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Research Article

Synchronized Chaos of a Three-Dimensional System with Quadratic Terms

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In this paper, a novel chaotic new three-dimensional system has been studied by Zhang et al. in 2012. In the system, there are three control parameters and three different nonlinear terms which governed equations. Zhang et al. studied elementary (preliminary) dynamic properties of the chaotic new three-dimensional system by means of bifurcation diagram, maximum Lyapunov exponent, phase portraits, dynamics behaviors by changing some parameters etc., using all possible theoretical analysis and numerical simulation. In this paper, we have demonstrated its complete synchronization. The proposed results are verified by numerical simulations.

1. Introduction

Since the first chaotic attractor in a three-dimensional autonomous system was found by Loranz in 1963 [1], many researchers and groups have concentrated in these three-dimensional systems recently. After the Loranz chaotic system, several chaotic systems have been found in the past few decades, for example, the Rösslor system [2], Lü and Chen system [3], Chua et al.'s system [4], Chen and Ueta system [5], and many others. Some proposed systems have studied by adding a nonlinear term in each equation (see [6], for more details). Zhang et al. [7] set up a novel three-dimensional autonomous system. In this system, they added to the second equation of the Rucklidge system [8] one cross-product nonlinear term only. The additional term created complex dynamics behaviors that can be shown in the numerical simulation. Zhang et al.'s system shows similar behavior as the Lorenz chaotic system by varying the control parameters. The study of the system covers most of the elementary dynamic properties, including phase portraits, Lyapunov exponent spectra, and bifurcation diagram, in each study cases.

In this paper, the synchronized chaos of Zhang et al.'s system has been provided which can finish the dynamical study of the system. Chaotic behaviors are very sensitive to initial conditions. If the nearby trajectories start, then we may have a huge difference in the future. We can synchronize the evolution of two nearby trajectories using a suitable control with the presence of a time lag. Literature reviews about synchronization schemes exist in numerous applications such as [9–12]. As pointed by [13], the concomitant unpredictability along with chaos poses certain difficulties in decoding the chaotic systems. Successful chaos synchronisation schemes can be found such as adaptive control [14], back stepping design [15], active control [16, 17], and nonlinear control [18, 19]. We herewith present a chaotic three-dimensional system, brief previous results from [7], and demonstrate how the active control can be employed to synchronize Zhang et al.'s system.

2. Preliminary Results

The new chaotic attractor proposed in this article is motivated by Zhang et al.'s system. It is given by the following system of ordinary differential equations:

$$\begin{aligned}\dot{x} &= -ax + by - yz,\\ \dot{y} &= x + xz,\\ \dot{z} &= -cz + y^2,\end{aligned} \tag{1}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}\tau$, x, y, and z are state of three variables, and a, b, and c are control parameters of system (1). In the topological point of view, Zhang et al. show that system (1) is not equivalent to the original Lorenz chaotic system or others similar to the Lorenz system. It has been presented using the effort of Ćelikovský and Chen [20–23] to the evident that system (1) is not topologically equivalent to the Lorenz chaotic system.

Preliminary results have been calculated. Equilibrium points can be obtained by setting $\dot{x} = \dot{y} = \dot{z} = 0$. Thus, we have

$$\begin{aligned}S_1 &= (0, 0, 0),\\ S_{2,3} &= (0, \pm\sqrt{bc}, b).\end{aligned} \tag{2}$$

The equilibrium point S_1 represents two eigenvalues with negative real part (stable) and one eigenvalue with non-negative real part (unstable) which is so-called saddle points of index one. However, the equilibrium point $S_{2,3}$ represent one eigenvalue with negative real part (stable) and two becoming a pair of complex conjugate eigenvalues with positive real parts which is so-called saddle points of index two.

System (1) is clearly dissipative. We have three Lyapunov exponents, and the divergence of the system is

$$f = \Delta V = \sum_{i=1}^{3} \mathrm{LE}_i = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(a + c), \tag{3}$$

where LE_i denotes the three Lyapunov exponents of the system. It is negative when $a, c \geq 0$. The exponent rate is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = e^{f} = e^{-(a+c)}. \tag{4}$$

From (4), it can be seen that a volume element V_0 is deceased by the flow into a volume element $V_0 e^{-\mathrm{ft}}$ in time t. This means that each volume containing the system trajectory tends to zero as $t \longrightarrow \infty$ at an exponential rate of $-f$.

The Lyapunov exponent is defined as the rate of divergence and convergence of nearby trajectories in the phase space [24,25]. According to the Jacobian method, the Lyapunov exponent can be calculated and presented. In system (1), we have seen one positive, one negative, and zero Lyapunov exponent for certain initial conditions, namely, $(x_0 = 7.2, y_0 = 7.8, z_0 = 2.3)$. Therefore, system (1) can be defined to be chaotic because we have at least one positive Lyapunov exponent, namely, we have $(l_1 = 2.46, l_2 = 0, l_3 = -14.45)$. In addition, the Lyapunov exponent dimension of system (1) is obtained by

$$D_l = j + \frac{\sum_{i=1}^{j} l_i}{\left|j_{j+1}\right|} = 2 + \frac{l_1 + l_2}{\left|l_3\right|} = 2.17. \tag{5}$$

Since the Lyapunov exponent dimension is fractal in system (1) and the system is dissipated, then a strange attractor is observed.

The dynamical studies of system (1) and the observation of the chaotic complex dynamics are well explained in [7], including the phase portrait, calculation of the maximum Lyapunov exponent, as well as the bifurcation diagram. All these dynamic properties are computed by fixing two control parameters and varying the third one. Thus, three complete cases have been studied.

3. Synchronization of the Chaotic System

In this section, we present synchronization of system (1). Consider the master system as

$$\begin{aligned}\dot{x}_1 &= -ax_1 + by_1 - y_1 z_1,\\ \dot{y}_1 &= x_1 + x_1 z_1,\\ \dot{z}_1 &= -cz_1 + y_1^2,\end{aligned} \tag{6}$$

and the slave system as

$$\begin{aligned}\dot{x}_2 &= -ax_2 + by_2 - y_2 z_2 + u_1,\\ \dot{y}_2 &= x_2 + x_2 z_2 + u_2,\\ \dot{z}_2 &= -cz_2 + y_2^2 + u_3,\end{aligned} \tag{7}$$

where $a = 4, b = 28$, and $c = 2$ and u_1, u_2, and u_3 are control terms to be determined. The initial conditions are chosen to be

$$x_1(0), y_1(0), z_1(0), x_2(0), y_2(0), z_2(0) = (0, -12, 33, -3, -14, 30). \tag{8}$$

Let us define the error functions as follows:

$$\begin{aligned}e_1 &= x_2 - x_1,\\ e_2 &= y_2 - y_1,\\ e_3 &= z_2 - z_1.\end{aligned} \tag{9}$$

The error system can be obtained by substituting equation (9) into systems (6) and (7) as follows:

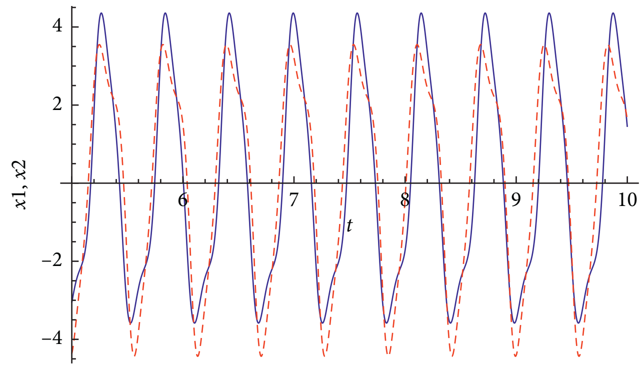

FIGURE 1: Time series of x_1 and x_2. The solid blue line represents x_1, and the dashed red line represents x_2.

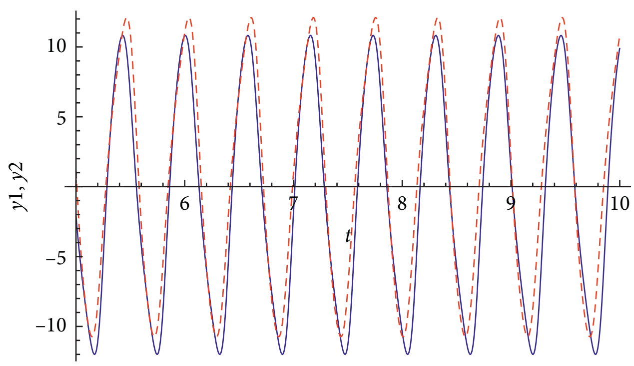

FIGURE 2: Time series of y_1 and y_2. The solid blue line represents y_1, and the dashed red line represents y_2.

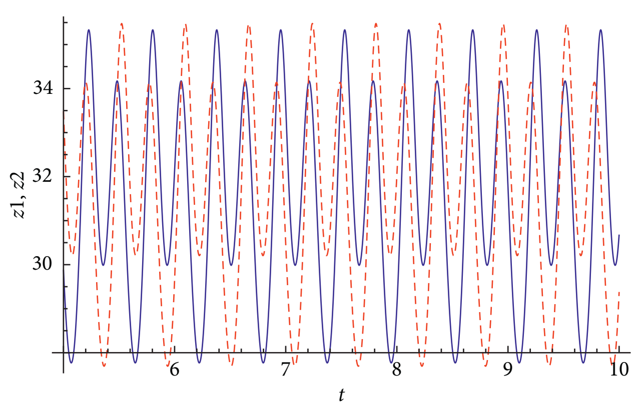

FIGURE 3: Time series of z_1 and z_2. The solid blue line represents z_1, and the dashed red line representsz_2.

$$\begin{aligned} \dot{e}_1 &= -ae_1 + be_2 + \left(y_2 z_2 - y_1 z_1\right) + u_1, \\ \dot{e}_2 &= e_1 + \left(x_2 z_2 - x_1 z_1\right) + u_2, \\ \dot{e}_3 &= -ce_3 + \left(y_2^2 - y_1^2\right) + u_3, \end{aligned} \tag{10}$$

with some choice of initial conditions $e_1(0), e_2(0)$, and $e_3(0)$.

The control functions u_i have been chosen to be stable system (10) which means that $\lim_{t\longrightarrow\infty} e_i(t) = 0$. Suppose

$$\begin{aligned} u_1 &= y_1 z_1 - y_2 z_2, \\ u_2 &= x_1 z_1 - x_2 z_2, \\ u_3 &= y_1^2 - y_2^2, \end{aligned} \tag{11}$$

System (10) can be written as

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -a & b & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{12}$$

From the coefficient matrix of error system (12), we can calculate the eigenvalues which are

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}\left(\pm \sqrt{a^2 + 4b} - a \right), \\ \lambda_3 &= -c. \end{aligned} \tag{13}$$

The system is stable as all the eigenvalues are negative, and synchronization is achieved when $b = 0$ and $a \geq 0$. From Figures 1–3, we show the synchronization for $x(t)$, $y(t)$, and $z(t)$, respectively. A solid blue line represents the response system and red lines represent to the drive system.

4. Conclusion

To sum up, a new chaotic system by Zhang et al. [7] presented a three-dimensional autonomous chaotic system with quadratic terms. The basic properties of the system have been already investigated in more detail by using the method of fixing the parameters and varying only one. We have observed the dynamic behavior for the system and calculated the preliminary dynamical studies, including bifurcation diagram, maximum Lyapunov exponent, and phase portraits. Since we have the chaotic attractors which are very important in some applications and some related areas, we have provided synchronization of the chaotic system using the active control method. On the other hand, we hope the synchronization of the chaotic system finds some interesting problems to apply.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this manuscript.

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