Research Article

Certain Generating Relations Involving the Generalized Multi-Index Bessel–Maitland Function

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Generating relations involving the special functions have already proved their important role in mathematics and other fields of sciences. In this paper, we aim to provide some presumably new generating relations in connection with the generalized multi-index Bessel–Maitland function $J_{(\lambda)_m, (\gamma)_m, q}^{\lambda}(\cdot)$. The main results presented here, being very general, can yield a number of particular or equivalent identities, some of which are explicitly demonstrated.

1. Introduction and Preliminaries

Here and elsewhere, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive real numbers, positive integers, and nonpositive integers, respectively.

The Bessel–Maitland function $J_{\nu}^{\lambda}(z)$ is defined as (see Marichev [1])

$$J_{\nu}^{\lambda}(z) = \sum_{r=0}^{\infty} \frac{(-z)^r}{\Gamma(\lambda r + \nu + 1)r!}, \quad \lambda \in \mathbb{R}^+, z \in \mathbb{C}. \quad (1)$$

Pathak [2] gave the following more generalized form of generalized Bessel–Maitland function (1):

$$J_{\nu, q}^{\lambda, \gamma}(z) = \sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\Gamma(\lambda r + \nu + 1)} \frac{(-z)^r}{r!}, \quad (2)$$

$$(\lambda, \nu, \gamma \in \mathbb{C}, \Re(\lambda) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0, q \in (0, 1) \cup \mathbb{N}). \quad (3)$$

Remark 1. Even though Pathak excluded $q = 0$ in (2), the case $q = 0$ yields (1).

If $q = 1$, $\gamma = 1$, ν is replaced by $\nu - 1$, and z is replaced by $-z$ in (2), then generalized Bessel–Maitland function reduces to the Mittag–Leffler function which was studied by Wiman [3] as follows:

$$J_{\nu-1, 1}^{\lambda, 1}(-z) = E_{\lambda, \nu}(z), \quad \Re(\lambda) > 0, \Re(\nu) > 0. \quad (4)$$

If ν is replaced by $\nu - 1$ and z is replaced by $-z$ in (2), then the generalized Bessel–Maitland function reduces to the well-known generalized Mittag–Leffler function $E_{\lambda, \nu}^{\gamma, q}(z)$ which was introduced by Shukla and Prajapati [4] as follows:

$$J_{\nu-1, q}^{\lambda, \gamma}(-z) = E_{\lambda, \nu}^{\gamma, q}(z), \quad (5)$$

$$(\Re(\lambda) > 0, \Re(\nu) > 0, \Re(\gamma) > 0; q \in (0, 1) \cup \mathbb{N}). \quad (6)$$

Jain and Agarwal [5] generalized Bessel–Maitland function $J_{\nu}^{\lambda}(z)$ (1) as follows:

$$J_{\nu, \mu}^{\lambda}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{\nu+2\mu+2r}}{\Gamma(\lambda r + \nu + \mu + 1)\Gamma(\mu + r + 1)}, \quad (7)$$

$$(\lambda \in \mathbb{R}^+, \nu, \mu \in \mathbb{C}, z \in \mathbb{C} \setminus (-\infty, 0]). \quad (8)$$

Choi and Agarwal [6] investigated the following generalized multi-index Bessel function:

$$J_{(\lambda_j)_{m,\gamma}}^{(\nu_j)_{m,q}}(z) = \sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\prod_{j=1}^m \Gamma(\lambda_j r + \nu_j + 1)} \frac{(-z)^r}{r!}, \quad (9)$$

where $m \in \mathbb{N}$ and $\lambda_j, \nu_j, \gamma, q$, and $z \in \mathbb{C}$ ($j = 1, \dots, m$) such that

$$\sum_{j=1}^m \Re(\lambda_j) > \max\{0, \Re(q) - 1\}, \quad \Re(\nu_j) > -1, \quad \Re(\gamma) > 0, \quad q \in (0, 1) \cup \mathbb{N}. \quad (10)$$

Remark 2. It is easily found that generalized multi-index Bessel–Maitland function (9) is equivalent to the generalized multi-index Mittag–Leffler function defined and studied by Saxena and Nishimoto [7] (see also [8]).

Pohlen [9] introduced the Hadamard product (or the convolution) $f * g$ of two analytic functions f and g as follows:

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z), \quad (|z| < R), \quad (11)$$

where $R \geq R_f \cdot R_g$. Here, $f(z)$ and $g(z)$ are analytic at $z = 0$ whose Maclaurin series with their respective radii of convergence R_f and R_g are

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z^n, \quad (|z| < R_f), \\ g(z) &= \sum_{n=0}^{\infty} b_n z^n, \quad (|z| < R_g). \end{aligned} \quad (12)$$

The concept of the Hadamard product has turned out to be useful, particularly, in factorizing a newborn function, which is usually expressed as a Maclaurin series, into two known functions (see, e.g., [10–13]).

The k -th derivative of the function $f(p) = p^{-\lambda-n\xi}$ ($\lambda, \xi \in \mathbb{C}, n \in \mathbb{N}$) is easily found to be given in terms of gamma function as follows:

$$f^{(k)}(p) = (-1)^k p^{-\lambda-n\xi-k} \frac{\Gamma(\lambda+n\xi+k)}{\Gamma(\lambda+n\xi)}, \quad (k \in \mathbb{N}_0). \quad (13)$$

Generating functions have been widely used in exploring certain properties and formulas involving sequences and polynomials in a wide range of research subjects. Many researchers have developed a remarkably large number of generating functions associated with a variety of special functions. For some works on this subject, one may refer, for example, to an extensive monograph [14–25] and the literature cited therein. In this search, we aim to provide some presumably new generating relations in connection with generalized multi-index Bessel–Maitland function (9). The main results developed here, being very general, can be reduced to produce a large

number of presumably new and potentially useful generating relations for other known functions, some of which are demonstrated.

2. Generating Relations

We give two generating relations involving generalized multi-index Bessel–Maitland function (9) asserted by the following theorems.

Theorem 1. Let $m \in \mathbb{N}$ and $\lambda_j, \nu_j, \gamma, q$, and $z \in \mathbb{C}$ ($j = 1, \dots, m$) such that

$$\sum_{j=1}^m \Re(\lambda_j) > \max\{0, \Re(q) - 1\}, \quad \Re(\nu_j) > -1, \quad \Re(\gamma) > 0, \quad q \in (0, 1) \cup \mathbb{N}. \quad (14)$$

Also, let $|t| < 1$. Then,

$$\begin{aligned} (1+t)^{-\sigma} J_{(\lambda_j)_{m,\gamma}}^{(\nu_j)_{m,q}}\left(\frac{z}{1+t}\right) \\ = \sum_{k=0}^{\infty} (-1)^k (\sigma)_k J_{(\lambda_j)_{m,\gamma}}^{(\nu_j)_{m,q}}(z) * {}_1F_1(\sigma+k; \sigma; -z) \frac{t^k}{k!}. \end{aligned} \quad (15)$$

Proof. We replace $1+t$ by s in the left-hand side of (15) and denote the resulting expression by $g(s)$. Then, using form (9), on expanding the function in series, gives

$$g(s) = s^{-\sigma} J_{(\lambda_j)_{m,\gamma}}^{(\nu_j)_{m,q}}\left(\frac{z}{s}\right) = \sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\prod_{j=1}^m \Gamma(\lambda_j r + \nu_j + 1)} \frac{(-z)^r}{r!} s^{-\sigma-r}. \quad (16)$$

Differentiating k times both sides of (16) with respect to s with the aid of (13) (term-by-term differentiation can be verified under the given conditions), we find

$$g^{(k)}(s) = (-1)^k s^{-\sigma-k} \sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\prod_{j=1}^m \Gamma(\lambda_j r + \nu_j + 1)} \frac{\Gamma(\sigma+r+k)}{\Gamma(\sigma+r)} \left(\frac{-z}{s}\right)^r \frac{1}{r!}, \quad (17)$$

which is simplified to yield

$$g^{(k)}(s) = (-1)^k s^{-\sigma-k} (\sigma)_k \sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\prod_{j=1}^m \Gamma(\lambda_j r + \nu_j + 1)} \frac{(\sigma+k)_r}{(\sigma)_r} \left(\frac{-z}{s}\right)^r \frac{1}{r!}. \quad (18)$$

Decomposing series (18) into Hadamard product (11), we obtain

$$g^{(k)}(s) = (-1)^k s^{-\sigma-k} (\sigma)_k J_{\left(\frac{\lambda_j}{\nu_j}\right)_{m,q}}^{\left(\frac{\lambda_j}{\nu_j}\right)_{m,\gamma}} \left(\frac{z}{s}\right) * {}_1F_1\left(\sigma+k; \sigma; -\frac{z}{s}\right). \quad (19)$$

Expanding $g(s+t)$ as the Taylor series gives

$$g(s+t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} g^{(k)}(s). \quad (20)$$

Combining (16), (19), and (20), we obtain

$$(s+t)^{-\sigma} J_{\left(\frac{\lambda_j}{\nu_j}\right)_{m,q}}^{\left(\frac{\lambda_j}{\nu_j}\right)_{m,\gamma}} \left(\frac{z}{s+t}\right) = \sum_{k=0}^{\infty} \frac{(-t)^k s^{-\sigma-k}}{k!} (\sigma)_k J_{\left(\frac{\lambda_j}{\nu_j}\right)_{m,q}}^{\left(\frac{\lambda_j}{\nu_j}\right)_{m,\gamma}} \left(\frac{z}{s}\right) * {}_1F_1\left(\sigma+k; \sigma; -\frac{z}{s}\right). \quad (21)$$

Finally, setting $s = 1$ yields desired result (15). $\square$

Theorem 2. Let $m \in \mathbb{N}$ and $\lambda_j, \nu_j, \gamma, q$, and $z \in \mathbb{C}$ ($j = 1, \dots, m$) such that

$$\sum_{j=1}^m \Re(\lambda_j) > \max\{0, \Re(q) - 1\}, \quad \Re(\nu_j) > -1, \quad \Re(\gamma) > 0, \quad q \in (0, 1) \cup \mathbb{N}. \quad (22)$$

Also, let $|t| < 1$. Then,

$$\sum_{k=0}^{\infty} \binom{\gamma+k-1}{k} J_{\left(\frac{\lambda_j}{\nu_j}\right)_{m,q}}^{\left(\frac{\lambda_j}{\nu_j}\right)_{m,\gamma+k}}(z) t^k = (1-t)^{-\gamma} J_{\left(\frac{\lambda_j}{\nu_j}\right)_{m,q}}^{\left(\frac{\lambda_j}{\nu_j}\right)_{m,\gamma}} \left(\frac{z}{(1-t)^q}\right). \quad (23)$$

Proof. Let J be the left-hand side of (23). Using (9), on expanding the function in series, gives

$$J = \sum_{k=0}^{\infty} \binom{\gamma+k-1}{k} \left\{ \sum_{r=0}^{\infty} \frac{(\gamma+k)_{qr}}{\prod_{j=1}^m \Gamma(\lambda_j r + \nu_j + 1)} \frac{(-z)^r}{r!} \right\} t^k. \quad (24)$$

Interchanging the order of summations in (24) and using the known identity (see, e.g., [26, p. 5])

$$\binom{\gamma}{k} = \frac{\Gamma(\gamma+1)}{k! \Gamma(\gamma-k+1)}, \quad k \in \mathbb{N}_0, \gamma \in \mathbb{C}, \quad (25)$$

we have

$$J = \sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\prod_{j=1}^m \Gamma(\lambda_j r + \nu_j + 1)} \left\{ \sum_{k=0}^{\infty} \binom{\gamma+qr+k-1}{k} t^k \right\} \frac{(-z)^r}{r!}. \quad (26)$$

Using the generalized binomial expansion, we find that the inner sum in (26) gives

$$\sum_{k=0}^{\infty} \binom{\gamma+qr+k-1}{k} t^k = (1-t)^{-(\gamma+qr)}, \quad |t| < 1. \quad (27)$$

Finally, interpreting (26) with the help of (27) yields desired result (23). $\square$

3. Further Remarks

Here, we choose to give some equivalent identities and particular cases of the results in Theorems 1 and 2. As noted in Remark 2, setting ν_j by $\nu_j - 1$ and z by $-z$ in (15) and (23) gives two corresponding generating relations involving the generalized multi-index Mittag-Leffler function $E_{(\lambda_j, \nu_j)_m}^{\gamma, q}(z)$, which are asserted, respectively, in Corollaries 1 and 2.

Corollary 1. Let $m \in \mathbb{N}$ and $\lambda_j, \nu_j, \gamma, q$, and $z \in \mathbb{C}$ ($j = 1, \dots, m$) such that

$$\sum_{j=1}^m \Re(\lambda_j) > \max\{0, \Re(q) - 1\}, \quad \Re(\nu_j) > 0, \quad \Re(\gamma) > 0, \quad q \in (0, 1) \cup \mathbb{N}. \quad (28)$$

Also, let $|t| < 1$. Then,

$$(1+t)^{-\sigma} E_{(\lambda_j, \nu_j)_m}^{\gamma, q} \left(\frac{z}{1+t}\right) = \sum_{k=0}^{\infty} (-1)^k (\sigma)_k E_{(\lambda_j, \nu_j)_m}^{\gamma, q}(z) * {}_1F_1(\sigma+k; \sigma; -z) \frac{t^k}{k!}. \quad (29)$$

Corollary 2. Let $m \in \mathbb{N}$ and $\lambda_j, \nu_j, \gamma, q$, and $z \in \mathbb{C}$ ($j = 1, \dots, m$) such that

$$\sum_{j=1}^m \Re(\lambda_j) > \max\{0, \Re(q) - 1\}, \quad \Re(\nu_j) > 0, \quad \Re(\gamma) > 0, \quad q \in (0, 1) \cup \mathbb{N}. \quad (30)$$

Also, let $|t| < 1$. Then,

$$\sum_{k=0}^{\infty} \binom{\gamma+k-1}{k} E_{(\lambda_j, \nu_j)_m}^{\gamma+k, q}(z) t^k = (1-t)^{-\gamma} E_{(\lambda_j, \nu_j)_m}^{\gamma, q} \left(\frac{z}{(1-t)^q}\right). \quad (31)$$

The particular cases of (15), (23), (29), and (31) when $m = 1$ give the following generating relations, stated, respectively, in Corollaries 3–6.

Corollary 3. Let $\sigma, \lambda, \nu, \gamma$, and $z \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, and $q \in (0, 1) \cup \mathbb{N}$. Also, let $|t| < 1$. Then,

$$(1+t)^{-\sigma} J_{\nu,q}^{\lambda,\gamma} \left(\frac{z}{1+t} \right) = \sum_{k=0}^{\infty} (-1)^k (\sigma)_k J_{\nu,q}^{\lambda,\gamma} (z) * {}_1F_1(\sigma+k; \sigma; -z) \frac{t^k}{k!}. \quad (32)$$

Corollary 4. Let $\sigma, \lambda, \nu, \gamma$, and $z \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, and $q \in (0, 1) \cup \mathbb{N}$. Also, let $|t| < 1$. Then,

$$\sum_{k=0}^{\infty} \binom{\gamma+k-1}{k} J_{\nu,q}^{\lambda,\gamma+k} (z) t^k = (1-t)^{-\gamma} J_{\nu,q}^{\lambda,\gamma} \left(\frac{z}{(1-t)^q} \right). \quad (33)$$

Corollary 5. Let $\sigma, \lambda, \nu, \gamma$, and $z \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\nu) \geq 0$, $\Re(\gamma) > 0$, and $q \in (0, 1) \cup \mathbb{N}$. Also, let $|t| < 1$. Then,

$$(1+t)^{-\sigma} E_{\lambda,\nu}^{\gamma,q} \left(\frac{z}{1+t} \right) = \sum_{k=0}^{\infty} (-1)^k (\sigma)_k E_{\lambda,\nu}^{\gamma,q} (z) * {}_1F_1(\sigma+k; \sigma; -z) \frac{t^k}{k!}. \quad (34)$$

Corollary 6. Let $\sigma, \lambda, \nu, \gamma$, and $z \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\nu) \geq 0$, $\Re(\gamma) > 0$, and $q \in (0, 1) \cup \mathbb{N}$. Also, let $|t| < 1$. Then,

$$\sum_{k=0}^{\infty} \binom{\gamma+k-1}{k} E_{\lambda,\nu}^{\gamma+k,q} (z) t^k = (1-t)^{-\gamma} E_{\lambda,\nu}^{\gamma,q} \left(\frac{z}{(1-t)^q} \right). \quad (35)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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