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Research Article

Hesitant Fuzzy Generalised Bonferroni Mean Operators Based on Archimedean Copula for Multiple-Attribute Decision-Making

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Information fusion is an important part of multiple-attribute decision-making, and aggregation operator is an important tool of decision information fusion. Integration operators in a variety of fuzzy information environments have a slight lack of consideration for the correlation between variables. Archimedean copula provides information fusion patterns that rely on the intrinsic relevance of information. This paper extends the Archimedean copula to the aggregation of hesitant fuzzy information. Firstly, the Archimedean copula is used to generate the operation rules of the hesitant fuzzy elements. Secondly, the hesitant fuzzy copula Bonferroni mean operator and hesitant fuzzy weighted copula Bonferroni mean operator are propounded, and several properties are proved in detail. Furthermore, a decision-making method based on the operators is proposed, and the specific decision steps are given. Finally, an example is presented to illustrate the practical advantages of the method, and the sensitivity analysis of the decision results with the change of parameters is carried out.

1. Introduction

In the practical problem of multiple-attribute decision-making (MADM), the decision maker (DM) vacillates between several values on the evaluation of the alternative. So traditional fuzzy sets show limitations in describing this uncertain information. As an important generalisation of fuzzy set [1], hesitant fuzzy set (HFS) is proposed by Torra [2, 3]. It allows DM to give multiple possible values flexibility, which is better to solve the indecision of DM and difficulty of reaching consensus. As a new information description tool for uncertain decision-making, the theory and its application in decision-making have attracted the attention of scholars at home and abroad. Xu and Xia [4, 5] proposed the mathematical form of the HFS and studied the hesitant fuzzy (HF) integration operator, similarity measure, distance of the hesitant fuzzy elements (HFEs), and so on. The following researchers further studied the theory of HF and applied it to the MADM problem [6–11].

The aggregation operator is the basis of many decision methods in the MADM problems. Therefore, the study of aggregation operator is particularly important under the HF environment. Common aggregation operators include weighted average (WA) operator, weighted geometric (WG) operator, power average (PA) operator, power geometric (PG) operator, Choquet integral operator, Bonferroni mean (BM) operator, and so on. The researchers improved these operators and applied them to the aggregation of HF information. Xia and Xu [4] presented eight HF aggregation operators, such as hesitant fuzzy weighted averaging (HFWA) operator and hesitant fuzzy weighted geometric (HFWG) operator. Zhang [12] propounded ten operators like hesitant fuzzy power average (HFPA) operator and hesitant fuzzy power geometric (HFPG) operator. Wei [13] defined hesitant fuzzy Choquet ordered averaging (GHFCOA) operator and hesitant fuzzy Choquet ordered geometric (HFCOG) operator to solve the MADM problem with HF information. Zhu [14] developed hesitant fuzzy

Bonferroni mean (HFBM) operator and weighted hesitant fuzzy Bonferroni mean (WHFBM) operator, which can deal with the MADM problem well. He [15] combined the power average operator with the Bonferroni mean in hesitant fuzzy environments and developed HFPBM and HFPGBM for hesitant fuzzy multiple-attribute group decision-making.

Each operator has its own characteristics, which can solve the corresponding problems well. These operators are basically derived from the t-norms and t-conorms. Most of the above operators only consider the influence of the location of data, but the correlation between attributes is slightly deficient. Because many decision problems do not satisfy the independence principle, there is some connection between attributes. For example, a factory needs to purchase a kind of equipment according to the four attributes of product price, technology, after-sales service level, and supplier reputation. Price may depend upon technology and supplier reputation. The after-sales service level may influence the supplier reputation. On the basis of t-norms and s-norms, the correlation between variables has gradually become an important aspect of consideration in MADM.

The copulas and co-copulas [16–18] are the generalisations of various t-norms and t-conorms. At first, copulas and co-copulas were widely used in the fields of finance and insurance, etc. In recent years, the application of copulas and co-copulas in decision problems has widely been concerned by scholars. This is mainly based on the following points: (1) Copulas and co-copulas can reveal the dependence among attributes; (2) Copulas and co-copulas can prevent information losing in the midst of aggregation; and (3) Copulas and co-copulas are flexible because DMs can select different types of copulas and co-copulas to define the operations under fuzzy environment, and the results obtained from these operations are closed. Archimedean copulas (ACs) and co-copulas [19] have the advantages of symmetry, associability, and simple operation. On the basis of AC, Tao [20] studied a new computational model for unbalanced linguistic variables. Chen [21] defined new aggregation operators in linguistic neutrosophic set based on copulas and applied them to settle MCDM problems.

So we propose the more general and flexible aggregation operators on the basis of BM operator and AC to solve MADM problem. Hesitant fuzzy copula Bonferroni mean (HFCBM) operator and hesitant fuzzy weight copula Bonferroni mean (HFWCBM) operator can not only reflect the interrelation among attributes but also have more flexible and diversified forms. They can solve the MADM problem well and provide a preference choice for DM. For the above goals, the structure of this work is arranged as follows: some notions on HFS, BM, and AC are reviewed firstly in Section 2. The HFCBM and HFWCBM are given in Section 3, and different forms of aggregation operators are investigated based on different AC functions. In Section 4, the MADM approach based on the propounded operators is constructed, case analysis will be carried out, some comparisons with existing approaches under the HF environment, and merits of the proposed MADM approach based on HFCBM operators are analysed. The conclusion is obtained in Section 5.

2. Preliminaries

In this section, several basis knowledge such as HFS, BM operator, HFBM operator, and AC is succinctly retrospected.

2.1. Hesitant Fuzzy Sets

Definition 1 (see [2]). Let S be a finite reference set. A hesitant fuzzy set G on S is in terms of a function when applied to S returning a subset of $[0,1]$ denoted by

$$G=\{\langle s, g(h)\rangle \mid \forall s\in S\}, \tag{1}$$

where $g(h)$ is a collection of numbers h_i from $[0,1]$, indicating the possible membership degrees of $\forall s\in S$ to G. We call $g(h)$ a hesitant fuzzy element (HFE) and G the set of all HFEs.

To compare the HFEs, the comparison laws are defined as follows.

Definition 2 (see [4]). For a HFE $g(h)=\cup_{i=1}^{\text{è}g}\{h_i\}$, $\pi(g)=1/\text{è}g\sum_{i=1}^{\text{è}g}h_i$ is called the score function of $g(h)$, where èg is the number of possible elements in $g(h)$.

For two HFEs $g_1(h)$ and $g_2(h)$,

(i) if $\pi(g_1)>\pi(g_2)$, then $g_1\succ g_2$

(ii) if $\pi(g_1)=\pi(g_2)$, then $g_1=g_2$

Definition 3 (see [4]). Let $g_1(h)=\cup_{m_1=1}^{\text{è}g_1}\{h_{1m_1}\}$, $g_2(h)=\cup_{m_2=1}^{\text{è}g_2}\{h_{2m_2}\}$, and $g(h)=\cup_{i=1}^{\text{è}g}\{h_i\}$ be three HFEs, $\rho\geq 0$, and the novel operational rules of HFEs are given as follows:

$$\begin{aligned}
g_1\oplus g_2 &= \bigcup_{h_{1m_1}\in g_1 h_{2m_2}\in g_2}\{h_{1m_1}+h_{2m_2}-h_{1m_1}h_{2m_2}\},\\
g_1\otimes g_2 &= \bigcup_{h_{1m_1}\in g_1 h_{2m_2}\in g_2}\{h_{1m_1}h_{2m_2}\},\\
\rho g &= \bigcup_{h_i\in g}\{1-(1-h_i)^{\rho}\mid i=1,2,\ldots,\text{è}g\},\\
g^{\rho} &= \bigcup_{h_i\in g}\{h_i^{\rho}\mid i=1,2,\ldots,\text{è}g\},
\end{aligned}\tag{2}$$

where $m_1=1,2,\ldots,\text{è}g_1$ and $\text{m}_2=1,2,\ldots,\text{è}g_2$.

2.2. BM Operator and HFBM Operator

Definition 4 (see [22]). Assume $a_k\,(k=1,2,\ldots,n)$ is a family of positive real number and $\mu,\nu\geq 0$, then the aggregated mapping

$$\text{BM}^{\mu,\nu}(a_1,a_2,\ldots,a_n)=\left(\frac{1}{n(n-1)}\sum_{\substack{k,j=1\\ k\neq j}}^{n}a_k^{\mu}a_j^{\nu}\right)^{1/\mu+\nu}. \tag{3}$$

is named BM operator.

Definition 5 (see [14]). Let $g_i(h) = \cup_{m_i=1}^{\grave{e}g_i}\{h_{im_i}|it = n1, 2q, h\ldots, xn\}$ and $\mu, \nu \geq 0$. Then the hesitant fuzzy Bonferroni mean (HFBM) operator is expressed as

$$\mathrm{HFBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \left(\frac{1}{n(n-1)}\sum_{\substack{k,j=1\\k\neq j}}^{n} g_k^{\mu}\otimes g_j^{\nu}\right)^{1/\mu+\nu} = \cup_{h_{im_i}\in g_i h_{jm_j}\in g_j}\left\{\left(1-\prod_{i,j=1i\neq j}^{n}\left(1-h_{im_i}^{\mu}h_{jm_j}^{\nu}\right)^{1/n(n-1)}\right)^{1/\mu+\nu}\right\}. \quad (4)$$

Definition 6 (see [14]). Let $g_i(h) = \cup_{m_i=1}^{\grave{e}g_i}\{h_{im_i}|i = 1, 2, \ldots, n\}$ and $\mu, \nu \geq 0$. Then the hesitant fuzzy weight Bonferroni mean (HFWBM) operator is expressed as

$$\begin{aligned}\mathrm{HFWBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) &= \left(\frac{1}{n(n-1)}\sum_{k,j=1k\neq j}^{n}(n\omega_k g_k)^{\mu}\otimes\left(n\omega_j g_j\right)^{\nu}\right)^{1/\mu+\nu}\\ &= \cup_{h_{im_i}\in g_i h_{jm_j}\in g_j}\left\{\left(1-\prod_{i,j=1i\neq j}^{n}\left(1-\left(1-\left(1-h_{im_i}\right)^{n\omega_i}\right)^{\mu}\left(1-\left(1-h_{jm_j}\right)^{n\omega_j}\right)^{\nu}\right)^{1/n(n-1)}\right)^{1/\mu+\nu}\right\}.\end{aligned} \quad (5)$$

2.3. Archimedean Copula

Definition 7 (see [16]). A two-dimensional mapping F: $[0,1]\times[0,1]\longrightarrow[0,1]$ is called a copula, if it contents the following properties:

(1) $\hat{o}(s, 1) = \hat{o}(1, s) = s$;

(2) $\hat{o}(s, 0) = \hat{o}(0, u) = 0$;

(3) $\hat{o}(s_1, t_1) - \hat{o}(s_1, t_2) - \hat{o}(s_2, t_1) + \hat{o}(s_2, t_2) \geq 0$.

Definition 8 (see [17]). A copula $\hat{o}$: $[0,1]\times[0,1]\longrightarrow[0,1]$ is said to be the Archimedean copula if there exists a continuous and strictly decreasing function ς from $[0,1]$ to $[0,\infty)$ with $\varsigma(1) = 0$, and function τ from $[0,\infty)$ to $[0,1]$ is defined by

$$\tau(r) = \begin{cases}\varsigma^{-1}(r), & r\in[0,\varsigma(0)];\\ 0, & r\in[\varsigma(0),+\infty).\end{cases} \quad (6)$$

such that for any $(t_1, t_2)\in[0,1]\times[0,1]$,

$$\mathsf{F}(t_1, t_2) = \tau(\varsigma(t_1) + \varsigma(t_2)). \quad (7)$$

The AC is propounded based on the following statements: (1) If $\hat{o}$ is strictly increasing on $[0,1]\times[0,1]$, then we can obtain $\varsigma(0) = +\infty$; (2) If τ accords ς^{-1} on $[0,+\infty)$. Then equation (7) is transformed as

$$\mathsf{F}(t_1, t_2) = \varsigma^{-1}(\varsigma(t_1) + \varsigma(t_2)), \quad (8)$$

where the functions ς and $\hat{o}$ are named strict generator and Archimedean copulas, respectively.

Definition 9 (see [18]). Suppose F: $[0,1]\times[0,1]\longrightarrow[0,1]$ be a copula, then co-copula is expressed as

$$\mathsf{F}^{*}(t_1, t_2) = 1 - \mathsf{F}(1-t_1, 1-t_2). \quad (9)$$

3. New Operators Based on AC and BM

3.1. Generalised Operations for HFEs. To deal with HF information, we propose a generalised version of operational rules based on AC.

Definition 10 (see [23]). Let $g_1(h) = \cup_{m_1=1}^{\grave{e}g_1}\{h_{1m_1}\}$, $g_2(h) = \cup_{m_2=1}^{\grave{e}g_2}\{h_{2m_2}\}$, and $g(h) = \cup_{i=1}^{\grave{e}g}\{h_i\}$ be three HFEs, $\rho\geq 0$, and the novel operational rules of HFEs are given as follows:

$$\begin{aligned} g_1\oplus g_2 &= \cup_{h_{1m_1}\in g_1 h_{2m_2}\in g_2}\left\{1-\varsigma^{-1}\left(\varsigma\left(1-h_{1m_1}\right)+\varsigma\left(1-h_{2m_2}\right)\right)\right\},\\ g_1\otimes g_2 &= \cup_{h_{1m_1}\in g_1 h_{2m_2}\in g_2}\left\{\varsigma^{-1}\left(\varsigma\left(h_{1m_1}\right)+\varsigma\left(h_{2m_2}\right)\right)\right\},\\ \rho g &= \cup_{h_i\in g}\left\{1-\varsigma^{-1}\left(\rho\varsigma(1-h_i)\right)|i = 1, 2, \ldots, \grave{e}g\right\},\\ g^{\rho} &= \cup_{h_i\in g}\left\{\varsigma^{-1}\left(\rho\varsigma(h_i)\right)|i = 1, 2, \ldots, \grave{e}g\right\},\end{aligned} \quad (10)$$

where $m_1 = 1, 2, \ldots, \grave{e}g_1$ and $\mathrm{m}_2 = 1, 2, \ldots, \grave{e}g_2$.

3.2. HFCBM Operator. Based on Definitions 5 and 10, the HFCBM operator can be proposed.

Theorem 1. *Let* $g_i(h) = \cup_{m_i=1}^{\grave{e}g_i}\{h_{im_i}|it = n1, 2q, h\ldots, xn\}$, *and* $\mu, \nu\geq 0$. *Then the hesitant fuzzy copula Bonferroni mean (HFCBM) operator is expressed as*

$$\mathrm{HFCBM}^{\mu,\nu}(g_1,g_2,\ldots,g_n)=\bigcup_{\substack{h_{im_i}\in g_i\\ h_{jm_j}\in g_j}}\left\{\varsigma^{-1}\left(\frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)+\nu\varsigma\left(h_{jm_j}\right)\right)\right)\right)\right)\right)\right)\right\}. \tag{11}$$

Proof. Since

$$\begin{aligned} g_i^{\mu} &= \bigcup_{h_{im_i}\in g_i}\left\{\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)\right)\mid i=1,2,\ldots,\grave{\mathrm{e}}\mathrm{g_i}\right\},\\ g_j^{\nu} &= \bigcup_{h_{jm_j}\in g_j}\left\{\varsigma^{-1}\left(\nu\varsigma\left(h_{jm_j}\right)\right)\mid j=1,2,\ldots,\grave{\mathrm{e}}\mathrm{g_j}\right\}. \end{aligned} \tag{12}$$

□

Then

$$\begin{aligned} g_i^{\mu}\otimes g_j^{\nu} &= \bigcup_{\substack{h_{im_i}\in g_i\\ h_{jm_j}\in g_j}}\left\{\varsigma^{-1}\left(\varsigma\left(\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)\right)\right)+\varsigma\left(\varsigma^{-1}\left(\nu\varsigma\left(h_{jm_j}\right)\right)\right)\right)\mid m_i=1,2,\ldots,\grave{\mathrm{e}}\mathrm{g_i},\mathrm{m_j}=1,2,\ldots,\grave{\mathrm{e}}\mathrm{g_j}\right\},\\ \sum_{\substack{i,j=1\\ i\neq j}}^{n} g_i^{\mu}\otimes g_j^{\nu} &= \bigcup_{\substack{h_{im_i}\in g_i\\ h_{jm_j}\in g_j}}\left\{1-\varsigma^{-1}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)+\nu\varsigma\left(h_{jm_j}\right)\right)\right)\right)\right\}. \end{aligned} \tag{13}$$

Furthermore

$$\frac{1}{n(n-1)}\sum_{i,j=1 i\neq j}^{n} g_i^{\mu}\otimes g_j^{\nu}=\bigcup_{h_{im_i}\in g_i h_{jm_j}\in g_j}\left\{1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\varsigma\left(\varsigma^{-1}\left(\sum_{i,j=1 i\neq j}^{n}\varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)+\nu\varsigma\left(h_{jm_j}\right)\right)\right)\right)\right)\right)\right\}. \tag{14}$$

Accordingly

$$\left(\frac{1}{n(n-1)}\sum_{i,j=1 i\neq j}^{n} g_i^{\mu}\otimes g_j^{\nu}\right)^{1/\mu+\nu}=\bigcup_{h_{im_i}\in g_i h_{jm_j}\in g_j}\left\{\varsigma^{-1}\left(\frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\left(\sum_{i,j=1 i\neq j}^{n}\varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)+\nu\varsigma\left(h_{jm_j}\right)\right)\right)\right)\right)\right)\right)\right\}. \tag{15}$$

Accordingly, we can attain the establishment of Theorem 1.

In what follows, several desired characteristics of HFCBM operator are proved.

Theorem 2. *Let* $g_i(h)=\cup_{m_i=1}^{\grave{e}g_i}\left\{h_{im_i}|it=n1,2q,h\ldots,xn\right\}$, *and* $\mu,\nu\geq 0$, *then*

(1) *(Idempotency) If* $g_1=g_2=\cdots g_n=\{h\}$, *then* $HFCBM^{\mu,\nu}(g_1,g_2,\ldots,g_n)=\{h\}$

(2) *(Monotonicity) Let* $g_i^*(h)=\cup_{m_i=1}^{\grave{e}g_i^*}\left\{h_{im_i}^*|i=1,2,\ldots,n\right\}$, *if* $h_{im_i}\leq h_{im_i}^*$, *so*

$$\mathrm{HFCBM}^{\mu,\nu}(g_1,g_2,\ldots,g_n)\leq\mathrm{HFCBM}^{\mu,\nu}(g_1^*,g_2^*,\ldots,g_n^*). \tag{16}$$

(3) *(Boudedness) If* $h^-=\min_{i=1,2,\ldots,n}\left\{h_{im_i}\right\}$ $,h^+=\max_{i=1,2,\ldots,n}\left\{h_{im_i}\right\}$,

$$h^{-} \leq \mathrm{HFCBM}^{\mu,\nu}\left(g_1, g_2, \ldots, g_n\right) \leq h^{+}. \tag{17}$$

Proof. (1) If $g_1 = g_2 = \cdots g_n = \{h\}$,

$$\begin{aligned}
&\varsigma^{-1}\left(\frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varsigma\left(1-\varsigma^{-1}\left((\mu+\nu)\varsigma(h)\right)\right)\right)\right)\right)\right)\\
&= \varsigma^{-1}\left(\frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\left(n(n-1)\varsigma\left(1-\varsigma^{-1}\left((\mu+\nu)\varsigma(h)\right)\right)\right)\right)\right)\right)\\
&= \varsigma^{-1}\left(\frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\varsigma\left(1-\varsigma^{-1}\left((\mu+\nu)\varsigma(h)\right)\right)\right)\right)\right)\\
&= \varsigma^{-1}\left(\frac{1}{\mu+\nu}(\mu+\nu)\varsigma(\alpha)\right) = \varsigma^{-1}\varsigma(h) = h.
\end{aligned} \tag{18}$$

Then

$$\mathrm{HFCBM}^{\mu,\nu}\left(g_1, g_2, \ldots, g_n\right) = \bigcup_{\substack{h_{im_i}\in g_i\\ h_{jm_j}\in g_j}} \varsigma^{-1}\left(\frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\left(\sum_{\substack{k,j=1\\ k\neq j}}^{n}\varsigma\left(1-\varsigma^{-1}\left((\mu+\nu)\varsigma(h)\right)\right)\right)\right)\right)\right) = \{h\}. \tag{19}$$

Accordingly, the idempotency of the HFCBM operator is proved.

(2) Since $h_{im_i} \leq h^{*}_{im_i}$, $h_{jm_j} \leq h^{*}_{ijm_j}$, $\varsigma(t)$ and $\varsigma^{-1}(t)$ are decreasing functions and $1-\varsigma(t)$ and $1-\varsigma^{-1}(t)$ are increasing functions. Then, we have

$$\begin{aligned}
\mu\varsigma\left(h_{im_i}\right) + \nu\varsigma\left(h_{jm_j}\right) &\geq \mu\varsigma\left(h^{*}_{im_i}\right) + \nu\varsigma\left(h^{*}_{jm_j}\right),\\
1-\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right) + \nu\varsigma\left(h_{jm_j}\right)\right) &\geq 1-\varsigma^{-1}\left(\mu\varsigma\left(h^{*}_{im_i}\right) + \nu\varsigma\left(h^{*}_{jm_j}\right)\right).
\end{aligned} \tag{20}$$

Then,

$$\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)+\nu\varsigma\left(h_{jm_j}\right)\right)\right) \leq \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h^{*}_{im_i}\right)+\nu\varsigma\left(h^{*}_{jm_j}\right)\right)\right),\\
&\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)+\nu\varsigma\left(h_{jm_j}\right)\right)\right) \leq \frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h^{*}_{im_i}\right)+\nu\varsigma\left(h^{*}_{jm_j}\right)\right)\right).
\end{aligned} \tag{21}$$

Furthermore,

$$\begin{aligned}
&1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)+\nu\varsigma\left(h_{jm_j}\right)\right)\right)\right) \leq 1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h^{*}_{im_i}\right)+\nu\varsigma\left(h^{*}_{jm_j}\right)\right)\right)\right),\\
&\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)+\nu\varsigma\left(h_{jm_j}\right)\right)\right)\right)\right) \geq \varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h^{*}_{im_i}\right)+\nu\varsigma\left(h^{*}_{jm_j}\right)\right)\right)\right)\right).
\end{aligned} \tag{22}$$

Moreover,

$$\begin{aligned}
&\frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)+\nu\varsigma\left(h_{jm_j}\right)\right)\right)\right)\right)\\
&\geq \frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h^{*}_{im_i}\right)+\nu\varsigma\left(h^{*}_{jm_j}\right)\right)\right)\right)\right).
\end{aligned} \tag{23}$$

Hence

$$\begin{aligned}
&\varsigma^{-1}\left(\frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h_{im_i}\right)+\nu\varsigma\left(h_{jm_j}\right)\right)\right)\right)\right)\right)\\
&\leq \varsigma^{-1}\left(\frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varsigma\left(1-\varsigma^{-1}\left(\mu\varsigma\left(h^{*}_{im_i}\right)+\nu\varsigma\left(h^{*}_{jm_j}\right)\right)\right)\right)\right)\right).
\end{aligned} \tag{24}$$

Accordingly, the monotonicity of the HFCBM operator is proved.

(3) Let $g^- = \{h^-\}$ and $g^+ = \{h^+\}$; according to the monotonicity, we can acquire

$$\begin{aligned}\mathrm{HFCBM}^{\mu,\nu}(g^-, g^-, \ldots, g^-) &\leq \mathrm{HFCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n)\\ &\leq \mathrm{HFCBM}^{\mu,\nu}(g^+, g^+, \ldots, g^+).\end{aligned} \tag{25}$$

According to idempotency, we can acquire

$$\begin{aligned}\mathrm{HFCBM}^{\mu,\nu}(g^-, g^-, \ldots, g^-) &= \{h^-\};\\ \mathrm{HFCBM}^{\mu,\nu}(g^+, g^+, \ldots, g^+) &= \{h^+\}.\end{aligned} \tag{26}$$

Accordingly, we can acquire $h^- \leq \mathrm{HFCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) \leq h^+$. □

3.3. HFWCBM Operator. To overcome the disadvantage of the HFCBM operator not considering attribute weight, the HFWCBM operator is given as follows.

Theorem 3. *Let $g_i(h) = \cup_{m_i=1}^{\grave{e}g_i}\{h_{im_i} | it = n1, 2q, h \ldots, xn\}$, and $\mu, \nu \geq 0$. Then the hesitant fuzzy weight copula Bonferroni mean (HFWCBM) operator is expressed as*

$$\mathrm{HFWCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \bigcup_{\substack{h_{im_i}\in g_i\\ h_{jm_j}\in g_j}} \varsigma^{-1}\left(\frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varsigma\left(1-\varsigma^{-1}(\zeta)\right)\right)\right)\right)\right), \tag{27}$$

where

$$\begin{aligned}\zeta =& \mu\varsigma\left(1-\varsigma^{-1}\left(n\omega_k\varsigma\left(1-h_{km_k}\right)\right)\right)\\ &+ \nu\varsigma\left(1-\varsigma^{-1}\left(n\omega_j\varsigma\left(1-h_{jm_j}\right)\right)\right).\end{aligned} \tag{28}$$

Proof. The proof of Theorem 3 is similar to Theorem 1, we can utilize the proof of $(n\omega_k g_k)^{\mu} \otimes (n\omega_j g_j)^{\nu}$ to replace $(\omega_k g_k)^{\mu} \otimes (\omega_j g_j)^{\nu}$ in equation (1).

Then

$$\begin{aligned}n\omega_k g_k &= 1-\varsigma^{-1}\left(n\omega_k\varsigma\left(1-h_{km_k}\right)\right),\\ n\omega_j g_j &= 1-\varsigma^{-1}\left(n\omega_j\varsigma\left(1-h_{jm_j}\right)\right),\\ (n\omega_k g_k)^{\mu} &= \varsigma^{-1}\left(\mu\varsigma\left(1-\varsigma^{-1}\left(n\omega_k\varsigma\left(1-h_{km_k}\right)\right)\right)\right),\\ (n\omega_j g_j)^{\nu} &= \varsigma^{-1}\left(\nu\varsigma\left(1-\varsigma^{-1}\left(n\omega_j\varsigma\left(1-h_{jm_j}\right)\right)\right)\right).\end{aligned} \tag{29}$$

Then

$$\begin{aligned}(n\omega_k g_k)^{\mu} \otimes (n\omega_j g_j)^{\nu} = \varsigma^{-1}\Big(&\mu\varsigma\left(1-\varsigma^{-1}\left(n\omega_k\varsigma\left(1-h_{km_k}\right)\right)\right)\\ &+ \nu\varsigma\left(1-\varsigma^{-1}\left(n\omega_j\varsigma\left(1-h_{jm_j}\right)\right)\right)\Big).\end{aligned} \tag{30}$$

Hence

$$\left(\frac{1}{n(n-1)}\sum_{\substack{k,j=1\\ k\neq j}}^{n}(n\omega_k g_k)^{\mu}\otimes(n\omega_j g_j)^{\nu}\right)^{1/\mu+\nu} = \bigcup_{\substack{h_{im_i}\in g_i\\ h_{jm_j}\in g_j}} \varsigma^{-1}\left(\frac{1}{\mu+\nu}\varsigma\left(1-\varsigma^{-1}\left(\frac{1}{n(n-1)}\left(\sum_{\substack{k,j=1\\ k\neq j}}^{n}\varsigma\left(1-\varsigma^{-1}(\zeta)\right)\right)\right)\right)\right), \tag{31}$$

where

$$\zeta = \mu\varsigma\left(1-\varsigma^{-1}\left(n\omega_k\varsigma\left(1-h_{km_k}\right)\right)\right)+\nu\varsigma\left(1-\varsigma^{-1}\left(n\omega_j\varsigma\left(1-h_{jm_j}\right)\right)\right). \quad (32)$$

So we can attain the establishment of Theorem 3. □

In what follows, several desired characteristics of HFWCBM operator are proved. □

Theorem 4. *Let* $g_i(h) = \cup_{m_i=1}^{\grave{e}g_i}\left\{h_{im_i} | it = n1, 2q, h\ldots, xn\right\}$, *and* $\mu, \nu \geq 0$, *then*

(1) *(Idempotency) If* $g_1 = g_2 = \cdots g_n = \{h\}$, *HFWCBM*$^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \{h\}$

(2) *(Monotonicity) Let* $g_i^*(h) = \cup_{m_i=1}^{\grave{e}g_i^*}\left\{h_{im_i}^* | it = n1, 2q, h\ldots, xn\right\}$; *if* $h_{im_i} \leq h_{im_i}^*$,

$$\mathrm{HFWCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) \leq \mathrm{HFWCBM}^{\mu,\nu}(g_1^*, g_2^*, \ldots, g_n^*); \quad (33)$$

(3) *(Boudedness) If* $h^- = \min_{i=1,2,\ldots,n}\left\{h_{im_i}\right\}$, $h^+ = \max_{i=1,2,\ldots,n}\left\{h_{im_i}\right\}$,

$$h^- \leq \mathrm{HFWCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) \leq h^+. \quad (34)$$

Theorem 4 is similar to Theorem 2, so the proofs are omitted here.

3.4. Different Forms of HFCBM and HFWCBM. Next, we will explore sever particular cases through taking diverse AC function $\varsigma(t)$.

Case 1. When $\varsigma(t) = (-\ln t)^{\kappa}\,(\kappa \geq 1)$, then

$$\mathrm{HFCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \bigcup_{h_{im_i}\in g_i h_{jm_j}\in g_j}\left\{e^{-\left(1/\mu+\nu\left(-\ln\left(1-e^{-\zeta_1^{1/\kappa}}\right)\right)^{\kappa}\right)^{1/\kappa}}\right\}, \quad (35)$$

where

$$\zeta_1 = \frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(-\ln\left(1-e^{-\zeta_1'^{1/\kappa}}\right)\right)^{\kappa},$$

$$\zeta_1' = \mu\left(-\ln h_{im_i}\right)^{\kappa} + \nu\left(-\ln h_{jm_j}\right)^{\kappa}, \quad (36)$$

$$\mathrm{HFWCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \bigcup_{h_{im_i}\in g_i h_{jm_j}\in g_j}\left\{e^{-\left(1/\mu+\nu\left(-\ln\left(1-e^{-\varphi_1^{1/\kappa}}\right)\right)^{\kappa}\right)^{1/\kappa}}\right\},$$

where

$$\varphi_1 = \frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(-\ln\left(1-e^{-\varphi_1'^{1/\kappa}}\right)\right)^{\kappa},$$

$$\varphi_1' = \mu\left(-\ln\left(1-e^{-\left(n\omega_k\left(-\ln\left(1-h_{im_i}\right)\right)^{\kappa}\right)^{1/\kappa}}\right)\right)^{\kappa} + \nu\left(-\ln\left(1-e^{-\left(n\omega_j\left(-\ln\left(1-h_{jm_j}\right)\right)^{\kappa}\right)^{1/\kappa}}\right)\right)^{\kappa}. \quad (37)$$

Specifically, when $\kappa = 1, \varsigma(t) = -\ln t$, then they are the operators proposed by Zhu [14].

Case 2. When $\varsigma(t) = t^{-\kappa} - 1$ with $\kappa \geq -1, \kappa \neq 0$, then

$$\mathrm{HFCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \bigcup_{h_{im_i}\in g_i h_{jm_j}\in g_j}\left\{\left(\frac{1}{\mu+\nu}\left(1-(\zeta_2+1)^{-1/\kappa}\right)^{-\kappa}\right)^{-1/\kappa}\right\}, \quad (38)$$

where

$$\zeta_2 = \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 - (\zeta_2' + 1)^{-1/\kappa} \right)^{-\kappa} - 1 \right),$$
$$\zeta_2' = \mu h_{im_i}^{-\kappa} + \nu h_{jm_j}^{-\kappa} - 2, \tag{39}$$

$$\mathrm{HFWCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \bigcup_{h_{im_i} \in g_i h_{jm_j} \in g_j} \left\{ \left(\frac{1}{\mu+\nu} \left(1 - (\varphi_2 + 1)^{-1/\kappa} \right)^{-\kappa} \right)^{-1/\kappa} \right\},$$

where

$$\varphi_2 = \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 - (\varphi_2' + 1)^{-1/\kappa} \right)^{-\kappa} - 1 \right),$$
$$\varphi_2' = \mu \left(1 - \left(n\omega_i \left(1 - h_{im_i} \right)^{-\kappa} \right)^{-1/\kappa} \right)^{-\kappa} + \nu \left(1 - \left(n\omega_j \left(1 - h_{jm_j} \right)^{-\kappa} \right)^{-1/\kappa} \right)^{-\kappa} - 2. \tag{40}$$

Case 3. When $\varsigma(t) = \ln(e^{-\kappa t} - 1/e^{-\kappa} - 1)\,(\kappa \neq 0)$, then

$$\mathrm{HFCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \bigcup_{h_{im_i} \in g_i h_{jm_j} \in g_j} \left\{ -\frac{1}{\kappa} \ln\left(\zeta_3 \left(e^{-\kappa} - 1 \right) + 1 \right) \right\}, \tag{41}$$

where

$$\zeta_3 = \left(\frac{e^{-\kappa\left(1+1/\kappa \ln\left(\zeta_3'(e^{-\kappa}-1)+1\right)\right)} - 1}{e^{-\kappa} - 1} \right)^{1/\mu+\nu},$$
$$\zeta_3' = \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(\frac{e^{-\kappa\left(1+1/\kappa \ln\left(\zeta_3''(e^{-\kappa}-1)+1\right)\right)} - 1}{e^{-\kappa} - 1} \right) \right)^{1/n(n-1)},$$
$$\zeta_3'' = \left(\frac{e^{-\kappa h_{im_i}} - 1}{e^{-\kappa} - 1} \right)^{\mu} \left(\frac{e^{-\kappa h_{jm_j}} - 1}{e^{-\kappa} - 1} \right)^{\nu}, \tag{42}$$

$$\mathrm{HFWCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \bigcup_{h_{im_i} \in g_i h_{jm_j} \in g_j} \left\{ -\frac{1}{\kappa} \ln\left(\varphi_3 \left(e^{-\kappa} - 1 \right) + 1 \right) \right\},$$

where

$$\varphi_3 = \left(\frac{e^{-\kappa\left(1+1/\kappa \ln\left(\varphi_3'(e^{-\kappa}-1)+1\right)\right)}-1}{e^{-\kappa}-1}\right)^{1/\mu+\nu},$$

$$\varphi_3' = \prod_{i,j=1 i\neq j}^{n} \left(\frac{e^{-\kappa\left(1+1/\kappa \ln\left(\varphi_3''(e^{-\kappa}-1)+1\right)\right)}-1}{e^{-\kappa}-1}\right)^{1/n(n-1)},$$

$$\varphi_3'' = \left(\frac{e^{-\kappa\left(\varphi_3^i\right)}-1}{e^{-\kappa}-1}\right)^{\mu}\left(\frac{e^{-\kappa\left(\varphi_3^j\right)}-1}{e^{-\kappa}-1}\right)^{\nu},$$

$$\varphi_3^i = 1+\frac{1}{\kappa}\ln\left(\left(\frac{e^{-\kappa\left(1-h_{im_i}\right)}-1}{e^{-\kappa}-1}\right)^{n\omega_k}\left(e^{-\kappa}-1\right)+1\right),$$

$$\varphi_3^j = 1+\frac{1}{\kappa}\ln\left(\left(\frac{e^{-\kappa\left(1-h_{jm_j}\right)}-1}{e^{-\kappa}-1}\right)^{n\omega_j}\left(e^{-\kappa}-1\right)+1\right). \tag{43}$$

Case 4. When $\varsigma(t) = \ln(1-\kappa(1-t)/t)$ with $\kappa \in [-1,1)$, we have

$$\mathrm{HFCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \bigcup_{h_{im_i}\in g_i h_{jm_j}\in g_j}\left\{\frac{1-\kappa}{\zeta_4-\kappa}\right\}, \tag{44}$$

where

$$\zeta_4 = \left(\frac{1-\kappa\left(1-\kappa/\zeta_4'-\kappa\right)}{1-\left(1-\kappa/\zeta_4'-\kappa\right)}\right)^{1/\mu+\nu},$$

$$\zeta_4' = \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(\frac{1-\kappa\left(1-\kappa/\zeta_4''-\kappa\right)}{1-\left(1-\kappa/\zeta_4''-\kappa\right)}\right)^{1/n(n-1)}, \tag{45}$$

$$\zeta_4'' = \left(\frac{1-\kappa\left(1-h_{im_i}\right)}{h_{im_i}}\right)^{\mu}\left(\frac{1-\kappa\left(1-h_{jm_j}\right)}{h_{jm_j}}\right)^{\nu},$$

$$\mathrm{HFWCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \bigcup_{h_{im_i}\in g_i h_{jm_j}\in g_j}\left\{\frac{1-\kappa}{\varphi_4-\kappa}\right\},$$

where

$$\varphi_4 = \left(\frac{1-\kappa\left(1-\kappa/\varphi_4'-\kappa\right)}{1-\left(1-\kappa/\varphi_4'-\kappa\right)}\right)^{1/\mu+\nu},$$

$$\varphi_4' = \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(\frac{1-\kappa\left(1-\kappa/\varphi_4''-\kappa\right)}{1-\left(1-\kappa/\varphi_4''-\kappa\right)}\right)^{1/n(n-1)},$$

$$\varphi_4'' = \left(\frac{1-\varsigma\left(1-\varsigma/\varphi_4^i-\varsigma\right)}{1-\left(1-\varsigma/\varphi_4^i-\varsigma\right)}\right)^{\mu}\left(\frac{1-\varsigma\left(1-\varsigma/\varphi_4^j-\varsigma\right)}{1-\left(1-\varsigma/\varphi_4^j-\varsigma\right)}\right)^{\nu},$$

$$\varphi_4^i = \left(\frac{1-\kappa h_{im_i}}{1-h_{im_i}}\right)^{n\omega_i},$$

$$\varphi_4^j = \left(\frac{1-\kappa h_{jm_j}}{1-h_{jm_j}}\right)^{n\omega_j}. \tag{46}$$

Case 5. When $\varsigma(t) = -\ln(1-(1-t)^{\kappa})$, $\kappa \geq 1$, then

$$\mathrm{HFCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \bigcup_{h_{im_i}\in g_i h_{jm_j}\in g_j}\left\{1-\left(1-\left(1-\zeta_5^{\kappa}\right)^{1/\mu+\nu}\right)^{1/\kappa}\right\}, \tag{47}$$

where

$$\zeta_5 = \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\zeta_5'\right)^{1/\kappa}\right)^{\kappa}\right)^{1/n(n-1)},$$

$$\zeta_5' = \left(1-\left(1-h_{im_i}\right)^{\kappa}\right)^{\mu}\left(1-\left(1-h_{jm_j}\right)^{\kappa}\right)^{\nu}, \tag{48}$$

$$\mathrm{HFWCBM}^{\mu,\nu}(g_1, g_2, \ldots, g_n) = \bigcup_{h_{im_i}\in g_i h_{jm_j}\in g_j}\left\{1-\left(1-\left(1-\varphi_5^{\kappa}\right)^{1/\mu+\nu}\right)^{1/\kappa}\right\},$$

where

$$\varphi_5 = \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1-\left(1-\left(1-\varphi_5'\right)^{1/\kappa}\right)^{\kappa}\right)^{1/n(n-1)},$$
$$\varphi_5' = \left(1-\left(1-\left(1-\left(1-h_{im_i}^{\kappa}\right)^{n\omega_k}\right)^{1/\kappa}\right)^{\kappa}\right)^{\mu}\left(1-\left(1-\left(1-\left(1-h_{jm_j}^{\kappa}\right)^{n\omega_j}\right)^{1/\kappa}\right)^{\kappa}\right)^{\nu}. \tag{49}$$

4. Novel MADM Approach Based on the Propounded Operators

In this section, we will propound a novel approach based on the presented operators for handling MADM issues.

Suppose that there are m alternatives $\Upsilon_i\,(i=1,2,\ldots,m)$, n attributes $G_j\,(j=1,2,\ldots,n)$. If DMs provide several values for the alternative Υ_i under the attribute $G_j\,(j=1,2,\ldots,n)$ with anonymity, these values can be considered as a HFE h_{ij}. In the case, if two DMs provide the same value, then the value emerges only once in h_{ij}. In what follows, the specific algorithm for MADM problems under the HF environment will be designed.

Step 1: DMs provide their evaluations about the alternative Υ_i under the attribute G_j , denoted by the hesitant fuzzy elements $h_{ij}, i=1,2,\ldots,m; j=1,2,\ldots,n$

Step 2: use the HFCBM or HFWCBM to fuse the HFE $y_i\,(i=1,2,\ldots,m)$ for $\Upsilon_i\,(i=1,2,\ldots,m)$

Step 3: the score values $\pi(y_i)\,(i=1,2,\ldots,m)$ of y_i are calculated using Definition 2 and compared

Step 4: the optimal alternatives $\Upsilon_i\,(i=1,2,\ldots,m)$ are made by ranking $\pi(y_i)\,(i=1,2,\ldots,m)$

4.1. Empirical Example

Example 1. Selection of Equipment Purchase. Assume that there are four short-term stocks $\Upsilon_1, \Upsilon_2, \Upsilon_3$, and Υ_4. The following four attributes: G_1, G_2, G_3, and G_4, should be considered. G_1: product price; G_2: technology; G_3: after-sales service level; G_4: supplier reputation.

Next, we use the developed method to find the ranking of the alternatives and the optimal choice.

Step 1: the decision matrix is given by expert and shown in Table 1

Step 2: use equation (35), let $\varsigma(t) = (-\ln t)^{1.2}$ and $\mu=1$, $\nu=1$ to aggregate the HFE $y_i\,(i=1,\ldots,4)$ for $\Upsilon_i\,(i=1,\ldots,4)$.

Step 3: compute the score values $\pi(y_i)\,(i=1,\ldots,4)$ of $y_i\,(i=1,\ldots,4)$ by Definition 2. We have $\pi(y_1)=0.5794$, $\pi(y_2)=0.5744$, $\pi(y_3)=0.5487$, and $\pi(y_4)=0.5738$.

Step 4: rank the alternatives $\Upsilon_i\,(i=1,\ldots,4)$, and select the desirable one in term of comparison rules. Since $\pi(y_1)>\pi(y_2)>\pi(y_4)>\pi(y_3)$, we can obtain the rank of alternatives $\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$, and Υ_1 is the best alternative.

4.2. Sensitivity Analysis.

4.2. Sensitivity Analysis. Next, we first analyse the sensitivity of sorting results obtained by diverse operations from AC. In addition, the effect of parameters μ, ν, and κ on the final sorting results of alternatives are also analysed in detail.

Firstly, we analyse the sorting results on the basis of different generators produced diverse copulas. The sorting results of alternatives using diverse generators are listed in Table 2. From Table 2, although the score values of alternatives obtained by diverse operations from AC, the final sorting relation are basically same, which can further illustrate that the diverse copulas are relatively stable for the ultimate decision outcomes.

Secondly, we research the effect of parameter κ in the HFCBM operator on the ultimate orders of alternatives. Table 3 shows the values of score functions with the different κ when $\mu=1$ and $\nu=1$. Figure 1 shows how the score functions obtained by Case 1 change with the parameter κ under different values of μ and ν. As can be seen from the results of the chart, the score values increase with the increase of κ for the same alternatives.

Thirdly, we explore the effect of parameters μ and ν in the HFCBM operator on the ranking results of alternatives. The ranking results of alternatives based on diverse values of μ and ν can be generated in Table 4 (we select Case 1 as an example and let $\kappa=2$). Figure 2 shows the variation of the score function with μ and ν. The results indicate that the scores of alternatives by the HFCBM operator increase as the parameter μ ranges from 0 to 10 and the parameter ν ranges from 0 to 10, but the ranking results of factors vary considerably, which provide DMs with flexible options. The DMs can adjust the value of μ and ν to get the desired ranking result.

4.3. Comparative Analysis. Next, we conduct a contrastive analysis between the propounded MADM method with the previous approaches to confirm the validity and to reveal the significant advantages of the propounded approach.

Table 1: Hesitant fuzzy decision matrix [4].

Alternatives	G_1	G_2	G_3	G_4
Υ_1	$\{0.2, 0.4, 0.7\}$	$\{0.2, 0.6, 0.8\}$	$\{0.2, 0.3, 0.6, 0.7, 0.9\}$	$\{0.3, 0.4, 0.5, 0.7, 0.8\}$
Υ_2	$\{0.2, 0.4, 0.7, 0.9\}$	$\{0.1, 0.2, 0.4, 0.5\}$	$\{0.3, 0.4, 0.6, 0.9\}$	$\{0.5, 0.6, 0.8, 0.9\}$
Υ_3	$\{0.3, 0.5, 0.6, 0.7\}$	$\{0.2, 0.4, 0.5, 0.6\}$	$\{0.3, 0.5, 0.7, 0.8\}$	$\{0.2, 0.5, 0.6, 0.7\}$
Υ_4	$\{0.3, 0.5, 0.6\}$	$\{0.2, 0.4\}$	$\{0.5, 0.6, 0.7\}$	$\{0.8, 0.9\}$

Table 2: Decision results obtained by the HFCBM operator with diverse copulas.

	Generator $\varsigma(t)$	$\pi(\Upsilon_1)$	$\pi(\Upsilon_2)$	$\pi(\Upsilon_3)$	$\pi(\Upsilon_4)$	Ranking
Case 1	$(-\ln t)^{\kappa}\,(\kappa = 1)$	0.5810	0.5778	0.5495	0.5791	$\Upsilon_1 \succ \Upsilon_4 \succ \Upsilon_2 \succ \Upsilon_3$
Case 2	$t^{-\kappa} - 1\,(\kappa = 1)$	0.8275	0.8236	0.7546	0.7976	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$
Case 3	$\ln(e^{-\kappa t} - 1/e^{-\kappa} - 1)\,(\kappa = 1)$	0.5782	0.5747	0.5476	0.5746	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$
Case 4	$\ln(1 - \kappa(1-t)/t)\,(\kappa = -1)$	0.5868	0.5833	0.5526	0.5862	$\Upsilon_1 \succ \Upsilon_4 \succ \Upsilon_2 \succ \Upsilon_3$
Case 5	$-\ln(1 - (1-t)^{\kappa})\,(\kappa = 1)$	0.5810	0.5778	0.5495	0.5791	$\Upsilon_1 \succ \Upsilon_4 \succ \Upsilon_2 \succ \Upsilon_3$

4.3.1. Testification of Validity. To demonstrate the validity of the propounded approach, we utilize three previously existing methodologies to address the aforementioned example and perform a comparison study. The previous methods include HFWA and HFWG operators proposed by Xia and Xu [4], HFBM operator proposed by Zhu and Xu [14], and HFFWA and HFFWG operators proposed by Qin et al. [24]. Using the data in Reference [24] and different operators, Table 5 is obtained.

From Table 5, we can observe that the orders are marginally diverse, but the optimal selections are all Υ_6. So the propounded approach in this paper is workable and efficient.

4.3.2. Further Comparison and Analysis

(i) Compared with the method based upon the HFWA (or HFWG) operator propounded by Xia and Xu [4], although the HFWA operator can fuse HF information and the computation is simple, it has the following weakness: (1) it assumes that all attributes are independent in the course of information integration; (2) it ignores the correction of dissimilar attribute assessment information; and (3) it lacks flexibility and robustness because its operational rules are constructed by the Algebraic operations. However, the proposed approach can effectively overcome these disadvantages through the following aspects: (a) the BM operator can take into consideration the interconnection of diverse attributes in this paper, and the operational laws are established based upon different copula with a flexible parameter, which can overcome the weakness (1) and (2); (b) the AC can make decision course more general and flexible because five copulas with their flexible parameters can be selected according to DMs' attitude, which can conquer the weakness (3).

(ii) Compared with the approach based upon the HFFWA operator propounded by Qin et al. [24], although it can aggregate vagueness information validly and make the aggregation process flexible by the adjustable parameter, it ignores the interrelationship of different attributes during the production of fusion. In comparison, the propounded method in this paper based on the HFCBM operator not only takes into account the correction of diverse attributes but also provides a more universal and flexible aggregation operator, and the HFCBM can regard five aggregation functions through assigning diverse copulas to it.

(iii)Compared with the method based upon the HFBM operator propounded by Zhu and Xu [14], it can consider the relevance of dissimilar attributes through the BM operator. But the operational laws of the HFBM operator are defined by the algebraic operation, which lacks flexibility and robustness in the course of information aggregation. In contrast, the HFBM operator supplies universal operations based on the Archimedean copula. Specially, when $\kappa = 1$ in the Gumbel copula and Joe copula, they shall degenerate into algebraic operations, i.e., the HFBM operator is a particular instance of HFCBM operator. Hence, the proposed method is more universal and flexible. Hence, the developed approach can utilize an ocean of practical decision issues and especially needs to consider the relevance of diverse attributes.

4.3.3. The Distinct Merits of the Propounded Approach in This Paper. A detailed contrastive analysis for aforementioned approaches is displayed in Table 5, and the propounded method based on the HFCBM operator has the following distinct advantages during the course of information fusion: (1) It provides a universal method through selection different copulas to attain diverse aggregation operator; (2) The flexibility and robustness of the presented methodology is shown through the parameter in each copula, and it supplies more selection to choose parameters that accord with their preferences attitude; and (3) The interconnection between diverse attributes is taken into consideration during the information aggregation proceeding.

Table 3: Decision results obtained by Case 1 with diverse κ.

κ	$\pi(\Upsilon_1)$	$\pi(\Upsilon_2)$	$\pi(\Upsilon_3)$	$\pi(\Upsilon_4)$	Ranking
2	0.5832	0.5689	0.5490	0.5606	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$
5	0.6194	0.5849	0.5629	0.5611	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_3 \succ \Upsilon_4$
7	0.6334	0.5950	0.5703	0.5689	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_3 \succ \Upsilon_4$
10	0.6460	0.6054	0.5780	0.5782	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$

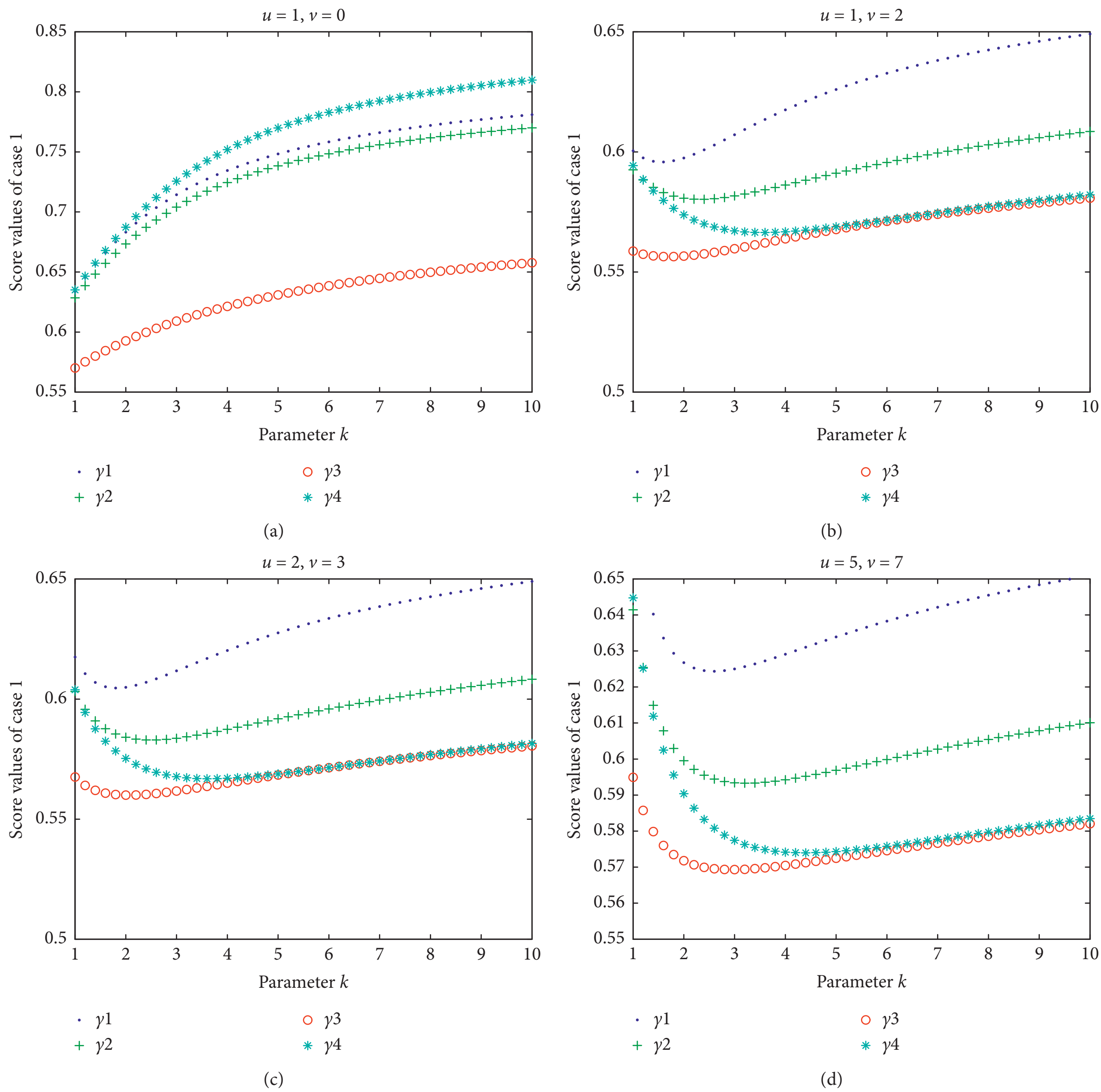

Figure 1: The score functions obtained by Case 1 with respect to parameter κ.

Table 4: Decision results obtained by Case 1 with diverse μ and ν.

Parameter μ	Parameter ν	$\pi(\Upsilon_1)$	$\pi(\Upsilon_2)$	$\pi(\Upsilon_3)$	$\pi(\Upsilon_4)$	Ranking
0	1	0.6832	0.6734	0.6386	0.6873	$\Upsilon_4 \succ \Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_3$
	3	0.7041	0.6922	0.6046	0.7104	$\Upsilon_4 \succ \Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_3$
	6	0.7186	0.7061	0.6139	0.7278	$\Upsilon_4 \succ \Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_3$
	10	0.7294	0.7169	0.6215	0.7415	$\Upsilon_4 \succ \Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_3$
1	1	0.5832	0.5689	0.5490	0.5606	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$
	3	0.6124	0.5944	0.5649	0.5903	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$
	6	0.6359	0.6265	0.5831	0.6300	$\Upsilon_1 \succ \Upsilon_4 \succ \Upsilon_2 \succ \Upsilon_3$
	10	0.6712	0.6535	0.5978	0.6640	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$
2	1	0.5975	0.5807	0.5566	0.5738	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$
	3	0.6378	0.5841	0.5600	0.5753	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$
	6	0.6294	0.6072	0.5739	0.6036	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$
	10	0.6537	0.6322	0.5879	0.6352	$\Upsilon_1 \succ \Upsilon_4 \succ \Upsilon_2 \succ \Upsilon_3$
5	1	0.6359	0.6173	0.5780	0.6185	$\Upsilon_1 \succ \Upsilon_4 \succ \Upsilon_2 \succ \Upsilon_3$
	3	0.6180	0.5939	0.5670	0.5854	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$
	6	0.6233	0.5964	0.5698	0.5867	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$
	10	0.6378	0.6103	0.5782	0.6038	$\Upsilon_1 \succ \Upsilon_2 \succ \Upsilon_4 \succ \Upsilon_3$

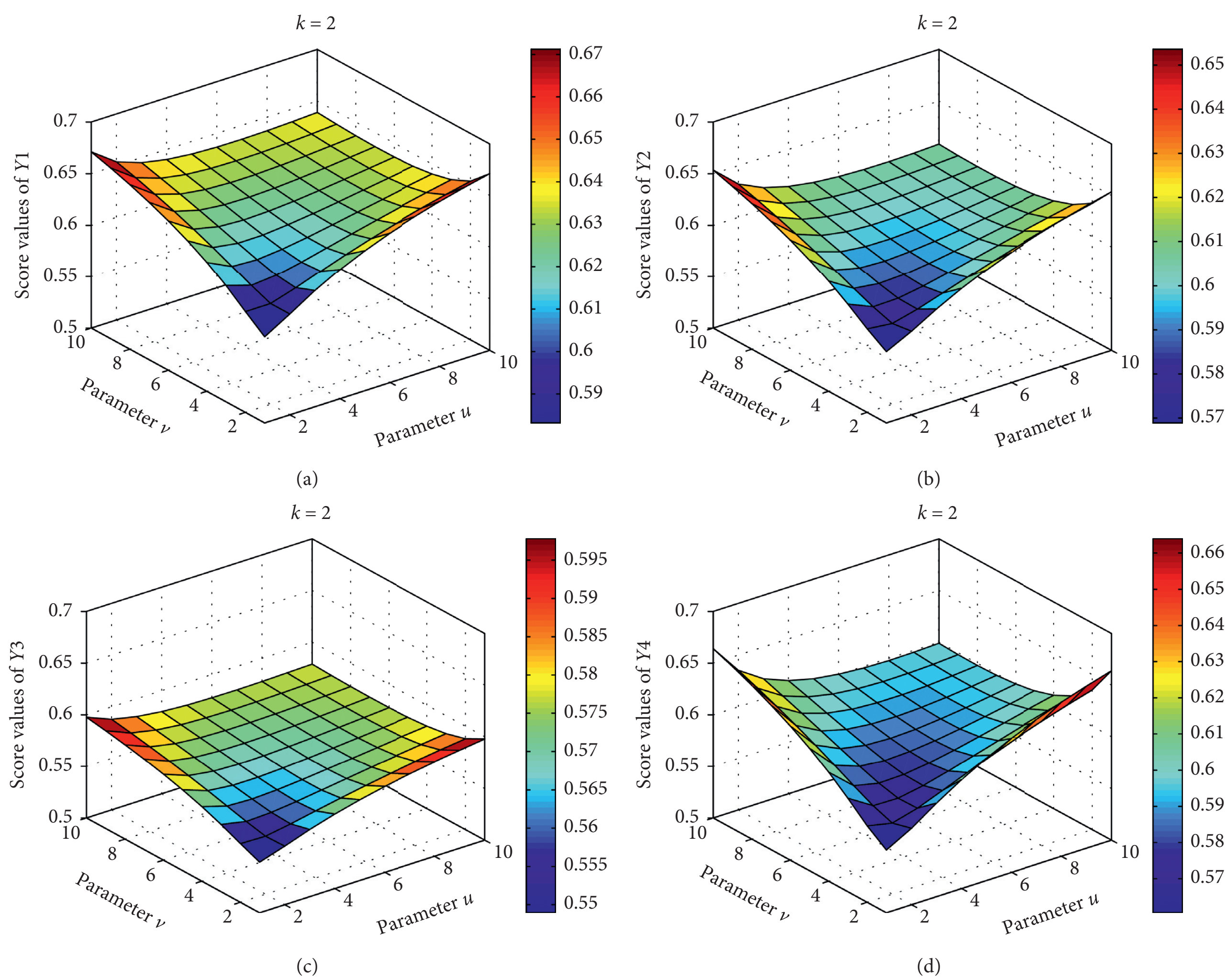

Figure 2: The score functions obtained by Case 1 with respect to μ and ν.

Table 5: The scores obtained by different operators.

Operator	Parameter	Need weight or not	Ranking
HFWA [4]	None	Yes	$A_6 \succ A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_4$
HFWG [4]	None	Yes	$A_6 \succ A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_4$
HFBM [14]	$p = 2, q = 1$	No	$A_6 \succ A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$
HFFWA [24]	$\lambda = 2$	Yes	$A_6 \succ A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_4$
HFFWG [24]	$\lambda = 2$	Yes	$A_6 \succ A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_4$
HFCBM (Case 1)	$\kappa = 2, \mu = 2, \nu = 1$	No	$A_6 \succ A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$
HFWCBM (Case 1)	$\kappa = 2, \mu = 2, \nu = 1$	Yes	$A_6 \succ A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$

5. Conclusions

In order to better accomplish the procedure of information aggregation and decrease information loss under the HF environment, this paper propounds HFCBM operator and HFWCBM operator to aggregate HF information and exert it to tackle MADM issues. The primarily contribution of our research was that the proposed operators can capture the connection between attributes and provide additional choice for DMs by using the generalised Archimedean copula functions. The comparative analysis showed that this method was more flexible, practical, general, and better than the existing methods. However, this article also has some shortcomings and limitations. First, the proposed method only solved a numerical example making the study incomplete. Second, we overlooked the scope of each Archimedean copula function making the study less rigorous. In the future, the propounded method will be applied to other actual application such as supplier selection, investment analysis, and pattern recognition. We will also continue to study the MADM problems under the HF environment [25–27]. Besides, we will explore other theories of copulas in different fuzzy settings [28–31], as well as the large-scale decision-making algorithm based upon linguistic assessment theory and methodology will become our focal point research target.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] V. Torra and Y. Narukawa, "On hesitant fuzzy sets and decision," in *Proceedings of the IEEE International Conference on Fuzzy Systems*, pp. 1378–1382, Jeju Island, South Korea, August 2009.

[3] V. Torra, "Hesitant fuzzy sets," *International Journal of Intelligent Systems*, vol. 25, pp. 529–539, 2010.

[4] M. Xia and Z. Xu, "Hesitant fuzzy information aggregation in decision making," *International Journal of Approximate Reasoning*, vol. 52, no. 3, pp. 395–407, 2011.

[5] Z. Xu and M. Xia, "Distance and similarity measures for hesitant fuzzy sets," *Information Sciences*, vol. 181, no. 11, pp. 2128–2138, 2011.

[6] R. Lin, X. Zhao, H. Wang, and G. Wei, "Hesitant fuzzy linguistic aggregation operators and their application to multiple attribute decision making," *Journal of Intelligent & Fuzzy Systems*, vol. 27, no. 1, pp. 49–63, 2014.

[7] F. Meng, X. Chen, and Q. Zhang, "Induced generalized hesitant fuzzy Shapley hybrid operators and their application in multi-attribute decision making," *Applied Soft Computing*, vol. 28, no. 1, pp. 599–607, 2015.

[8] Q. Yu, F. Hou, Y. Zhai, and Y. Du, "Some hesitant fuzzy Einstein aggregation operators and their application to multiple attribute group decision making," *International Journal of Intelligent Systems*, vol. 31, no. 7, pp. 722–746, 2016.

[9] Y. Lin, Y.-M. Wang, and S.-Q. Chen, "Hesitant fuzzy multiattribute matching decision making based on regret theory with uncertain weights," *International Journal of Fuzzy Systems*, vol. 19, no. 4, pp. 955–966, 2017.

[10] J. Hu, Y. Yang, X. Zhang, and X. Chen, "Similarity and entropy measures for hesitant fuzzy sets," *International Transactions in Operational Research*, vol. 25, no. 3, pp. 857–886, 2018.

[11] H. Xing, L. Song, and Z. Yang, "An evidential prospect theory framework in hesitant fuzzy multiple-criteria decision-making," *Symmetry*, vol. 11, no. 12, p. 1467, 2019.

[12] Z. Zhang, "Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making," *Information Sciences*, vol. 234, pp. 150–181, 2013.

[13] G. W. Wei, X. F. Zhao, H. J. Wang et al., "Hesitant fuzzy choquet integral aggregation operators and their applications to multiple attribute decision making £ Information," *Information: An International Interdisciplinary Journal*, vol. 15, no. 2, pp. 441–448, 2012.

[14] B. Zhu and Z. S. Xu, "Hesitant fuzzy Bonferroni means for multi-criteria decision making," *Journal of the Operational Research Society*, vol. 64, no. 12, pp. 1831–1840, 2013.

[15] Y. D. He, Z. He, G. D. Wang et al., "Hesitant fuzzy power Bonferroni means and their application to multiple attribute

decision making," *IEEE Transactions on Fuzzy Systems*, vol. 23, no. 5, pp. 1655–1668, 2015.

[16] M. Sklar, "Fonctions de rpartition n dimensions et leurs marges," *Universit Paris*, vol. 8, pp. 229–231, 1959.

[17] R. B. Nelsen, "An introduction to copulas," *Technometrics*, vol. 42, no. 3, p. 317, 2000.

[18] U. Cherubini, E. Luciano, and W. Vecchiato, *Copula Methods in Finance*, pp. 129–150, John Wiley and Sons Ltd., Hoboken, NJ, USA, 2004.

[19] C. Genest and J. MacKay, "The joy of copulas: bivariate distributions with uniform marginals," *The American Statistician*, vol. 40, no. 4, pp. 280–283, 1986.

[20] Z. Tao, B. Han, L. Zhou, and H. Chen, "The novel computational model of unbalanced linguistic variables based on archimedean copula," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 26, no. 4, pp. 601–631, 2018.

[21] T. Chen, S.-S. He, J.-Q. Wang, L. Li, and H. Luo, "Novel operations for linguistic neutrosophic sets on the basis of Archimedean copulas and co-copulas and their application in multi-criteria decision-making problems," *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 2, pp. 2887–2912, 2019.

[22] Z. S. Xu and R. R. Yager, "Intuitionistic fuzzy Bonferroni means," *IEEE Transactions on Systems*, vol. 41, no. 2, pp. 568–578, 2010.

[23] J. Wu, L. M. Mou, F. Liu et al., "Archimedean copula-based hesitant fuzzy information aggregation operators for multiple attribute decision making," *Mathematical Problems in Engineering*, vol. 2020, Article ID 6284245, 21 pages, 2020.

[24] J. Qin, X. Liu, and W. Pedrycz, "Frank aggregation operators and their application to hesitant fuzzy multiple attribute decision making," *Applied Soft Computing*, vol. 41, pp. 428–452, 2016.

[25] B. Bedregal, R. Reiser, H. Bustince, C. Lopez-Molina, and V. Torra, "Aggregation functions for typical hesitant fuzzy elements and the action of automorphisms," *Information Sciences*, vol. 255, pp. 82–99, 2014.

[26] H. Bustince, E. Barrenechea, M. Pagola et al., "A historical account of types of fuzzy sets and their relationships," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 1, pp. 179–194, 2015.

[27] R. M. Rodríguez, B. Bedregal, H. Bustince et al., "A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress," *Information Fusion*, vol. 29, pp. 89–97, 2016.

[28] M. Kong, Z. Pei, F. Ren, and F. Hao, "New operations on generalized hesitant fuzzy linguistic term sets for linguistic decision making," *International Journal of Fuzzy Systems*, vol. 21, no. 1, pp. 243–262, 2019.

[29] Y. Liu, J. Liu, and Y. Qin, "Pythagorean fuzzy linguistic Muirhead mean operators and their applications to multi-attribute decision-making," *International Journal of Intelligent Systems*, vol. 35, no. 2, pp. 300–332, 2020.

[30] P. Liu and P. Wang, "Multiple-attribute decision-making based on archimedean Bonferroni operators of q-rung orthopair fuzzy numbers," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 5, pp. 834–848, 2019.

[31] Y. Rong, Z. Pei, and Y. Liu, "Hesitant fuzzy linguistic hamy mean aggregation operators and their application to linguistic multiple attribute decision-making," *Mathematical Problems in Engineering*, vol. 2020, Article ID 3262618, 22 pages, 2020.