline{\partial}_t\rho^n\|_0 \\ &\leq Ch^4\|u^n\|_2^2+\varepsilon\|\overline{\partial}_t\rho^n\|_0^2+C\|\rho^n\|_h^2.\end{aligned} \tag{18}$$

Substituting (13)–(18) into (12), we get

$$\begin{aligned}&\frac{1}{2\Delta t}\left(\|\rho^n\|_h^2-\|\rho^{n-1}\|_h^2\right)+\frac{1}{2}\|\eta^n\|_h^2 \\ &\quad\leq Ch^4\left(\|p^n\|_3^2+\|u^n\|_2^2+\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\|u_t\|_3^2 d\tau\right) \\ &\qquad+C\|\rho^n\|_h^2+C\|\rho^{n-1}\|_h^2+\varepsilon\|\overline{\partial}_t\rho^n\|_0^2 \\ &\qquad+\left|\overline{\partial}_t\left(\sum_K\int_{\partial K}\frac{\partial u^n}{\partial n}\rho^n ds\right)\right|.\end{aligned} \tag{19}$$

Multiplying by $2\Delta t$ and then summing up the above inequality, by applying discrete Gronwall's lemma, we can obtain

$$\|\rho^n\|_h^2+2\Delta t\sum_{i=1}^n\|\eta^i\|_h^2\leq C\left(h^4+(\Delta t)^2\right)+4\varepsilon\Delta t\sum_{i=1}^n\|\overline{\partial}_t\rho^i\|_0^2. \tag{20}$$

Secondly, taking the difference between two time levels $n$ and $n-1$ of (11b) reduces to

$$\begin{aligned}&\left(\overline{\partial}_t\eta^n,q_h\right)-\left(\nabla\overline{\partial}_t\rho^n,\nabla q_h\right)_h=-\left(\overline{\partial}_t\xi^n,q_h\right) \\ &\quad+\left(\overline{\partial}_t\left(f(u^n)-f(U_h^n)\right),q_h\right)-\sum_K\int_{\partial K}\frac{\partial\left(\overline{\partial}_t u^n\right)}{\partial n}q_h ds.\end{aligned} \tag{21}$$

Choosing $v_h=\overline{\partial}_t\rho^n$ in (11a) and $q_h=\eta^n$ in (21) and then adding them, we can get

$$\begin{aligned}&\|\overline{\partial}_t\rho^n\|_0^2+\left(\overline{\partial}_t\eta^n,\eta^n\right)+\alpha\left(\overline{\partial}_t\rho^n,\rho^n\right) \\ &\quad=-\left(\overline{\partial}_t\theta^n,\overline{\partial}_t\rho^n\right)-\alpha\left(\theta^n,\overline{\partial}_t\rho^n\right)+\sum_K\int_{\partial K}\frac{\partial p^n}{\partial n}\overline{\partial}_t\rho^n ds \\ &\qquad-\left(R_1^n,\overline{\partial}_t\rho^n\right)-\left(\overline{\partial}_t\xi^n,\eta^n\right)+\left(\overline{\partial}_t\left(f(u^n)-f(U_h^n)\right),\eta^n\right) \\ &\qquad-\sum_K\int_{\partial K}\frac{\partial\left(\overline{\partial}_t u^n\right)}{\partial n}\eta^n ds \\ &\quad=\sum_{i=1}^7 B_i.\end{aligned} \tag{22}$$

It is not difficult to verify that

$$\begin{aligned}&\left(\overline{\partial}_t\eta^n,\eta^n\right)\geq\frac{1}{2\Delta t}\left(\|\eta^n\|_0^2-\|\eta^{n-1}\|_0^2\right), \\ &\alpha\left(\overline{\partial}_t\rho^n,\rho^n\right)\geq\frac{\alpha}{2\Delta t}\left(\|\rho^n\|_0^2-\|\rho^{n-1}\|_0^2\right), \\ &\quad|B_4|=\left|\left(R_1^n,\overline{\partial}_t\rho^n\right)\right|\leq C\Delta t\int_{t_{n-1}}^{t_n}\|u_{tt}\|_0^2 d\tau+\frac{1}{6}\|\overline{\partial}_t\rho^n\|_0^2.\end{aligned} \tag{23}$$

By (5) and (17), there holds

$$\begin{aligned}|B_1+B_2+B_5| &= \left|\left(\overline{\partial}_t\theta^n,\overline{\partial}_t\rho^n\right)+\alpha\left(\theta^n,\overline{\partial}_t\rho^n\right)+\left(\overline{\partial}_t\xi^n,\eta^n\right)\right| \\ &\le Ch^4\left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\left(\|u_t\|_2^2+\|p_t\|_2^2\right)d\tau+\|u^n\|_2^2\right)+\frac{1}{6}\|\overline{\partial}_t\rho^n\|_0^2+C\|\eta^n\|_0^2,\end{aligned} \tag{24}$$

$$\begin{aligned}|B_6| &= \left|\left(\overline{\partial}_t\left(f(u^n)-f(U_h^n)\right),\eta^n\right)\right| \\ &= \left(\overline{\partial}_t u^n\left(\frac{\theta^n+\theta^{n-1}}{2}+\frac{\rho^n+\rho^{n-1}}{2}\right)+\frac{U_h^n+U_h^{n-1}}{2}\left(\overline{\partial}_t\theta^n+\overline{\partial}_t\rho^n\right),\eta^n\right)-\left(2\left(\overline{\partial}_t\theta^n+\overline{\partial}_t\rho^n\right),\eta^n\right) \\ &\le Ch^4\left(\|u^n\|_2^2+\|u^{n-1}\|_2^2+\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\|u_t\|_2^2 d\tau\right)+\frac{1}{6}\|\overline{\partial}_t\rho^n\|_0^2+C\|\rho^n\|_0^2+C\|\rho^{n-1}\|_h^2+C\|\eta^n\|_0^2.\end{aligned} \tag{25}$$

From derivative transfer technique and (8), it can be proved that

$$\begin{aligned}|B_3| &= \left|\sum_K\int_{\partial K}\frac{\partial p^n}{\partial n}\overline{\partial}_t\rho^n ds\right| \\ &= \overline{\partial}_t\left(\sum_K\int_{\partial K}\frac{\partial p^n}{\partial n}\rho^n ds\right)-\sum_K\int_{\partial K}\frac{\partial\left(\overline{\partial}_t p^n\right)}{\partial n}\rho^{n-1}ds \\ &\le \frac{Ch^4}{\Delta t}\int_{t_{n-1}}^{t_n}\|p_t\|_3^2 d\tau+C\|\rho^{n-1}\|_h^2+\overline{\partial}_t\left(\sum_K\int_{\partial K}\frac{\partial p^n}{\partial n}\rho^n ds\right),\end{aligned} \tag{26}$$

$$|B_7| = \left|\sum_K\int_{\partial K}\frac{\partial\left(\overline{\partial}_t u^n\right)}{\partial n}\eta^n ds\right| \le \frac{Ch^4}{\Delta t}\int_{t_{n-1}}^{t_n}\|u_t\|_3^2 d\tau+\|\eta^n\|_h^2. \tag{27}$$

Then, substituting (23)–(27) into (22) reduces to

$$\begin{aligned}&\frac{1}{2}\|\overline{\partial}_t\rho^n\|_0^2+\frac{\alpha}{2\Delta t}\left(\|\rho^n\|_0^2-\|\rho^{n-1}\|_0^2\right)+\frac{1}{2\Delta t}\left(\|\eta^n\|_0^2-\|\eta^{n-1}\|_0^2\right) \\ &\le Ch^4\left[\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\left(\|u_t\|_3^2+\|p_t\|_3^2\right)d\tau+\|u^n\|_2^2+\|u^{n-1}\|_2^2\right] \\ &\quad + C(\Delta t)\int_{t_{n-1}}^{t_n}\|u_{tt}\|_0^2 d\tau+\overline{\partial}_t\left(\sum_K\int_{\partial K}\frac{\partial p^n}{\partial n}\rho^n ds\right) \\ &\quad + C\|\eta^n\|_0^2+C\|\rho^n\|_0^2+\|\eta^n\|_h^2+C\|\rho^{n-1}\|_h^2.\end{aligned} \tag{28}$$

Multiplying by $2\Delta t$ and summing up the above inequality and then plugging (28) into (20), by discrete Gronwall's lemma again, choosing appropriate $\Delta t$ and $\varepsilon>0$ such that $1-C\varepsilon\Delta t>0$, by applying (8) and noting that $\rho^0=0$, $\eta^0=0$, we can obtain

$$\Delta t\sum_{i=1}^{n}\|\overline{\partial}_t\rho^i\|_0^2+\|\rho^n\|_h^2+\Delta t\sum_{i=1}^{n}\|\eta^i\|_h^2 \le C\left(h^4+(\Delta t)^2\right). \tag{29}$$

At last, choosing $v_h=\overline{\partial}_t\eta^n$ in (11a) and $q_h=\overline{\partial}_t\rho^n$ in (21) and then substituting them, it yields

$$\begin{aligned}\left(\nabla\overline{\partial}_t\eta^n,\nabla\eta^n\right)_h+\|\overline{\partial}_t\rho^n\|_h^2 &= -\left(\overline{\partial}_t\theta^n,\overline{\partial}_t\eta^n\right)-\alpha\left(\theta^n,\overline{\partial}_t\eta^n\right) \\ &\quad -\alpha\left(\rho^n,\overline{\partial}_t\eta^n\right)+\sum_K\int_{\partial K}\frac{\partial p^n}{\partial n}\overline{\partial}_t\eta^n ds \\ &\quad -\left(R_1^n,\overline{\partial}_t\eta^n\right)+\left(\overline{\partial}_t\xi^n,\overline{\partial}_t\rho^n\right) \\ &\quad -\left(\overline{\partial}_t\left(f(u^n)-f(U_h^n)\right),\overline{\partial}_t\rho^n\right) \\ &\quad +\sum_K\int_{\partial K}\frac{\partial\left(\overline{\partial}_t u^n\right)}{\partial n}\overline{\partial}_t\rho^n ds \\ &= \sum_{i=1}^{8}D_i.\end{aligned} \tag{30}$$

Similar to the estimates of (20) and (29), we have

$$\left(\nabla\overline{\partial}_t\eta^n,\nabla\eta^n\right)_h \geq \frac{1}{2\Delta t}\left(\|\eta^n\|_h^2-\|\eta^{n-1}\|_h^2\right), \tag{31}$$

$$\begin{aligned}|D_1+D_2| &= \left|\left(\overline{\partial}_t\theta^n,\overline{\partial}_t\eta^n\right)+\alpha\left(\theta^n,\overline{\partial}_t\eta^n\right)\right| \\ &= \left|\overline{\partial}_t\left(\overline{\partial}_t\theta^n,\eta^n\right)+\alpha\overline{\partial}_t\left(\theta^n,\eta^n\right)-\left(\overline{\partial}_t\left(\overline{\partial}_t\theta^n\right),\eta^{n-1}\right)-\alpha\left(\overline{\partial}_t\theta^n,\eta^{n-1}\right)\right| \\ &\leq \left|\overline{\partial}_t\left[\left(\overline{\partial}_t\theta^n,\eta^n\right)+\alpha\left(\theta^n,\eta^n\right)\right]\right|+\frac{Ch^4}{\Delta t}\left(\int_{t_{n-2}}^{t_n}\|u_{tt}\|_2^2 d\tau+\int_{t_{n-1}}^{t_n}\|u_t\|_2^2 d\tau\right)+C\|\eta^{n-1}\|_0^2,\end{aligned} \tag{32}$$

$$\begin{aligned}|D_3+D_4+D_5| &= \left|-\alpha\left(\rho^n,\overline{\partial}_t\eta^n\right)+\sum_K\int_{\partial K}\frac{\partial p^n}{\partial n}\overline{\partial}_t\eta^n ds-\left(R_1^n,\overline{\partial}_t\eta^n\right)\right| \\ &\leq \left|\overline{\partial}_t\left[-\alpha\left(\rho^n,\eta^n\right)+\sum_K\int_{\partial K}\frac{\partial p^n}{\partial n}\eta^n ds-\left(R_1^n,\eta^n\right)\right]\right|+C\|\overline{\partial}_t\rho^n\|_0^2+C\|\overline{\partial}_t R_1^n\|_0^2+\frac{Ch^4}{\Delta t}\int_{t_{n-1}}^{t_n}\|p_t\|_3^2 d\tau+C\|\eta^{n-1}\|_h^2,\end{aligned} \tag{33}$$

$$\begin{aligned}|D_6+D_7| &= \left|\left(\overline{\partial}_t\xi^n,\overline{\partial}_t\rho^n\right)-\left(\overline{\partial}_t\left(f\left(u^n\right)-f\left(U_h^n\right)\right),\overline{\partial}_t\rho^n\right)\right| \\ &\leq Ch^4\left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\left(\|p_t\|_2^2+\|u_t\|_2^2\right)d\tau+\|u^n\|_2^2+\|u^{n-1}\|_2^2\right)+C\|\overline{\partial}_t\rho^n\|_0^2+C\|\rho^n\|_0^2+C\|\rho^{n-1}\|_0^2,\end{aligned} \tag{34}$$

$$|D_8| = \left|\sum_K\int_{\partial K}\frac{\partial\left(\overline{\partial}_t u^n\right)}{\partial n}\overline{\partial}_t\rho^n ds\right| \leq \frac{Ch^4}{\Delta t}\int_{t_{n-1}}^{t_n}\|u_t\|_3^2 d\tau+\frac{1}{2}\|\overline{\partial}_t\rho^n\|_h^2. \tag{35}$$

Substituting (31)–(35) into (30), we have

$$\begin{aligned}\frac{1}{2\Delta t}\left(\|\eta^n\|_h^2-\|\eta^{n-1}\|_h^2\right) \leq{}& Ch^4\left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\left(\|p_t\|_3^2+\|u_t\|_3^2+\|u_{tt}\|_2^2\right)d\tau+\|u^n\|_2^2+\|u^{n-1}\|_2^2\right) \\ &+\left|\overline{\partial}_t\left[\left(\overline{\partial}_t\theta^n,\eta^n\right)+\alpha\left(\theta^n,\eta^n\right)-\alpha\left(\rho^n,\eta^n\right)+\sum_K\int_{\partial K}\frac{\partial p^n}{\partial n}\eta^n ds-\left(R_1^n,\eta^n\right)\right]\right| \\ &+C\|\overline{\partial}_t\rho^n\|_0^2+C\|\rho^n\|_0^2+C\|\rho^{n-1}\|_0^2.\end{aligned} \tag{36}$$

Multiplying by $2\Delta t$ and summing up the above inequality, by discrete Gronwall's lemma and (29), there holds

$$\|\eta^n\|_h^2+\Delta t\sum_{i=1}^n\|\overline{\partial}_t\rho^i\|_h^2 \leq C\left(h^4+(\Delta t)^2\right). \tag{37}$$

With (29) and (37), the proof is completed.

Finally, we use mathematical induction to verify assumption (15) which is similar to the technique used in [22, 23, 26].

Let $\mu^n := u^n - U_h^n$. Initially, when $n=0$, we have $\|\mu^0\|_{0,\infty} = \|u^0 - I_h u^0\|_{0,\infty} \leq Ch < 1$, and the assumption is true.

Furthermore, we assume that when $n=k-1$, there holds $\|\mu^{k-1}\|_{0,\infty} < 1$. Then, by Theorem 1, we have $\|U_h^{k-1} - I_h u^{k-1}\|_h \leq C(h^2+\Delta t)$.

Additionally, we consider the situation at $n=k$. We know that $\|\mu(t)\|_{0,\infty}$ is continuous function about time $t$, so there exists $\delta > 0$, for $\forall\epsilon > 0$; when $|t_{k-1}-t_k| = \Delta t < \delta$, there holds

$$\left|\|\mu^{k-1}\|_{0,\infty}-\|\mu^k\|_{0,\infty}\right| < \epsilon. \tag{38}$$

Taking $\epsilon = \Delta t$ in (38), we have

$$\begin{aligned}\left\|\mu^{k}\right\|_{0,\infty} &\leq \left\|\mu^{k-1}\right\|_{0,\infty} + \Delta t \leq \left\|U_{h}^{k-1} - I_{h}u^{k-1}\right\|_{0,\infty} \\ &\quad + \left\|u^{k-1} - I_{h}u^{k-1}\right\|_{0,\infty} + \Delta t \\ &\leq Ch^{-1}\left\|U_{h}^{k-1} - I_{h}u^{k-1}\right\|_{0} + Ch\left|u^{k-1}\right|_{1,\infty} + \Delta t \\ &\leq C\left(h^{-1}\Delta t + h\right) + Ch + \Delta t < 1.\end{aligned} \tag{39}$$

In the last step of (39), we need the time step $\Delta t$ and space step $h$ satisfy the condition $\Delta t = O(h^{1+\alpha})\,(0 < \alpha \leq 1)$. Above all, choose appropriate $h_1$ to make that $Ch_1^{\alpha} < 1$, which ends the proof. □

## 4. Superconvergence Analysis for the Fully-Discrete Approximation Scheme

To obtain the superconvergence results, we combine the adjacent four elements $K_1, K_2, K_3$, and $K_4$ into a big element $\widetilde{K}$, i.e., $\widetilde{K} = \cup_{i=1}^{4} K_i$ (see Figure 1). The corresponding subdivision is defined by $T_{2h}$.

Then, construct the interpolation postprocessing operator $I_{2h}$ on $\widetilde{K}$ as in [27–29] which satisfies

$$\begin{cases} I_{2h}u\,|_{\widetilde{K}} \in P_2(\widetilde{K}), \\ \displaystyle\int_{L_i} (I_{2h}u - u)\mathrm{d}s = 0, \quad i = 1,2,3,4, \forall \widetilde{K} \in T_{2h}, \\ \displaystyle\int_{K_1 \cup K_3} (I_{2h}u - u)\mathrm{d}x\,\mathrm{d}y = 0, \\ \displaystyle\int_{K_2 \cup K_4} (I_{2h}u - u)\mathrm{d}x\,\mathrm{d}y = 0, \end{cases} \tag{40}$$

where $P_2(\widetilde{K})$ denotes the space of polynomial on $\widetilde{K}$ with degree less than or equal to 2 and $L_i$ $(i = 1,2,3,4)$ are the four sides of $\widetilde{K}$.

From [27], the interpolation postprocessing operator $I_{2h}$ satisfies

$$\begin{gathered} I_{2h}I_h u = I_{2h}u, \\ \left\|I_{2h}u - u\right\|_h \leq Ch^2 \|u\|_3, \\ \left\|I_{2h}v_h\right\|_h \leq C\left\|v_h\right\|_h, \\ \forall v_h \in V_h. \end{gathered} \tag{41}$$

**Theorem 2.** *Under the assumption of Theorem 1, there holds*

$$\left\|u^n - I_{2h}U_h^n\right\|_h + \left\|p^n - I_{2h}P_h^n\right\|_h \leq C\left(h^2 + \Delta t\right). \tag{42}$$

*Proof.* From triangle inequality, (10), and (41), there holds

$$\begin{aligned}\left\|u^n - I_{2h}U_h^n\right\|_h &= \left\|u^n - I_{2h}I_h u^n + I_{2h}I_h u^n - I_{2h}U_h^n\right\|_h \\ &\leq \left\|u^n - I_{2h}u^n\right\|_h + C\left\|I_h u^n - U_h^n\right\|_h \leq C\left(h^2 + \Delta t\right).\end{aligned} \tag{43}$$

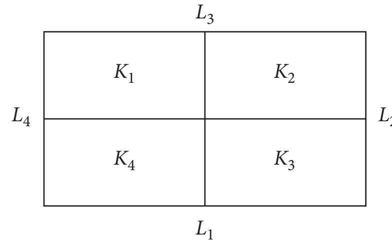

Figure 1: The big element $\widetilde{K}$.

The superconvergence result of $p$ can be obtained similarly, which completes the proof. □

*Remark 1.*

(i) From the analysis of Theorems 1 and 2, the superclose and superconvergence results for the two variables $u$ and $p$ in the broken $H^1$-norm are derived, which are one order higher than the convergence results in [1].

(ii) The conclusions of this paper are applicable to other nonconforming elements such as the $Q_1^{\mathrm{rot}}$ element [30], rectangular constrained $Q_1^{\mathrm{rot}}$ element [31], quasi-Wilson element [32, 33], modified quasi-Wilson element [34], and quasi-Carey element [35].

(iii) For nonconforming linear triangle Crouzeix–Raviart element [36], Carey element [37], and Wilson element [38], the consistency errors are only of order $O(h)$; for nonconforming $P_1$ element [39] and $p_1^{\mathrm{mod}}$ element [40], though the consistency errors reach to $O(h^3)$ order, the interpolation error term can be estimated as $(\nabla(u - I_h u), \nabla v_h)_h \leq Ch|u|_2\|v_h\|_h$. Therefore, the superconvergence results are unable to get.

## 5. Numerical Example

Numerical simulation results are presented in this section. The Newton iterative algorithm is used to solve the nonlinear system.

Consider the following problem [1]:

$$\begin{cases} u_t + \Delta^2 u + \alpha u = \Delta f(u) + g(X), & (X,t) \in \Omega \times [0,T], \\ u(X,0) = u^0(X), & X \in \Omega, \end{cases} \tag{44}$$

where $\Omega = [0,1] \times [0,1]$, $\alpha = 0.5, T = 1$.

Let $p = f(u) - \Delta u$; then, the mixed variational formulation for (44) is find $(u,p) \in V \times V$ such that

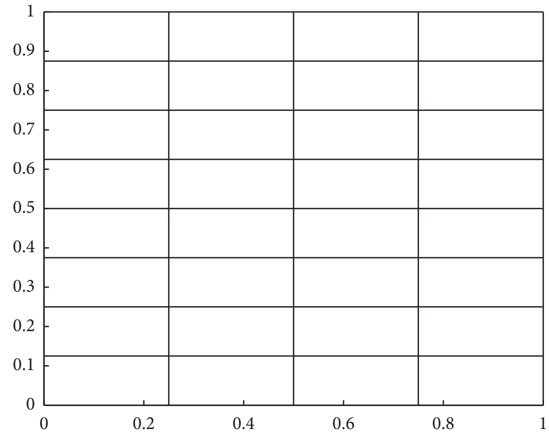


FIGURE 2: The rectangular mesh.

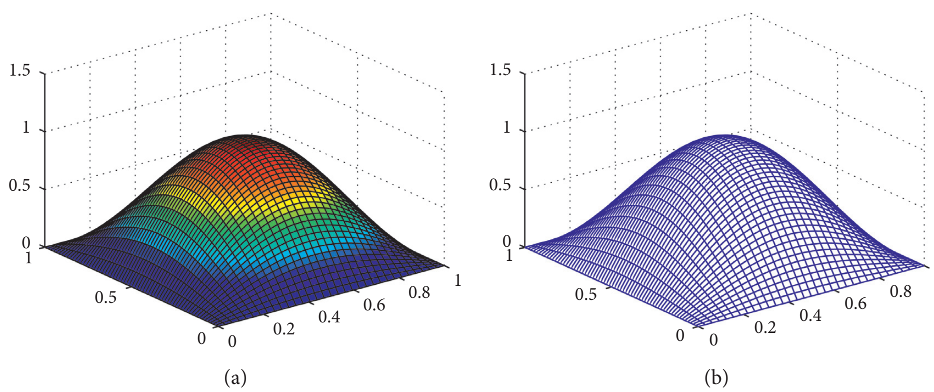


FIGURE 3: Exact solution $u$ (a) and numerical solution $U_h$ (b) on $16 \times 32$ meshes at $t = 0.5$.

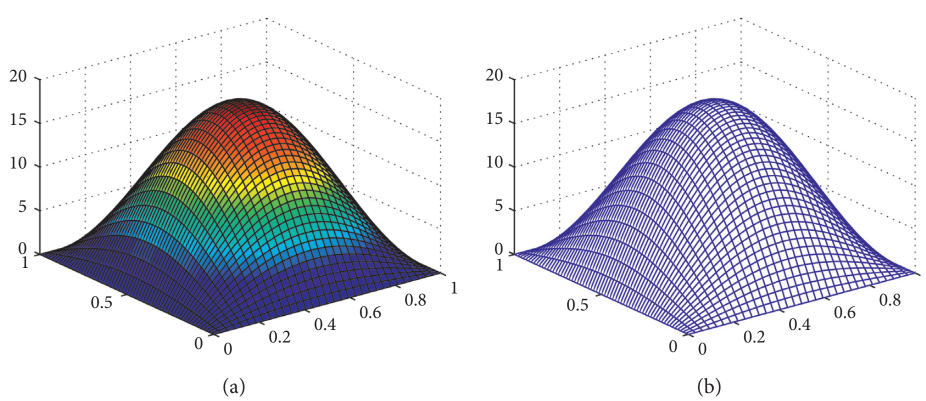


FIGURE 4: Exact solution $p$ (a) and numerical solution $P_h$ (b) on $16 \times 32$ meshes at $t = 0.5$.

$$\begin{cases} (u_t, v) + (\nabla p, \nabla v) + \alpha (u, v) = (g(X), v), & \forall v \in V, \\ (p, q) - (\nabla u, \nabla q) = (f(u), q), & \forall q \in V, \\ u(X, 0) = u^0(X), & p(X, 0) = p^0(X) = f\left(u^0(X)\right) - \Delta u^0(X). \end{cases} \tag{45}$$

Table 1: Numerical results of $u$ at $t = 0.1$.

| $m \times n$ | $\|u^n - U_h^n\|_h$ | Order | $\|U_h^n - I_h u^n\|_h$ | Order | $\|u^n - I_{2h} U_h^n\|_h$ | Order |
|---|---|---|---|---|---|---|
| $4 \times 8$ | $4.2013e-01$ | — | $1.4228e-01$ | — | $2.6226e-01$ | — |
| $8 \times 16$ | $2.0201e-01$ | 1.0564 | $3.6301e-02$ | 1.9707 | $6.7246e-02$ | 1.9634 |
| $16 \times 32$ | $9.9915e-02$ | 1.0156 | $9.1169e-03$ | 1.9933 | $1.6911e-02$ | 1.9914 |
| $32 \times 64$ | $4.9818e-02$ | 1.0040 | $2.2818e-03$ | 1.9983 | $4.2340e-03$ | 1.9978 |

Table 2: Numerical results of $u$ at $t = 0.5$.

| $m \times n$ | $\|u^n - U_h^n\|_h$ | Order | $\|U_h^n - I_h u^n\|_h$ | Order | $\|u^n - I_{2h} U_h^n\|_h$ | Order |
|---|---|---|---|---|---|---|
| $4 \times 8$ | $4.2408e-01$ | — | $1.3655e-01$ | — | $2.6207e-01$ | — |
| $8 \times 16$ | $2.0484e-01$ | 1.0497 | $3.5024e-02$ | 1.9630 | $6.7315e-02$ | 1.9609 |
| $16 \times 32$ | $1.0147e-01$ | 1.0134 | $9.2067e-03$ | 1.9275 | $1.7147e-02$ | 1.9729 |
| $32 \times 64$ | $5.0596e-02$ | 1.0039 | $2.3042e-03$ | 1.9983 | $4.2931e-03$ | 1.9978 |

Table 3: Numerical results of $p$ at $t = 0.1$.

| $m \times n$ | $\|p^n - P_h^n\|_h$ | Order | $\|P_h^n - I_h p^n\|_h$ | Order | $\|p^n - I_{2h} P_h^n\|_h$ | Order |
|---|---|---|---|---|---|---|
| $4 \times 8$ | 7.2813 | — | 1.2981 | — | 4.1865 | — |
| $8 \times 16$ | 3.6128 | 1.0110 | $3.2303e-01$ | 2.0066 | 1.0770 | 1.9586 |
| $16 \times 32$ | 1.8031 | 1.0026 | $8.0602e-02$ | 2.0027 | $2.7098e-01$ | 1.9908 |
| $32 \times 64$ | $9.0114e-01$ | 1.0006 | $2.0140e-02$ | 2.0007 | $6.7852e-02$ | 1.9977 |

Table 4: Numerical results of $p$ at $t = 0.5$.

| $m \times n$ | $\|p^n - P_h^n\|_h$ | Order | $\|P_h^n - I_h p^n\|_h$ | Order | $\|p^n - I_{2h} P_h^n\|_h$ | Order |
|---|---|---|---|---|---|---|
| $4 \times 8$ | 7.3758 | — | 1.1901 | — | 4.2129 | — |
| $8 \times 16$ | 3.66791 | 1.0078 | $2.9887e-01$ | 1.9935 | 1.0854 | 1.9565 |
| $16 \times 32$ | 1.8318 | 1.0016 | $8.2041e-02$ | 1.8651 | $2.7536e-01$ | 1.9788 |
| $32 \times 64$ | $9.1551e-01$ | 1.0006 | $2.0500e-02$ | 2.0006 | $6.8948e-02$ | 1.9977 |

The exact solutions are $u(x, y, t) = (1 + t^6)\sin \pi x \sin \pi y$ and $p(x, y, t) = (1/2)(1 + t^6)^2 (\sin \pi x)^2 (\sin \pi y)^2 + 2(\pi^2 - 1)(1 + t^6)\sin \pi x \sin \pi y$.

The fully-discrete approximation scheme for (45) is find $(U_h^n, P_h^n) \in V_h \times V_h$ such that

$$\begin{cases} \left(\overline{\partial}_t U_h^n, v_h\right) + \left(\nabla P_h^n, \nabla v_h\right)_h + \alpha\left(U_h^n, v_h\right) = \left(g(X), v_h\right), & \forall v_h \in V_h, \\ \left(P_h^n, q_h\right) - \left(\nabla U_h^n, \nabla q_h\right)_h = \left(f\left(U_h^n\right), q_h\right), & \forall q_h \in V_h, \\ U_h^0 = I_h u^0, & P_h^0 = I_h p^0. \end{cases} \tag{46}$$

We divide the domain $\Omega$ into $m \times n$ rectangular meshes (see Figure 2). Then, the FE solutions $U_h^n$, $P_h^n$ can be calculated according to (46); here, we choose $\Delta t = \sqrt{(1/m^2) + (1/n^2)}$.

For simplicity and concreteness, we just plot the exact solutions $u$, $p$ and the numerical solutions $U_h$, $P_h$ on $16 \times 32$ meshes at $t = 0.5$ (see Figures 3 and 4), respectively.

Then, the convergence, superclose, and superconvergence results of $u$ and $p$ in the broken $H^1$-norm at time $t = 0.1$ and 0.5 are listed in Tables 1–4, respectively.

From Tables 1 and 2, we can see that $\|u^n - U_h^n\|_h$ are convergent at order $O(h)$ and $\|U_h^n - I_h u^n\|_h$ and $\|u^n - I_{2h} U_h^n\|_h$ are convergent at order $O(h^2)$, which coincide with the theoretical analysis. Meanwhile, the results of $\|u^n - I_{2h} U_h^n\|_h$ are better than $\|u^n - U_h^n\|_h$, which indicate the superiority of the superconvergence algorithm. The results of $p$ in Tables 3 and 4 are consistent with those of $u$ in Tables 1 and 2.

## 6. Conclusions

In this work, we study the nonconforming MFEM for fourth-order nonlinear Sivashinsky equation. The superconvergence results of the relevant variables in the broken $H^1$-norm are obtained, which are one order higher than those of convergence. Furthermore, a numerical example demonstrates the efficiency of the theoretical analysis.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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