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Research Article

Degenerate Analogues of Euler Zeta, Digamma, and Polygamma Functions

Fuli He[1] **Ahmed Bakhet**[1,2] **Mohamed Akel**[3] **and Mohamed Abdalla**[3,4]

[1]School of Mathematics and Statistics, Central South University, Changsha 410083, China
[2]Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt
[3]Department of Mathematics, Faculty of Science, South Valley University, Qena 83523, Egypt
[4]Department of Mathematics, College of Science, King Khalid University, Abha 61471, Saudi Arabia

Correspondence should be addressed to Mohamed Abdalla; moabdalla@kku.edu.sa

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In recent years, much attention has been paid to the role of degenerate versions of special functions and polynomials in mathematical physics and engineering. In the present paper, we introduce a degenerate Euler zeta function, a degenerate digamma function, and a degenerate polygamma function. We present several properties, recurrence relations, infinite series, and integral representations for these functions. Furthermore, we establish identities involving hypergeometric functions in terms of degenerate digamma function.

1. Introduction

The gamma, digamma, and polygamma functions have an increasing and recognized role in fractional differential equations, mathematical physics, the theory of special functions, statistics, probability theory, and the theory of infinite series. The reader may refer, for example, to [1–9]. These functions are directly connected with a variety of special functions such as zeta function, Clausen's function, and hypergeometric functions. The evaluations of series involving Riemann zeta function $\zeta(s)$ and related functions have a long history that can be traced back to Christian Goldbach (1690–1764) and Leonhard Euler (1707–1783) (see, for details, [10]). The Euler zeta function and its generalizations and extensions have been widely studied [11–15].

Later on, these functions arise in the study of matrix-valued special functions and in the theory of matrix-valued orthogonal polynomials, see e.g., [16–23] and the references therein.

Motivated by this great importance of these functions, their investigations and generalizations to the degenerate framework have been widely considered in the literature, for instance, [24–27].

In this section, we present some basic properties and well-known results on a degenerate gamma function which we need in this work. In Section 2, we introduce a degenerate Euler zeta function and discuss its region of convergence, integral representation, and infinite series representation. In Section 3, we define a degenerate digamma function along with its region of convergence and integral representation. We also give certain recurrence relations and formulae satisfied by the degenerate digamma function. In Section 4, we define a degenerate polygamma function and describe its convergence conditions. Some recurrence relations satisfied by the degenerate polygamma function are also given here. Finally, in Section 5, the hypergeometric functions are expressed in terms of the degenerate digamma function.

In [26], a degenerate gamma function, denoted Γ_λ^*, has been defined by

$$\Gamma_\lambda^*(z) = \int_0^\infty (1+\lambda)^{-t/\lambda} t^{z-1} \mathrm{d}t, \quad \lambda \in (0,1), \mathrm{Re}(z) > 0. \tag{1}$$

The basic results of this function, given in [26], can be summarized in the following lemma.

Lemma 1. *Let $\lambda \in (0,1)$. Then, for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0, \Gamma_\lambda^*(z)$ satisfies*

$$\Gamma_\lambda^*(z+1) = \frac{\lambda z}{\log(1+\lambda)}\Gamma_\lambda^*(z),$$
$$\Gamma_\lambda^*(1) = \frac{\lambda}{\log(1+\lambda)}, \tag{2}$$

$$\Gamma_\lambda^*(z+1) = \frac{\lambda^{k+1} z(z-1)\cdots(z-k)}{(\log(1+\lambda))^{k+1}}\Gamma_\lambda^*(z-k), \quad k \geq 0, \tag{3}$$

$$\Gamma_\lambda^*(k+1) = \frac{\lambda^{k+1}k!}{(\log(1+\lambda))^{k+1}}, \quad k \in \mathbb{N}. \tag{4}$$

Also, we can easily show that

Corollary 1. *Let $\lambda \in (0,1)$. Then, $\Gamma_\lambda^*(z)$ satisfies*

$$\Gamma_\lambda^*(z) = \left[\frac{\lambda}{\log(1+\lambda)}\right]^z \Gamma(z), \quad z \in \mathbb{C}, \mathrm{Re}(z) > 0, \tag{5}$$

where $\Gamma(z)$ is the gamma function. Moreover, for $m, n \in \mathcal{N}$, we have

$$\Gamma_\lambda^*(m)\Gamma_\lambda^*(n) = B(m,n)\Gamma_\lambda^*(m+n), \tag{6}$$

where $B(.,.)$ is the beta function.

2. Degenerate Euler Zeta Function

The Euler zeta function in two complex variables s, z such that $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(z) > 0$ is defined by (see [12, 24])

$$\zeta_E(s,z) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+z)^s}. \tag{7}$$

An integral representation of $\zeta_E(s,z)$ is given as

$$\zeta_E(s,z) = \Gamma^{-1}(s)\int_0^\infty F(-t,z)t^{s-1}\mathrm{d}t, \tag{8}$$

where

$$F(t,z) = \frac{2e^{zt}}{1+e^t} = \sum_{n=0}^{\infty} E_n(z)\frac{t^n}{n!}, \tag{9}$$

where $E_n(z)$ is the Euler polynomial of degree n. When $z = 0$, $E_n = E_n(0)$ are Euler numbers (see, [12, 14]). Kim in [14] obtained that $\zeta_n(-n,z) = E_n(z), n \geq 0$.

In this section, we consider a degenerate analogue of the Euler zeta function which is given as

$$\zeta_{E_\lambda}(s,z) = \Gamma^{-1}(s)\int_0^\infty F_\lambda(-t,z)t^{s-1}\mathrm{d}t, \tag{10}$$

where $\lambda \in (0,1), s, z \in \mathbb{C}$ with $\mathrm{Re}(s) > 0, \mathrm{Re}(z) > 0$, and

$$F_\lambda(t,z) = \frac{2}{1+(1+\lambda)^{t/\lambda}}(1+\lambda)^{zt/\lambda} = \sum_{n=0}^{\infty} \mathscr{E}_n^\lambda(z)\frac{t^n}{n!}. \tag{11}$$

By (9) and (11), it follows that

$$\mathscr{E}_n^\lambda(z) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^n E_n(z), \tag{12}$$

which is the degenerate Euler polynomial of degree n.

From (10) and (11), we obtain that

$$\begin{aligned}
\Gamma^{-1}(s)\int_0^\infty F_\lambda(-t,z)t^{s-1}\mathrm{d}t &= \Gamma^{-1}(s)\int_0^\infty 2\sum_{m=0}^{\infty}(-1)^m(1+\lambda)^{-(m+z)t/\lambda}t^{s-1}\mathrm{d}t \\
&= 2\Gamma^{-1}(s)\sum_{m=0}^{\infty}(-1)^m\int_0^\infty (1+\lambda)^{-\tau/\lambda}\frac{\tau^{s-1}}{(m+z)^s}\mathrm{d}\tau \\
&= 2\frac{\Gamma_\lambda^*(s)}{\Gamma(s)}\sum_{m=0}^{\infty}(-1)^m\frac{1}{(m+z)^s}.
\end{aligned} \tag{13}$$

Thus, using (10) and (13), we conclude the following result.

Theorem 1. *For $s, z \in \mathbb{C}$ with $\mathrm{Re}(s) > 0, \mathrm{Re}(z) > 0$, and $\lambda \in (0,1)$, the degenerate Euler zeta function $\zeta_{E_\lambda}(s,z)$ defined in (10) has the following infinite series representation:*

$$\zeta_{E_\lambda}(s,z) = 2\frac{\Gamma_\lambda^*(s)}{\Gamma(s)}\sum_{m=0}^{\infty}(-1)^m\frac{1}{(m+z)^s}. \tag{14}$$

Moreover, in view of (5), we have

$$\zeta_{E_\lambda}(s,z) = \zeta_E(s,z)\left(\tfrac{\lambda}{\ln(1+\lambda)}\right)^s, \tag{15}$$

where $\zeta_E(s,z)$ is the Euler zeta function defined by (7).

Furthermore, from (11), it follows

$$\Gamma^{-1}(s)\int_0^\infty F_\lambda(-t,z)t^{s-1}\mathrm{d}t = \Gamma^{-1}(s)\sum_{m=0}^{\infty}\mathscr{E}_n^\lambda(z)\frac{(-1)^m}{m!}\int_0^\infty t^{s+m-1}\mathrm{d}t. \tag{16}$$

Hence, we obtain the following results.

Theorem 2. *For $s, z \in \mathbb{C}$ with $\operatorname{Re}(s) > 0, \operatorname{Re}(z) > 0$, and $\lambda \in (0,1)$, the degenerate Euler zeta function $\zeta_{E_\lambda}(s,z)$, defined in (10), satisfies*

$$\zeta_{E_\lambda}(s,z) = \Gamma^{-1}(s) \sum_{m=0}^{\infty} \mathscr{E}_n^{\lambda}(z) \frac{(-1)^m}{m!} \int_0^{\infty} t^{s+m-1} \mathrm{d}t. \tag{17}$$

And for $n \in \mathbb{N} \cup \{0\}$,

$$\zeta_{E_\lambda}(-n,z) = \frac{2\pi i (-1)^n}{n!\Gamma(-n)} \mathscr{E}_n^{\lambda} = \mathscr{E}_n^{\lambda}(z). \tag{18}$$

Remark 1. Note that $\zeta_{E_\lambda}(s,z)$ is an entire function in the complex s–plane.

Remark 2.

$$\lim_{\lambda \longrightarrow 0} \zeta_{E_\lambda}(-n,z) = E_n(z) = \zeta_E(-n,z). \tag{19}$$

3. Degenerate Digamma Function

The digamma function, denoted by $\psi(z)$, is the logarithmic derivative of the gamma function given by [6, 16, 28]:

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{20}$$

In this section, we define a degenerate digamma function as follows:

$$\psi_\lambda^*(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma_\lambda^*(z) = \frac{\Gamma_\lambda^{*'}(z)}{\Gamma_\lambda^*(z)}, \tag{21}$$

where $\Gamma_\lambda^*(z)$ is the degenerate gamma function defined by (1). Now, we are going to obtain certain functional equations involving the degenerate digamma function $\psi_\lambda^*(z)$. Using (2) and (21), it follows that

$$\begin{aligned}\psi_\lambda^*(z+1) &= \frac{\Gamma_\lambda^{*'}(z+1)}{\Gamma_\lambda^*(z+1)} = \frac{(z\Gamma_\lambda^*(z))'}{z\Gamma_\lambda^*(z)} \\ &= \frac{\Gamma_\lambda^{*'}(z)}{\Gamma_\lambda^*(z)} + \frac{1}{z} = \psi_\lambda^*(z) + \frac{1}{z},\end{aligned} \tag{22}$$

$\operatorname{Re}(z) > 0.$

Generally, we have the following.

Theorem 3. *For $n \in \mathbb{N}$, $z \in \mathbb{C}$, and $\operatorname{Re}(z) > 0$, we have*

$$\psi_\lambda^*(z+n) = \psi_\lambda^*(z) + \sum_{m=0}^{n-1} \frac{1}{z+m}. \tag{23}$$

Furthermore, using relation (5), we find that

$$\psi_\lambda^*(z) = \psi(z) + \log\left(\frac{\lambda}{\log(1+\lambda)}\right), \tag{24}$$

where ψ is the digamma function defined by (20). According to Batir [28], we have

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \frac{z-1}{(n+1)(n+z)}, \tag{25}$$

where

$$\gamma = \lim_{n\longrightarrow\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) = -0.577215 \tag{26}$$

is the Euler–Mascheroni constant. Hence, substituting (25) into (24), one gets the following.

Theorem 4. *For $z \in \mathbb{C}$, $\operatorname{Re}(z) > 0$, and $\lambda \in (0,1)$,*

$$\psi_\lambda^*(z) = \log\left(\frac{\lambda}{\log(1+\lambda)}\right) - \lim_{n\longrightarrow\infty}\left[\log n - \sum_{j=0}^{n} \frac{1}{z+j}\right], \tag{27}$$

$$\psi_\lambda^*(z) = \log\left(\frac{\lambda}{\log(1+\lambda)}\right) - \gamma + (z-1)\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+z)}, \tag{28}$$

$$\psi_\lambda^*(z+1) = \log\left(\frac{\lambda}{\log(1+\lambda)}\right) - \gamma + z\sum_{n=1}^{\infty} \frac{1}{n(n+z)}. \tag{29}$$

Next, the degenerate digamma function $\psi_\lambda^*(z)$ defined by (21) can be expressed as a series expression in terms of Riemann's zeta function. Using

$$(n+z)^{-1} = n^{-1} \sum_{m=0}^{\infty} \left(\frac{-z}{n}\right)^m, \tag{30}$$

equation (29) can be rewritten as

$$\psi_\lambda^*(z+1) = \log\left(\frac{\lambda}{\log(1+\lambda)}\right) - \gamma - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^{-(m+1)}(-z)^m. \tag{31}$$

Thus, one gets the following.

Theorem 5. *For $z \in \mathbb{C}, \operatorname{Re}(z) > 0$, and $\lambda \in (0,1)$,*

$$\psi_\lambda^*(z+1) = \log\left(\frac{\lambda}{\log(1+\lambda)}\right) - \gamma - \sum_{m=1}^{\infty} \zeta(m+1)(-z)^m. \tag{32}$$

Note that these series converge absolutely for $|z| < 1$. Using the Legendre duplication formula [29]

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{33}$$

and (5), one can simply find

$$\Gamma_\lambda^*\left(\frac{1}{2}\right)\Gamma_\lambda^*(2z) = 2^{2z-1}\Gamma_\lambda^*(z)\Gamma_\lambda^*\left(z+\frac{1}{2}\right), \tag{34}$$

$$\psi_\lambda^*(2z) = \log 2 + \frac{1}{2}\psi_\lambda^*(z) + \frac{1}{2}\psi_\lambda^*\left(z+\frac{1}{2}\right), \quad \operatorname{Re}(z) > 0. \tag{35}$$

Equation (35) can be extended to an arbitrary integral multiplication of z as follows.

Theorem 6. *For $z \in \mathbb{C}, \operatorname{Re}(z) > 0$, and $\lambda \in (0,1)$,*

$$\psi_\lambda^*(mz) = \log m + \frac{1}{m}\sum_{j=1}^{m}\psi_\lambda^*\left(z + \frac{j-1}{m}\right), \quad \operatorname{Re}(z) > 0. \tag{36}$$

Figures 1–3 illustrate the degenerate digamma function $\psi_\lambda^*(z)$ in (24) at different values for $\lambda \in (0,1)$.

Remark 3. Its worth to mention here that all plotted functions in the below figures were multiplied by $\sin x$, since Fourier space, for the sake of clarify the results to the reader.

Now, we are going to find the integral representations for the degenerate digamma function $\psi_\lambda^*(z)$, defined by (21), as follows. Note that

$$\begin{aligned}\int_0^1 \left(1 - t^{z-1}\right)(1-t)^{-1}\mathrm{d}t &= \sum_{n=0}^{\infty}\int_0^1 \left(1 - t^{z-1}\right)t^n \mathrm{d}t \\ &= (z-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+z)}.\end{aligned} \tag{37}$$

Hence, using (28) and (37), it can be shown that

$$\psi_\lambda^*(z) = -\gamma + \log\left(\frac{\lambda}{\log(1+\lambda)}\right) + \int_0^1 \left(1 - t^{z-1}\right)(1-t)^{-1}\mathrm{d}t. \tag{38}$$

Now, substituting $t = (1+\lambda)^{-s/\lambda}$ in (37) gives

$$\begin{aligned}\psi_\lambda^*(z) &= -\gamma + \log\left(\frac{\lambda}{\log(1+\lambda)}\right) + \frac{\log(1+\lambda)}{\lambda} \\ &\quad \times \int_0^\infty \left[(1+\lambda)^{-t/\lambda} - (1+\lambda)^{-zt/\lambda}\right]\left[1 - (1+\lambda)^{-t/\lambda}\right]^{-1}\mathrm{d}t.\end{aligned} \tag{39}$$

Since

$$z^{-1} = \frac{\log(1+\lambda)}{\lambda}\int_0^\infty (1+\lambda)^{-zt/\lambda}\mathrm{d}t \tag{40}$$

and by integrating from 1 to n, it follows that

$$\begin{aligned}\log n &= \int_0^\infty \int_1^n (1+\lambda)^{-zt/\lambda}\cdot \log(1+\lambda)^{1/\lambda}\mathrm{d}z\mathrm{d}t \\ &= \int_0^\infty \int_1^n \frac{1}{t}\mathrm{d}_z(1+\lambda)^{-zt/\lambda}\mathrm{d}t \\ &= \int_0^\infty \frac{1}{t}\left[(1+\lambda)^{-t/\lambda} - (1+\lambda)^{-nt/\lambda}\right]\mathrm{d}t.\end{aligned} \tag{41}$$

Inserting (41) and

$$(z+j)^{-1} = \frac{\log(1+\lambda)}{\lambda}\int_0^\infty (1+\lambda)^{-(z+j)t/\lambda}\mathrm{d}t \tag{42}$$

in (27), we get

$$\begin{aligned}\psi_\lambda^*(z) &= \log\left(\frac{\lambda}{\log(1+\lambda)}\right) + \lim_{n\to\infty}\int_0^\infty \left[\left((1+\lambda)^{-t/\lambda} - (1+\lambda)^{-nt/\lambda}\right)\frac{1}{t} - \sum_{j=0}^{n}\frac{\log(1+\lambda)}{\lambda}(1+\lambda)^{-(z+j)t/\lambda}\right]\mathrm{d}t \\ &= \log\left(\frac{\lambda}{\log(1+\lambda)}\right) + \lim_{n\to\infty}\int_0^\infty \left\{(1+\lambda)^{-\frac{t}{\lambda}}t^{-1} - \frac{\log(1+\lambda)}{\lambda}(1+\lambda)^{-\frac{zt}{\lambda}}\left[1 - (1+\lambda)^{-\frac{t}{\lambda}}\right]^{-1}\right\}\mathrm{d}t \\ &\quad - \lim_{n\to\infty}\int_0^\infty (1+\lambda)^{-\frac{nt}{\lambda}}\left\{t^{-1} - \frac{\log(1+\lambda)}{\lambda}(1+\lambda)^{-\frac{zt}{\lambda}}\left[1 - (1+\lambda)^{-\frac{t}{\lambda}}\right]^{-1}\right\}\mathrm{d}t.\end{aligned} \tag{43}$$

Since the last limit equals to zero, it follows

$$\begin{aligned}\psi_\lambda^*(z) &= \log\left(\frac{\lambda}{\log(1+\lambda)}\right) + \int_0^\infty \left[\frac{1}{t}(1+\lambda)^{-t/\lambda}\right. \\ &\quad \left. - \frac{\log(1+\lambda)}{\lambda}\left(1 - (1+\lambda)^{-t/\lambda}\right)^{-1}(1+\lambda)^{-zt/\lambda}\right]\mathrm{d}t.\end{aligned} \tag{44}$$

The following theorem summarizes the above results.

Theorem 7. *For $z \in \mathbb{C}$, $\operatorname{Re}(z) > 0$, and $\lambda \in (0,1)$, the degenerate digamma function $\psi_\lambda^*(z)$, defined by (10), can be expressed as (38), (39) as well as (44).*

4. Degenerate Polygamma Function

The polygamma function of order m is obtained by taking the $(m+1)$th derivative of the logarithm of gamma function (cf. [28]). Thus,

$$\psi^{(m)}(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m}\psi(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}\log\Gamma(z), \quad \operatorname{Re}(z) > 0. \tag{45}$$

In this section, we define the degenerate polygamma function of order m as

$$\psi_\lambda^{*(m)}(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m}\psi_\lambda^*(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}\log\Gamma_\lambda^*(z), \quad \operatorname{Re}(z) > 0, \tag{46}$$

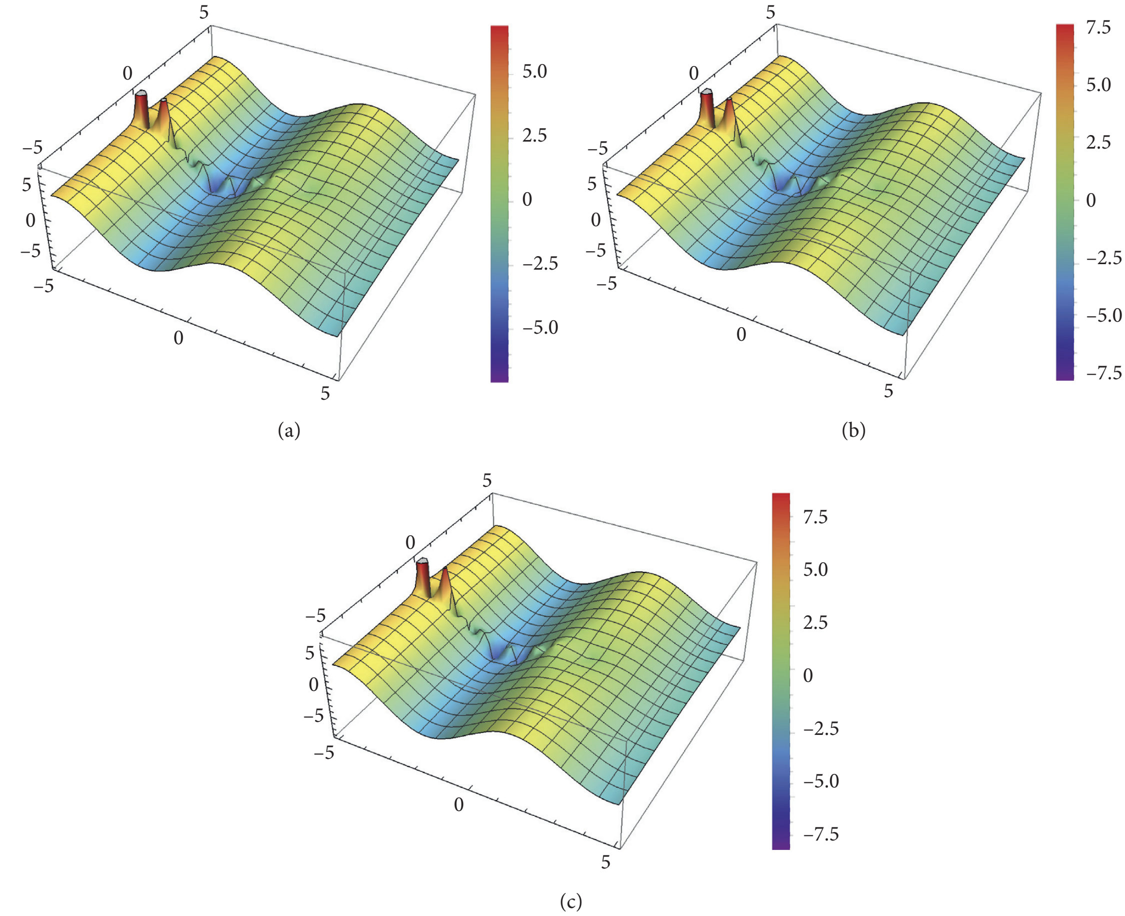


Figure 1: Absolute plots of the degenerate digamma function. (a) $\lambda = 0.1$. (b) $\lambda = 0.5$. (c) $\lambda = 1.0$.

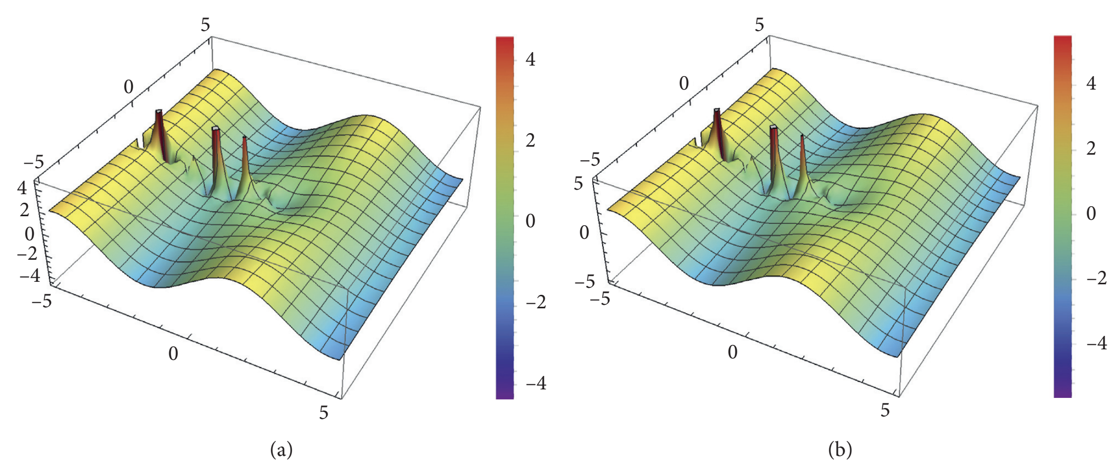
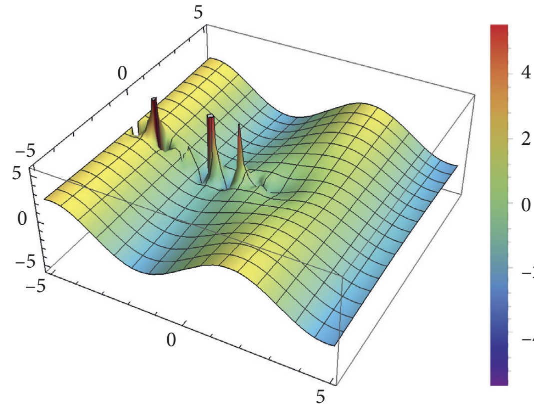
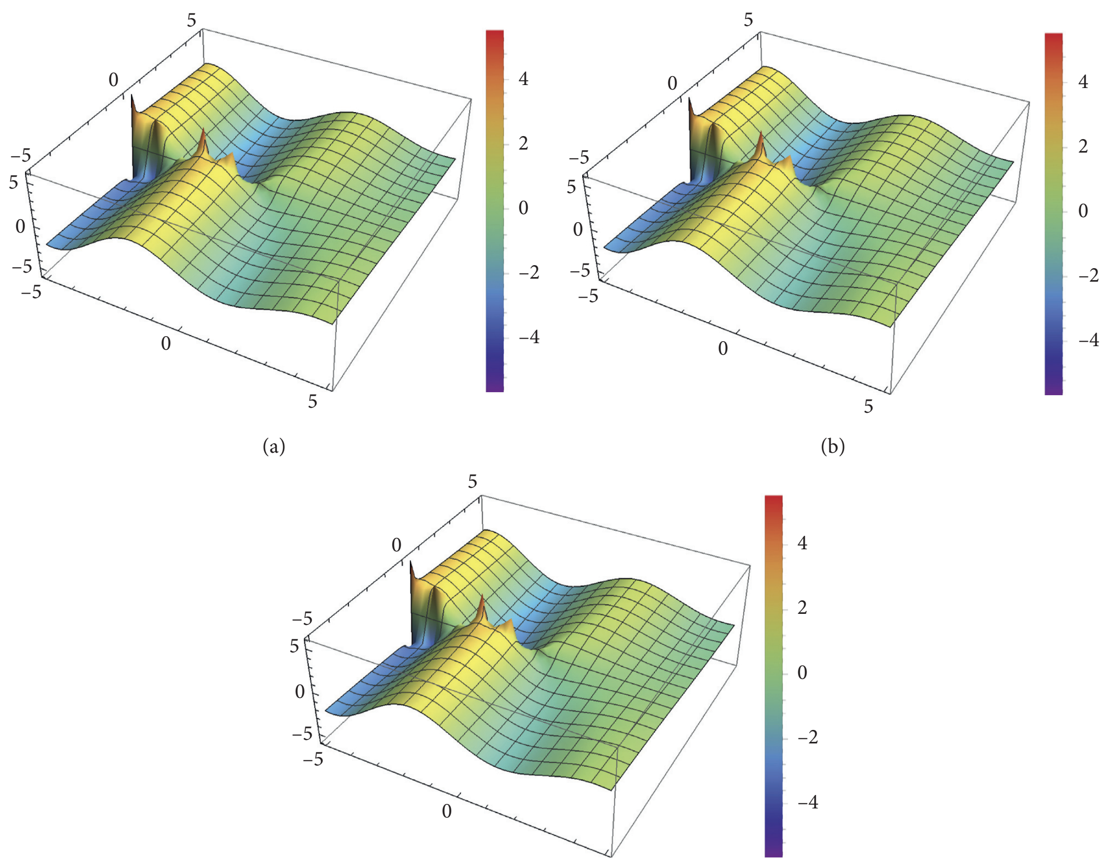


Figure 2: Continued.



(c)

Figure 2: Real-part plots of the degenerate digamma function. (a) $\lambda = 0.1$. (b) $\lambda = 0.5$. (c) $\lambda = 1.0$.



(a) (b) (c)

Figure 3: Imagery-part plots of the degenerate digamma function. (a) $\lambda = 0.1$. (b) $\lambda = 0.5$. (c) $\lambda = 1.0$.

where $\Gamma_\lambda^*(z)$ is the degenerate gamma function defined by (1) and $\psi_\lambda^*(z)$ is the degenerate digamma function defined by (21).

By (24), it follows that

$$\psi_\lambda^{*\,(m)}(z) = \psi^{(m)}(z), \quad \operatorname{Re}(z) > 0. \tag{47}$$

Using (44), an integral representation for $\psi_\lambda^{*\,(m)}(z)$, given in the next theorem, can be obtained.

Theorem 8. *Let $\lambda \in (0,1)$ and $m \in \mathbb{N}$. Then, for $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$, the degenerate polygamma function $\psi_\lambda^{*\,(m)}(z)$, defined by (46), can be expressed as*

$$\psi_\lambda^{*\,(m)}(z) = (-1)^m\left(\frac{\log(1+\lambda)}{\lambda}\right)^{m+1} \times \int_0^\infty t^m\left[1-(1+\lambda)^{-t/\lambda}\right]^{-1}(1+\lambda)^{-zt/\lambda}\mathrm{d}t. \tag{48}$$

The following recurrence relations for the degenerate polygamma function $\psi_\lambda^{*\,(m)}(z)$ defined by (47) can be obtained from (22)–(24), (35), and (36) as the following.

Theorem 9. *For $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$, $\lambda \in (0,1)$, and $m \in \mathbb{N}$, the recurrence relations hold true:*

$$\begin{aligned}
\psi_\lambda^{*\,(m)}(z+1) &= \psi_\lambda^{*\,(m)}(z) + \frac{(-1)^m\Gamma(m+1)}{z^{m+1}},\\
\psi_\lambda^{*\,(m)}(1-z) &= (-1)^m\psi_\lambda^{*\,(m)}(z) + (-1)^m\pi\left(\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^m\cot(\pi z),\\
\psi_\lambda^{*\,(m)}(z+n) &= \psi_\lambda^{*\,(m)}(z) + \sum_{k=0}^{n-1}\frac{(-1)^m\Gamma(m+1)}{(z+k)^{m+1}},\\
\psi_\lambda^{*\,(m)}(2z) &= \frac{1}{4}\psi_\lambda^{*\,(m)}(z) + \frac{1}{4}\psi_\lambda^{*\,(m)}\left(z+\frac{1}{2}\right),\\
\psi_\lambda^{*\,(m)}(nz) &= \frac{1}{n^{m+1}}\sum_{k=1}^{n}\psi_\lambda^{*\,(m)}\left(z+\frac{k-1}{n}\right), \quad \mathrm{Re}(z)>0.
\end{aligned} \tag{49}$$

From (25), a series representation of the degenerate polygamma function $\psi_\lambda^{*\,(m)}(z)$ is given in the following result.

Theorem 10. *For $z \in \mathbb{C}, \mathrm{Re}(z) > 0, \lambda \in (0,1)$, and $m \in \mathbb{N}$, we have*

$$\psi_\lambda^{*\,(m)}(z) = (-1)^{m+1}\Gamma(m+1)\sum_{n=0}^{\infty}\frac{1}{(z+n)^{m+1}}. \tag{50}$$

Remark 4. The degenerate polygamma function $\psi_\lambda^{*\,(m)}(z)$ can be expressed in terms of the generalized zeta function

$$\zeta(m,z) = \sum_{n=0}^{\infty}(z+n)^{-m} \tag{51}$$

as

$$\psi_\lambda^{*\,(m)}(z) = (-1)^m\Gamma(m+1)\zeta(m+1,z). \tag{52}$$

Finally, using (32), a series representation in terms of the Riemann zeta function can be obtained, see the following result.

Theorem 11. *For $z \in \mathbb{C}, \mathrm{Re}(z) > 0, \lambda \in (0,1)$, and $m \in \mathbb{N}$, we have*

$$\psi_\lambda^{*\,(m)}(z+1) = \sum_{n=0}^{\infty}(-1)^{m+n+1}\Gamma(m+n+1)\zeta(m+n+1)\frac{z^n}{n!}, \quad m,n \in \mathbb{N}. \tag{53}$$

5. Applications

Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and $n \in \mathbb{N}$. Then, it can be verified that

$${}_3F_2\left[\begin{matrix}(-n+2), z+1, 1\\ z+(n+1), 2\end{matrix}; 1\right] = \frac{z+n}{z(-n+1)} \times \left({}_2F_1\left[\begin{matrix}(-n+1), z\\ z+1\end{matrix}; 1\right] - 1\right). \tag{54}$$

Now, we can directly use the integral transform of Gauss hypergeometric function (see [29]) and the formulae:

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \quad \mathrm{Re}(z) > 0, \tag{55}$$

$$2F_1\left[\begin{matrix}(-n+2), z\\ z+1\end{matrix}; 1\right] = 2^{-z}\frac{\Gamma(z+n)\Gamma(n-(1/2))}{\Gamma((z/2)+n)\Gamma((z/2)+(n-(1/2)))}, \quad \mathrm{Re}(z) > 0. \tag{56}$$

Using (54) in (56) and L'Hôpital rule for complex numbers with applying equation (24) yields the following identity in terms of the degenerate digamma function:

$${}_3F_2\left[\begin{matrix}(-n+2), z+1, 1\\ z+(n+1), 2\end{matrix}; 1\right] = \frac{z+1}{z} \times \left[\psi_\lambda^*\left(\frac{1}{2}\right) + \psi_\lambda^*(z+1) - \psi_\lambda^*\left(\frac{1}{2}(z+1)\right) - \psi_\lambda^*\left(\frac{1}{2}z+1\right)\right], \quad \mathrm{Re}(z) > 0. \tag{57}$$

Similarly, we can present another identity involving hypergeometric function in terms of our degenerate digamma function in the following form:

$$4F_3\left[\begin{matrix}1,1,1,-n\\ 2,2,z+1\end{matrix};1\right]\times\left[\left(\psi_\lambda^*(n+2)-\log\left(\frac{\lambda}{\log(1+\lambda)}\right)\right)\left(\psi_\lambda^*(z+n+1)-\psi_\lambda^*(z)\right)\right.$$
$$\left.-\sum_{s=1}^{n}\frac{\psi_\lambda^*(s+1)-\log(\lambda/\log(1+\lambda))}{z+1}\right],\quad \mathrm{Re}(z)>0. \tag{58}$$

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly to writing this article. All the authors read and approved the final manuscript.

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