Research Article

Positive Solutions for Two-Point Boundary Value Problems for Fourth-Order Differential Equations with Fully Nonlinear Terms

Yixin Zhang¹ and Yujun Cui²

¹School of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, China

²State Key Laboratory of Mining Disaster Prevention and Control Co-founded by Shandong Province and the Ministry of Science and Technology, Shandong University of Science and Technology, Qingdao 266590, China

Correspondence should be addressed to Yujun Cui; cuj720201@163.com

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In this paper, we consider the existence of positive solutions for the fully fourth-order boundary value problem
$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & 0 \leq t \leq 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$
 where $f: [0, 1] \times [0, +\infty) \times (-\infty, +\infty) \times (-\infty, 0) \times (-\infty, +\infty) \rightarrow [0, +\infty]$ is continuous. This equation can simulate the deformation of an elastic beam simply supported at both ends in a balanced state. By using the fixed-point index theory and the cone theory, we discuss the existence of positive solutions of the fully fourth-order boundary value problem. We transform the fourth-order differential equation into a second-order differential equation by order reduction method. And then, we examine the spectral radius of linear operators and the equivalent norm on continuous space. After that, we obtain the existence of positive solutions of such BVP.

1. Introduction

In this paper, we study the existence of positive solutions for the fully fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & 0 \leq t \leq 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \quad (1)$$

where $f: [0, 1] \times [0, +\infty) \times (-\infty, +\infty) \times (-\infty, 0) \times (-\infty, +\infty) \rightarrow [0, +\infty]$ is continuous. This boundary value problem can simulate the deformation of an elastic beam, whose one end is fixed and the other end is free in a balanced state. In mechanics, BVP (1) is called a cantilever beam equation. In this equation, each derivative of $u(t)$ has its physical meaning: $u'(t)$ is the slope, $u''(t)$ is the bending moment stiffness, $u'''(t)$ is the shear force stiffness, and $u^{(4)}(t)$ is the load density stiffness. The nonlinear fourth-order differential equation boundary value problem can

simulate the deformation of an elastic beam under external force, and different boundary value conditions can show its force under different conditions. Because of its importance in mechanics, many scholars have done a lot of research on the existence of solutions for fourth-order ordinary differential equations using various nonlinear methods [1–11].

As the nonlinear term does not contain the derivative term of the unknown function, equation (1) becomes

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), & 0 \leq t \leq 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \quad (2)$$

If $f(t, u)$ is superlinear or sublinear growth on u , the authors in [1] used the fixed-point theorem on the cone to obtain the existence of the positive solution of equation (2). In [3], the author used the fixed-point theorem and topological degree theory to study the existence of one or two positive solutions for the fourth-order differential equation boundary value problem:

$$\begin{cases} u^{(4)}(t) - \lambda f(t, u(t)) = 0, & 0 \leq t \leq 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \quad (3)$$

Under the fixed-point index method on the cone, the authors of [11] discussed the existence of positive solutions for fourth-order boundary value problems with two parameters. Among them, the assumption condition of the nonlinear term f is related to the first eigenvalue of the corresponding linear operator. It is noteworthy that the nonlinear term in the abovementioned boundary value problems does not include the higher-order derivative u''' . When the nonlinear term contains the higher-order derivative of the unknown function, the authors of [8, 9] used the upper and lower solution method to study the existence of solutions for fully fourth-order nonlinear boundary value problems with nonlinear boundary conditions. In [10], the author discussed a fourth-order boundary value problem with fully form:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & 0 \leq t \leq 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \quad (4)$$

When the nonlinear term satisfies superlinear growth and sublinear growth, the author used the fixed-point index method, combined with the positivity of linear operators and spectral radius, to get the positive solutions for the boundary value problem. But the linear operator in [10] does not involve the first and second derivatives of unknown functions.

In this paper, by using cone theory and the fixed-point index, combined with the spectral radius of linear integral operators, and the application of equivalent norms, we discuss the existence of positive solutions for boundary value problems (1).

2. Preliminaries

In this section, we give some assumptions that are important to our main results:

- (i) (H_1) : $f: [0, 1] \times [0, +\infty) \times (-\infty, +\infty) \times (-\infty, 0) \times (-\infty, +\infty) \rightarrow [0, +\infty]$ is continuous
- (ii) (H_2) : there exist nonnegative constants $a_1, c_1 \geq 0$ and $r > 0$ such that $a_1 x_1 + c_1 x_3 \leq f(t, x_1, x_2 - x_3, x_4)$, $x_1, x_3 \in [0, r]$, $x_2 \in [-r, r]$, and $x_4 \in (-\infty, +\infty)$.
- (iii) (H_3) : there exist nonnegative constants $a, b, c, d, M \geq 0$ and $0 < q < 1$ such that

$$\begin{aligned} f(t, x_1, x_2, -x_3, x_4) &\leq ax_1 + b|x_2| + cx_3 + d|x_4|^q + M, x_1, \\ x_3 &\in [0, +\infty], x_2, x_4 \in (-\infty, +\infty). \end{aligned} \quad (5)$$

Let $C[0, 1]$ denote the Banach space of continuous functions from $[0, 1]$ into $\mathbb{R}$ with norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Let

$P = \{u \in C[0, 1] \mid u(t) \text{ is concave on } [0, 1], u(0) = u(1) = 0\}$. Then, P is a positive cone on $C[0, 1]$.

The functions on cone P have the following properties:

Lemma 1 (see [12]). *Every function $u \in P$ on the cone P is differentiable almost everywhere on $(0, 1)$ and satisfies*

$$\begin{aligned} u(t) &\geq t(1-t)\|u\|, \quad t \in [0, 1], \\ |u'(t)| &\leq \frac{u(t)}{t(1-t)}, \quad \text{a.e. on } [0, 1]. \end{aligned} \quad (6)$$

Let $v(t) = -u''(t)$, then the differential equation BVP (1) can be transformed into the following two second-order differential equations:

$$\begin{cases} u''(t) = -v(t), \\ u(0) = u(1) = 0, \end{cases} \quad (7)$$

$$\begin{cases} -v''(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \\ v(0) = v(1) = 0. \end{cases} \quad (8)$$

Let $G(t, s) = \begin{cases} s(1-t) & 0 \leq s \leq t \leq 1 \\ t(1-s) & 0 \leq t \leq s \leq 1 \end{cases}$, which is the corresponding Green's function of BVP (7). Thus, BVP (7) can be transformed into an equivalent integral equation:

$$u(t) = \int_0^1 G(t, s)v(s)ds. \quad (9)$$

By using (9), BVP (8) can be reduced to

$$\begin{cases} -v''(t) = f(t, (Av)(t), (Av)'(t), -v(t), -v'(t)), \\ v(0) = v(1) = 0, \end{cases} \quad (10)$$

where operator A is defined as $(Av)(t) = \int_0^1 G(t, s)v(s)ds$. Thus, BVP (10) can be reduced to the equivalent integral equation:

$$v(t) = \int_0^1 G(t, s)f(s, (Av)(s), (Av)'(s), -v(s), -v'(s))ds. \quad (11)$$

From the expression of Green's function $G(t, s)$, we know that $G(t, s)$ is continuous on $[0, 1] \times [0, 1]$, and we have

$$t(1-t)s(1-s) \leq G(t, s) \leq s(1-s) \text{ or } t(1-t), \quad 0 \leq t, s \leq 1. \quad (12)$$

From the standard proof, we can easily obtain the following statement.

Lemma 2. $A: L[0, 1] \rightarrow C[0, 1]$ is a completely continuous operator.

For $\forall v \in P$, we define an operator $(Fv)(t) = f(t, (Av)(t), (Av)'(t), -v(t), -v'(t))$. It follows from Lemma 1 and condition (H_3) that

$$\begin{aligned}
|(Fv)(t)| &= |f(t, (Av)(t), (Av)'(t), -v(t), -v'(t))| \\
&\leq a(Av)(t) + b|(Av)'(t)| + cv(t) + d|v'(t)|^q + M \\
&\leq a \int_0^1 G(t, s)v(s)ds + b \int_0^1 |G_t(t, s)|v(s)ds + cv(t) + d \frac{\|v\|^q}{t^q(1-t)^q} + M.
\end{aligned} \tag{13}$$

It shows that F is an operator which is defined on P to $L(0, 1)$. It is easy to conclude that F is the continuous bounded operator mapping from P to $L(0, 1)$. Therefore, $T = A^\circ F: P \longrightarrow P$ is a completely continuous operator. So, BVP (1) is equivalent to the operator equation $u = Tu$, where the operator $T = P \longrightarrow P$ is given by

$$(Tv)(t) = \int_0^1 G(t, s)f(s, (Av)(s), (Av)'(s), -v(s), -v'(s))ds. \tag{14}$$

Let $\tilde{a} = (a, b, c)$ with $a, b, c \geq 0$. We define an operator $T_{\tilde{a}}$ on cone P :

$$\begin{aligned}
(T_{\tilde{a}}u)(t) &= a \int_0^1 G(t, s) \int_0^1 G(s, \tau)u(\tau)d\tau ds \\
&+ b \int_0^1 G(t, s) \int_0^1 |G_s(s, \tau)|u(\tau)d\tau ds \\
&+ c \int_0^1 G(t, s)u(s)ds, \quad u \in P.
\end{aligned} \tag{15}$$

It is easy to see that $T_{\tilde{a}}: P \longrightarrow P$ is a linear operator. Let $u_0(t) = t(1-t)$, by (15), we obtain

$$\begin{aligned}
(T_{\tilde{a}}u_0)(t) &\geq at(1-t) \int_0^1 s(1-s) \int_0^1 G(s, \tau)\tau(1-\tau)d\tau ds + bt(1-t) \\
&\cdot \int_0^1 s(1-s) \int_0^1 |G_s(s, \tau)|\tau(1-\tau)d\tau ds + ct(1-t) \int_0^1 s(1-s)s(1-s)ds = t(1-t) \\
&\cdot \left[a \int_0^1 s(1-s) \int_0^1 G(s, \tau)\tau(1-\tau)d\tau ds + b \int_0^1 s(1-s) \int_0^1 |G_s(s, \tau)|\tau(1-\tau)d\tau ds + c \int_0^1 s^2(1-s)^2 ds \right].
\end{aligned} \tag{16}$$

Then, if scalars $a, b, c \geq 0$, not all equal to zero, there exists a constant $\alpha > 0$ satisfies $T_{\tilde{a}}u_0 \geq \alpha u_0$. Thus, from Krein–Rutman theorem, we know $r(T_{\tilde{a}}) > 0$, and there is $v_{\tilde{a}} \in P \setminus \{\theta\}$ such that $\lambda_{\tilde{a}} T_{\tilde{a}} v_{\tilde{a}} = v_{\tilde{a}}$, where $\lambda_{\tilde{a}} = (r(T_{\tilde{a}}))^{-1}$ is the first eigenvalue of the operator $T_{\tilde{a}}$.

Now, we estimate the range of $\lambda_{\tilde{a}} = (r(T_{\tilde{a}}))^{-1}$. For any $u \in P$, from (15), we get

$$\begin{aligned}
|(T_{\tilde{a}}u)(t)| &\leq a \int_0^1 s(1-s) \int_0^1 G(s, \tau)u(\tau)d\tau ds \\
&+ b \int_0^1 s(1-s) \int_0^1 |G_s(s, \tau)|u(\tau)d\tau ds \\
&+ c \int_0^1 s(1-s)u(s)ds, \quad a, b, c \geq 0, \tilde{a} = (a, b, c),
\end{aligned} \tag{17}$$

thereby

$$\begin{aligned}
\|T_{\tilde{a}}u\| &\leq a \int_0^1 s(1-s) \int_0^1 G(s, \tau)u(\tau)d\tau ds \\
&+ b \int_0^1 s(1-s) \int_0^1 |G_s(s, \tau)|u(\tau)d\tau ds \\
&+ c \int_0^1 s(1-s)u(s)ds.
\end{aligned} \tag{18}$$

By the expression of Green's function $G(t, s)$ and (15), we infer that

$$\begin{aligned}
(T_{\tilde{a}}u)(t) &\geq t(1-t) \left[a \int_0^1 s(1-s) \int_0^1 G(s, \tau)u(\tau)d\tau ds \right. \\
&+ b \int_0^1 s(1-s) \int_0^1 |G_s(s, \tau)|u(\tau)d\tau ds \\
&\left. + c \int_0^1 s(1-s)u(s)ds \right], \quad a, b, c \geq 0, \tilde{a} = (a, b, c).
\end{aligned} \tag{19}$$

According to the above two inequalities, we have

$$(T_{\tilde{a}}u)(t) \geq t(1-t) \|T_{\tilde{a}}u\|. \tag{20}$$

Recalled that $u_0(t) = t(1-t)$. By (20), we have

$$T_{\tilde{a}}u_0 \geq \|T_{\tilde{a}}u_0\|u_0. \tag{21}$$

Note the positivity of operator $T_{\tilde{a}}$, we have $T_{\tilde{a}}^2 u_0 \geq \|T_{\tilde{a}}u_0\|^2 u_0$. Hence, for any $n \in \mathbb{N}$, using the recursion method to the above inequality, we get $T_{\tilde{a}}^n u_0 \geq \|T_{\tilde{a}}u_0\|^n u_0$. Therefore, we have

$$\|T_{\tilde{a}}^n u_0\| \geq \|T_{\tilde{a}}u_0\|^n \|u_0\|. \tag{22}$$

Thus,

$$\|T_a^-\|^n \geq \frac{\|T_a^m u_0\|}{\|u_0\|} \geq \|T_a^- u_0\|^n. \quad (23)$$

From this inequality and Gelfand formula on spectral radius, we obtain

$$r(T_a^-) = \lim_{n \rightarrow \infty} \sqrt[n]{\|T_a^m u_0\|} \geq \|T_a^- u_0\| > 0. \quad (24)$$

From (24), we get

$$\|T_a^-\| \geq r(T_a^-) \geq \|T_a^- u_0\|. \quad (25)$$

By $\lambda_a^- = (r(T_a^-))^{-1}$, we can get

$$\frac{1}{\|T_a^- u_0\|} \leq \lambda_a^- \leq \frac{1}{\|T_a^-\|}. \quad (26)$$

Now, we calculate $\|T_a^-\|$ and $\|T_a^- u_0\|$ as follows. We have known that

$$G(s, \tau) = \begin{cases} s(1-\tau), & 0 \leq s \leq \tau \leq 1, \\ \tau(1-s), & 0 \leq \tau \leq s \leq 1, \end{cases} \quad (27)$$

$$G_s(s, \tau) = \begin{cases} 1-\tau, & 0 \leq s \leq \tau \leq 1, \\ -\tau, & 0 \leq \tau < s \leq 1. \end{cases}$$

We first calculate $\|T_a^-\|$. From (15), we get

$$\begin{aligned} \|T_a^-\| &= \max_{t \in [0,1]} a \int_0^1 G(t, s) \int_0^1 G(s, \tau) u(\tau) d\tau ds \\ &\quad + b \int_0^1 G(t, s) \int_0^1 |G_s(s, \tau)| u(\tau) d\tau ds \\ &\quad + c \int_0^1 G(t, s) u(s) ds. \end{aligned} \quad (28)$$

Through integration, we have

$$\|T_a^-\| = \max_{t \in [0,1]} \left[t(1-t) \left(\frac{1}{12}a + \frac{1}{3}b + \frac{1}{2}c \right) \right] = \frac{1}{48}a + \frac{1}{12}b + \frac{1}{8}c. \quad (29)$$

Next, we calculate $\|T_a^- u_0\|$. Because

$$\begin{aligned} \|T_a^- u_0\| &= \max_{t \in [0,1]} \left[a \int_0^1 G(t, s) \int_0^1 G(s, \tau) u_0(\tau) d\tau ds \right. \\ &\quad + b \int_0^1 G(t, s) \int_0^1 |G_s(s, \tau)| u_0(\tau) d\tau ds \\ &\quad \left. + c \int_0^1 G(t, s) u_0(s) ds \right]. \end{aligned} \quad (30)$$

Through integration, we have

$$\begin{aligned} \|T_a^- u_0\| &= \max_{t \in [0,1]} \left[t(1-t) \left(\frac{1}{60}a + \frac{1}{15}b \right) + \left(\frac{1}{12}t - \frac{1}{6}t^3 + \frac{1}{12}t^4 \right) c \right] \\ &= \frac{1}{240}a + \frac{1}{60}b + \frac{5}{192}c. \end{aligned} \quad (31)$$

Combining this with (29), we can obtain

$$\frac{48}{a+4b+6c} \leq \lambda_a^- \leq \frac{960}{4a+16b+25c}, \quad a, b, c > 0, \tilde{a} = (a, b, c). \quad (32)$$

Now, we consider the special case with $\tilde{b} = (a_1, 0, c_1)$. We make a definition that

$$\begin{aligned} (T_b^- u)(t) &= a_1 \int_0^1 G(t, s) \int_0^1 G(s, \tau) u(\tau) d\tau ds \\ &\quad + c_1 \int_0^1 G(t, s) u(s) ds, \quad \tilde{b} = (a_1, 0, c_1). \end{aligned} \quad (33)$$

Because of $\lambda_b^- T_b^- v_b^-(t) = v_b^-(t)$, it is equivalent to λ_b^- and $v_b^-(t)$ satisfying the differential equation:

$$\begin{cases} u^{(4)}(t) = \lambda_b^- [a_1 u(t) - c_1 u''(t)], & t \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \quad (34)$$

We notice that the function $v_b^-(t)$ is nonnegative. After solving the above differential equations, we may take $v_b^-(t) = \sin \pi t$. Hence, we conclude the value of λ_b^- is $\lambda_b^- = \pi^4/a_1 + \pi^2 c_1$.

To prove our main results, we also need the two following lemmas.

Lemma 3 (see [13, 14]). *Let E is a Banach space, P is a cone in E , and $\Omega(P)$ is a bounded open set in P . Assume that $A: \overline{\Omega(P)} \rightarrow P$ is a completely continuous operator. If there exists $x_0 \in P/\{\theta\}$ such that*

$$x - Ax \neq \mu x_0, \quad \forall x \in \partial\Omega(P), \mu \geq 0. \quad (35)$$

Then, $i(A, \Omega(P), P) = 0$.

Lemma 4 (see [13, 14]). *Let E is a Banach space, P is a cone in E , $\Omega(P)$ is a bounded open set in P , and $\theta \in \Omega(P)$. Assume that $A: \overline{\Omega(P)} \rightarrow P$ is a completely continuous operator. If*

$$Ax \neq \mu x, \quad \forall x \in \partial\Omega(P), \mu \geq 1, \quad (36)$$

then $i(A, \Omega(P), P) = 1$.

3. Main Results

Theorem 1. *Suppose that conditions (H_1) , (H_2) , and (H_3) are satisfied, and*

$$\lambda_a^- > 1, \quad (37)$$

$$\lambda_b^- < 1. \quad (38)$$

λ_a^- and λ_b^- are the first eigenvalues of operator T_a^- and operator T_b^- , respectively. Then, BVP (1) has at least one positive solution.

Proof. It follows from (H_1) and (H_2) that, for some $r > 0$, the function f satisfies

$$\begin{aligned}
f(t, x_1, x_2, -x_3, x_4) &\geq a_1 x_1 + c_1 x_3, \\
x_1, x_3 &\in [0, r], \\
x_2 &\in [-r, r], \\
x_4 &\in (-\infty, +\infty).
\end{aligned} \tag{39}$$

It is known that v_b^- is the positive eigenvector of T_b^- corresponding to λ_b^- , and we have $v_b^- = \lambda_b^- T_b^- v_b^-$. According to (14), (38), and (39), for any $v \in \partial B_r \cap P$, we have

$$\begin{aligned}
|(Av)(t)| &= \int_0^1 G(t, s) |v(s)| ds \leq r \int_0^1 G(t, s) ds < r, \\
|(Av)'(t)| &= \int_0^1 |G_t(t, s)| |v(s)| ds \leq r \int_0^1 |G_t(t, s)| ds < r.
\end{aligned} \tag{40}$$

Thus, we get

$$\begin{aligned}
(Tv)(t) &= \int_0^1 G(t, s) f(s, (Av)(s), (Av)'(s), -v(s), -v'(s)) ds \\
&\geq \int_0^1 G(t, s) (a_1 (Av)(s) + c_1 v(s)) ds \\
&= a_1 \int_0^1 G(t, s) \int_0^1 G(s, \tau) v(\tau) d\tau ds \\
&\quad + c_1 \int_0^1 G(t, s) v(s) ds = (T_b^- v)(t), \quad t \in [0, 1].
\end{aligned} \tag{41}$$

Let us suppose that T has no fixed point on $\partial B_r \cap P$; otherwise, Theorem 1 is proved.

Now, we prove

$$v - Tv \neq \zeta v_b^-, \quad \forall v \in \partial B_r \cap P, \zeta \geq 0, \tag{42}$$

where $B_r = \{x \in C[0, 1] \mid \|x\| \leq r\}$. If otherwise, there exists $v_0 \in \partial B_r \cap P$ and $\zeta_0 \geq 0$ such that $v_0 - Tv_0 = \zeta_0 v_b^-$. Then, we have $\zeta_0 > 0$ and $v_0 = Tv_0 + \zeta_0 v_b^- \geq \zeta_0 v_b^-$. Let $\zeta^* = \sup \{\zeta \mid v_0 \geq \zeta v_b^-\}$, we can know $\zeta^* \geq \zeta_0 > 0$ and $v_0 \geq \zeta^* v_b^-$. By $T_b^-(P) \subset P$, we get $\lambda_b^- T_b^- v_0 \geq \zeta^* \lambda_b^- T_b^- v_b^- = \zeta^* v_b^-$. Based on (41), we have

$$v_0 = Tv_0 + \zeta_0 v_b^- \geq T_b^- v_0 + \zeta_0 v_b^- \geq \left(\frac{\zeta^*}{\lambda_b^-} + \zeta_0 \right) v_b^-. \tag{43}$$

Then, by the definition of ζ^* , we conclude that $\zeta^* \geq \zeta^* / \lambda_b^- + \zeta_0$. By (15), we notice that $\lambda_b^- < 1$ and $\zeta_0 > 0$. So, the above inequality contradicts the definition of ζ^* . Hence, (42) holds. By Lemma 3, we have

$$i(T, B_r \cap P, P) = 0. \tag{44}$$

From (H_3) , we have

$$\begin{aligned}
(Tv)(t) &\leq \int_0^1 G(t, s) \left(a \int_0^1 G(s, \tau) v(\tau) d\tau + b \int_0^1 |G_s(s, \tau)| |v(\tau)| d\tau + cv(s) + d|v'(s)|^q + M \right) ds \\
&\leq a \int_0^1 G(t, s) \int_0^1 G(s, \tau) v(\tau) d\tau ds + b \int_0^1 G(t, s) \int_0^1 |G_s(s, \tau)| |v(\tau)| d\tau ds + c \int_0^1 G(t, s) v(s) ds + d \int_0^1 G(t, s) \frac{|v(s)|^q}{s^q(1-s)^q} ds + M \\
&= (T_a^- v)(t) + d \int_0^1 G(t, s) \frac{|v(s)|^q}{s^q(1-s)^q} ds \leq (T_a^- v)(t) + d \int_0^1 G(t, s) \frac{|v(s)|^q}{s^q(1-s)^q} ds + M \\
&\leq (T_a^- v)(t) + d \int_0^1 s(1-s) \frac{|v(s)|^q}{s^q(1-s)^q} ds + M \leq (T_a^- v)(t) + d \int_0^1 |v(s)|^q ds + M.
\end{aligned} \tag{45}$$

Note that $(r(T_a^-))^{-1} > 1$. Take

$$\varepsilon = \frac{1 - r(T_a^-)}{2}. \tag{46}$$

According to Gelfand formula, $r(T_a^-) = \lim_{n \rightarrow \infty} \sqrt[n]{\|T_a^{-n}\|}$, and there is a natural number N , so that, for $n \geq N$, $\|T_a^{-n}\| \leq [r(T_a^-) + \varepsilon]^n$.

For any $v \in C[0, 1]$, we define

$$\|v\|^* = \sum_{i=1}^N [r(T_a^-) + \varepsilon]^{N-i} \|T_a^{i-1} v\|, \tag{47}$$

where $T_1^0 = I$ is an identity operator. It is not difficult to verify that $\|\cdot\|^*$ is a norm in $C[0, 1]$, and we have

$$\left\| \int_0^1 v^q(s) ds \right\|^* \leq K \|v\|^{*q}, \quad v \in C[0, 1], \tag{48}$$

where K is a constant. Take

$$r_1 > \max \left\{ 1, r, 4 \|M\|^* \varepsilon^{-1}, (4dK\varepsilon^{-1})^{1/(1-q)} \right\}. \tag{49}$$

Because of $\|v\|^* \geq [r(T_a^-) + \varepsilon]^{N-1} \|v\|$, we can take $r_2 > r_1$ so that when $\|v\| \geq r_2$, we have $\|v\|^* > r_1$.

Next, we prove

$$Tv \neq \mu v, \quad \forall v \in \partial B_{r_2} \cap P, \mu \geq 1. \quad (50)$$

If there exists $v_1 \in \partial B_{r_2} \cap P$ and $\mu_0 \geq 1$, such that $Tv_1 = \mu_0 v_1$, by Lemma 1 and (45), we can obtain

$$\begin{aligned} (Tv_1)(t) &= \int_0^1 G(t, s) f(s, (Av_1)(s), (Av_1)'(s), v_1(s), v_1'(s)) ds \\ &\leq (T_{\tilde{a}} v_1)(t) + d \int_0^1 v_1^q(s) ds + M. \end{aligned} \quad (51)$$

Hence, we have

$$\begin{aligned} 0 \leq \mu_0 v_1(t) = (Tv_1)(t) &\leq (T_{\tilde{a}} v_1)(t) + d \int_0^1 v_1^q(s) ds + M, \\ &\forall t \in [0, 1]. \end{aligned} \quad (52)$$

Furthermore, $T_{\tilde{a}}(P) \subset P$, and we have

$$\begin{aligned} 0 \leq (T_{\tilde{a}}^j(Tv_1))(t) &\leq \left(T_{\tilde{a}}^j \left(T_{\tilde{a}} v_1 + d \int_0^1 v_1^q(s) ds + M \right) \right)(t), \\ &\forall t \in [0, 1], j = 1, 2, \dots, N-1, \end{aligned} \quad (53)$$

and we also get

$$\begin{aligned} \|T_{\tilde{a}}^j(Tv_1)\| &\leq \left\| T_{\tilde{a}}^j \left(T_{\tilde{a}} v_1 + d \int_0^1 v_1^q(s) ds + M \right) \right\|, \\ j &= 1, 2, \dots, N-1. \end{aligned} \quad (54)$$

Therefore, from (47), we have

$$\begin{aligned} \|Tv_1\|^* &= \sum_{i=1}^N [r(T_{\tilde{a}}) + \varepsilon]^{N-i} \|T_{\tilde{a}}^{i-1}(Tv_1)\| \\ &\leq \sum_{i=1}^N [r(T_{\tilde{a}}) + \varepsilon]^{N-i} \left\| T_{\tilde{a}}^{i-1} \left(T_{\tilde{a}} v_1 + d \int_0^1 v_1^q(s) ds + M \right) \right\| \\ &= \left\| T_{\tilde{a}} v_1 + d \int_0^1 v_1^q(s) ds + M \right\|^*. \end{aligned} \quad (55)$$

When $\|v_1\| = r_2$, we have $\|v_1\|^* > r_1$. Hence, from (48) and (55), it follows that

$$\begin{aligned} \mu_0 \|v_1\|^* &= \|Tv_1\|^* \leq \left\| T_{\tilde{a}} v_1 + d \int_0^1 v_1^q(s) ds + M \right\|^* \leq \|T_{\tilde{a}} v_1\|^* + \left\| d \int_0^1 v_1^q(s) ds \right\|^* \\ &\leq \|T_{\tilde{a}} v_1\|^* + dK \|v_1\|^{*q} + \frac{\varepsilon}{4} \|v_1\|^* = \sum_{i=1}^{N-1} [r(T_{\tilde{a}}) + \varepsilon]^{N-i} \|T_{\tilde{a}}^i v_1\| + dK \|v_1\|^{*q} + \frac{\varepsilon}{4} \|v_1\|^* \\ &\leq [r(T_{\tilde{a}}) + \varepsilon] \sum_{i=1}^{N-1} [r(T_{\tilde{a}}) + \varepsilon]^{N-i-1} \|T_{\tilde{a}}^i v_1\| + \|T_{\tilde{a}}^N v_1\| + \frac{\varepsilon}{4} \|v_1\|^{*q} r_1^{1-q} + \frac{\varepsilon}{4} \|v_1\|^* \\ &\leq [r(T_{\tilde{a}}) + \varepsilon] \sum_{i=1}^{N-1} [r(T_{\tilde{a}}) + \varepsilon]^{N-i-1} \|T_{\tilde{a}}^i v_1\| [r(T_{\tilde{a}}) + \varepsilon]^N \|v_1\| + \frac{\varepsilon}{4} \|v_1\|^* + \frac{\varepsilon}{4} \|v_1\|^* \\ &\leq [r(T_{\tilde{a}}) + \varepsilon] \sum_{i=1}^{N-1} [r(T_{\tilde{a}}) + \varepsilon]^{N-i} \|T_{\tilde{a}}^{i-1} v_1\| + \frac{\varepsilon}{2} \|v_1\|^* \\ &= [r(T_{\tilde{a}}) + \varepsilon] \|v_1\|^* + \frac{\varepsilon}{2} \|v_1\|^* \leq \left[r(T_{\tilde{a}}) + \frac{3\varepsilon}{2} \right] \|v_1\|^*. \end{aligned} \quad (56)$$

By $\mu_0 \geq 1$ and (56), we can get $1 \leq r(T_{\tilde{a}}) + 3\varepsilon/2$. But it is contradictory to the definition of $\varepsilon = 1 - r(T_{\tilde{a}})/2$. Hence, (50) holds. According to Lemma 4, we obtain that

$$i(T, B_{r_2} \cap P, P) = 1. \quad (57)$$

Now, from (44) and (57), it follows that

$$i\left(T, \frac{(B_{r_2} \cap P)}{(\overline{B_r} \cap P)}, P\right) = i(T, B_{r_2} \cap P, P) - i(T, B_r \cap P, P) = 1. \quad (58)$$

Hence, T has at least one fixed point in $(B_{r_2} \cap P)/(\overline{B_r} \cap P)$. That is to say, BVP (1) has at least one positive solution. The proof of Theorem 1 is completed.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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