Research Article

Second-Order Differential Equation: Oscillation Theorems and Applications

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Differential equations of second order appear in a wide variety of applications in physics, mathematics, and engineering. In this paper, necessary and sufficient conditions are established for oscillations of solutions to second-order half-linear delay differential equations of the form $(\varsigma(y)(u'(y))^a)' + p(y)u^c(\vartheta(y)) = 0$, for $y \geq y_0$, under the assumption $\int_{y_1}^{\infty} (\varsigma(\eta))^{-(1/a)} = \infty$. Two cases are considered for $a < c$ and $a > c$, where a and c are the quotients of two positive odd integers. Two examples are given to show the effectiveness and applicability of the result.

1. Introduction

Differential equations (DEs) have received a lot of attention, and it is an active research area among scientists and engineers [1–8]. The DEs have ability to formulate many complex phenomena in various fields such as biology, fluid mechanics, plasma physics, fluid mechanics, and optics; many exact and numerical schemes have been being derived such as [9–15]. Differential equation of second order appears in models as well as in physical applications such as fluid dynamics, electromagnetism, acoustic vibrations and quantum mechanics, biological, physical and chemical phenomena, optimization, mathematics of networks, and dynamical systems (see [16–24]).

In this article, we consider the differential equation

$$(\varsigma(y)(u'(y))^a)' + p(y)u^c(\vartheta(y)) = 0, \quad \text{for } y \geq y_0, \quad (1)$$

where a and c are the quotient of two positive odd integers, and the functions p , ς , and ϑ are continuous that satisfy the conditions stated below:

- (A1) $\vartheta \in C([0, \infty), \mathbb{R})$, $\vartheta(y) < y$, $\lim_{y \rightarrow \infty} \vartheta(y) = \infty$.
- (A2) $\varsigma \in C^1([0, \infty), \mathbb{R})$, $p \in C([0, \infty), \mathbb{R})$; $0 < \varsigma(y)$, $0 \leq p(y)$ for all $y \geq 0$; $p(y)$ is not identically zero in any interval $[b, \infty)$.
- (A3) $Y(y) = \int_{y_1}^y \varsigma^{-1/a}(\eta) d\eta$ with $\lim_{y \rightarrow \infty} Y(y) = \infty$.
- (A4) the existence of a differentiable function ϑ_0 such that $0 < \vartheta_0(y) \leq \vartheta(y)$, for $\vartheta'_0(y) \geq \vartheta_0 > 0$, for $y \geq y_0$.

In [25, 26], Baculiková and Džurina have considered

$$\begin{aligned} &(\varsigma(y)(z'(y))^a)' + p(y)u^c(\vartheta(y)) = 0, \\ &z(y) = u(y) + q(y)u(\tau(y)), \quad y \geq y_0, \end{aligned} \quad (2)$$

and obtained oscillation criteria for the solutions of (2) using comparison techniques when $a = c = 1$, $0 \leq q(y) < \infty$, and $\lim_{y \rightarrow \infty} Y(y) = \infty$. In the same technique, Džurina and Džurina [27] have studied the oscillatory behavior of the solutions of (2) under the assumptions $0 \leq q(y) < \infty$ and $\lim_{y \rightarrow \infty} Y(y) = \infty$. In [28], Bohner et al. have studied the oscillatory behavior of solutions of (2) under $a = c$, $\lim_{y \rightarrow \infty} Y(y) < \infty$, and $0 \leq q(y) < 1$. Grace et al. [29] have studied the oscillatory behavior of (2) when $a = c$ and $\lim_{y \rightarrow \infty} Y(y) < \infty$ and $\lim_{y \rightarrow \infty} Y(y) = \infty$ and $0 \leq q(y) < 1$. In [30], Ali has studied the oscillatory behavior of the solutions of (2), under the assumptions $\lim_{y \rightarrow \infty} Y(y) < \infty$ and $q(y) \geq 0$. Karpuz and Santra [31] have studied the oscillatory behavior of

$$(\zeta(y)(u(y) + q(y)u(\tau(y)))')' + p(y)f(u(\vartheta(y))) = 0, \quad (3)$$

by considering the assumptions $\lim_{y \rightarrow \infty} Y(y) < \infty$ and $\lim_{y \rightarrow \infty} Y(y) = \infty$ for different ranges of the neutral coefficient q .

For further work on the oscillation of this type of equations, we refer the readers to the references. Note that the majority of works consider only sufficient conditions, and merely a few consider both necessary and sufficient conditions. Hence, the objective of this work is to establish both necessary and sufficient conditions for the oscillation of solutions of (1) without using the comparison and the Riccati techniques. In this paper, we restrict our attention to the study (1), which includes the class of functional differential equations of neutral type.

Remark 1. When the domain is not specified explicitly, all functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all y large enough.

2. Necessary and Sufficient Conditions

Lemma 1. *Let (A1)–(A3) hold and that u is an eventually positive solution of (1). Then, there exist $y_1 \geq y_0$ and $d > 0$ such that*

$$0 < u(y) \leq dY(y), \quad (4)$$

$$Y(y) \left[\int_y^\infty p(\zeta)u^c(\vartheta(\zeta))d\zeta \right]^{1/a} \leq u(y), \quad (5)$$

for $y \geq y_1$.

Proof. Let u be an eventually positive solution of (1). Then, by (A1), there exists a y^* such that $u(y) > 0$ and $u(\vartheta(y)) > 0$ for all $y \geq y^*$. From (1) it follows that

$$(\zeta(y)(u'(y))^a)' = -p(y)u^c(\vartheta(y)) \leq 0. \quad (6)$$

Therefore, $\zeta(y)(u'(y))^a$ is nonincreasing for $y \geq y^*$. Next, we show that $\zeta(y)(u'(y))^a$ is positive. By contradiction, assume that $\zeta(y)(u'(y))^a \leq 0$ at a certain time

$y \geq y^*$. Using that p is not identically zero on any interval $[b, \infty)$ and by (6), there exists $y_1 \geq y^*$ such that

$$\zeta(y)(u'(y))^a \leq \zeta(y_1)(u'(y_1))^a < 0, \quad \text{for all } y \geq y_1. \quad (7)$$

Recall that a is the quotient of two positive odd integers. Then,

$$u'(y) \leq \left(\frac{\zeta(y_1)}{\zeta(y)} \right)^{1/a} u'(y_1), \quad \text{for } y \geq y_1. \quad (8)$$

Integrating from y_2 to y , we have

$$u(y) \leq u(y_1) + (\zeta(y_1))^{1/a} u'(y_1) Y(y). \quad (9)$$

By (A3), the right-hand side approaches $-\infty$; then, $\lim_{y \rightarrow \infty} u(y) = -\infty$. This is a contradiction to the fact that $u(y) > 0$. Therefore, $\zeta(y)(u'(y))^a > 0$ for all $y \geq y^*$. From $\zeta(y)(u'(y))^a$ being nonincreasing, we have

$$u'(y) \leq \left(\frac{\zeta(y_1)}{\zeta(y)} \right)^{1/a} u'(y_1), \quad \text{for } y \geq y_1. \quad (10)$$

Integrating this inequality from y_1 to y and using that u is continuous,

$$u(y) \leq u(y_1) + (\zeta(y_1))^{1/a} u'(y_1) Y(y). \quad (11)$$

Since $\lim_{y \rightarrow \infty} Y(y) = \infty$, there exists a positive constant d such that (4) holds.

Since $\zeta(y)(u'(y))^a$ is positive and nonincreasing, $\lim_{y \rightarrow \infty} \zeta(y)(u'(y))^a$ exists and is nonnegative. Integrating (1) from y to b , we have

$$\zeta(b)(u'(b))^a - \zeta(y)(u'(y))^a + \int_y^\infty p(\eta)u^c(\vartheta(\eta))d\eta = 0. \quad (12)$$

Letting limit as $b \rightarrow \infty$, we get

$$\zeta(y)(u'(y))^a \geq \int_y^\infty p(\eta)u^c(\vartheta(\eta))d\eta. \quad (13)$$

Then,

$$u'(y) \geq \left[\frac{1}{\zeta(y)} \int_y^\infty p(\eta)u^c(\vartheta(\eta))d\eta \right]^{1/a}. \quad (14)$$

Since $u(y_1) > 0$, integrating the above inequality yields

$$u(y) \geq \int_{y_1}^y \left[\frac{1}{\zeta(\eta)} \int_\eta^\infty p(\zeta)u^c(\vartheta(\zeta))d\zeta \right]^{1/a} d\eta. \quad (15)$$

Since the integrand is positive, we can increase the lower limit of integration from η to y and then use the definition of $Y(y)$ to obtain

$$u(y) \geq Y(y) \left[\int_y^\infty p(\zeta)u^c(\vartheta(\zeta))d\zeta \right]^{1/a}, \quad (16)$$

which yields (5). $\square$

Theorem 1. *Assume that there exists a constant b_1 , the quotient of two positive odd integers, such that $0 < c < b_1 < a$.*

If (A1)–(A3) hold, then each solution of (1) is oscillatory if and only if

$$\int_0^\infty p(\zeta)Y^c(\vartheta(\zeta))d\zeta = \infty. \quad (17)$$

Proof. On the contrary, we assume that u is eventually positive solution. So, Lemma 1 holds, and then there exists $y_1 \geq y_0$ such that

$$u(y) \geq Y(y)w^{1/a}(y) \geq 0, \quad \text{for } y \geq y_1, \quad (18)$$

where

$$w(y) = \int_y^\infty p(\zeta)u^c(\vartheta(\zeta))d\zeta. \quad (19)$$

Computing the derivative of w , we have

$$w'(y) = -p(y)u^c(\vartheta(y)). \quad (20)$$

Thus, w is nonnegative and nonincreasing. Since $u > 0$, by (A2), it follows that $p(y)u^c(\vartheta(y))$ cannot be identically zero in any interval $[b, \infty)$; thus, w' cannot be identically zero, and w cannot be constant on any interval $[b, \infty)$. Therefore, $w(y) > 0$ for $y \geq y_1$. Computing the derivative, we have

$$(w^{1-b_1/a}(y))' = \left(1 - \frac{b_1}{a}\right)w^{-b_1/a}(y)w'(y). \quad (21)$$

Integrating (21) from y_2 to y and using that $w > 0$, we have

$$\begin{aligned} w^{1-b_1/a}(y_2) &\geq \left(1 - \frac{b_1}{a}\right) \left[- \int_{y_2}^y w^{-b_1/a}(\zeta)w'(\zeta)d\zeta \right] \\ &= \left(1 - \frac{b_1}{a}\right) \left[\int_{y_2}^y w^{-b_1/a}(\zeta)(p(\zeta)u^c(\vartheta(\zeta)))d\zeta \right]. \end{aligned} \quad (22)$$

Next, we find a lower bound for the right-hand side of (25), independent of the solution u . By (4) and (19), we have

$$\begin{aligned} u^c(y) &= u^{c-b_1}(y)u^{b_1}(y) \geq (dY(y))^{c-b_1}u^{b_1}(y) \\ &\geq (dY(y))^{c-b_1}(Y(y)w^{1/a}(y))^{b_1} \\ &= d^{c-b_1}Y^c(y)w^{b_1/a}(y), \quad \text{for } y \geq y_2. \end{aligned} \quad (23)$$

Since w is nonincreasing, $b_1/a > 0$, and $\vartheta(\eta) < \eta$, it follows that

$$\begin{aligned} u^c(\vartheta(\eta)) &\geq d^{c-b_1}Y^c(\vartheta(\eta))w^{b_1/a}(\vartheta(\eta)) \\ &\geq d^{c-b_1}Y^c(\vartheta(\eta))w^{b_1/a}(\eta). \end{aligned} \quad (24)$$

Going back to (22), we have

$$w^{1-b_1/a}(y_2) \geq \left(1 - \frac{b_1}{a}\right) d^{c-b_1} \left[\int_{y_2}^y p(\eta)Y^c(\vartheta(\eta))d\eta \right]. \quad (25)$$

Since $(1 - b_1/a) > 0$, by (17) the right-hand side approaches $+\infty$ as $y \rightarrow \infty$. This contradicts (25) and completes the proof of sufficiency for eventually positive solutions.

The eventually negative solution can be dealt similarly by introducing the variables $v = -u$.

Next, we show the necessity part by a contrapositive argument. If (17) does not hold, then for each $\kappa > 0$ there exists $y_1 \geq y_0$ such that

$$\int_{\eta}^{\infty} p(\zeta)Y^c(\vartheta(\zeta))d\zeta \leq \frac{\kappa^{(1-c/a)}}{2}, \quad (26)$$

for all $\eta \geq y_1$. We define the set of continuous functions

$$S = \left\{ u \in C([0, \infty)): \left(\frac{\kappa}{2}\right)^{1/a} Y(y) \leq u(y) \leq \kappa^{1/a} Y(y), \quad y \geq y_1 \right\}. \quad (27)$$

We define an operator Ω on S by

$$(\Omega u)(y) = \begin{cases} 0, & \text{if } y \leq y_1, \\ \int_{y_1}^y \left[\frac{1}{\varsigma(\eta)} \left[\frac{\kappa}{2} + \int_{\eta}^{\infty} p(\zeta)u^c(\vartheta(\zeta))d\zeta \right] \right]^{1/a} d\eta, & \text{if } y > y_1. \end{cases} \quad (28)$$

Note that when u is continuous, Ωu is also continuous on $[0, \infty)$. If u is a fixed point of Ω , i.e., $\Omega u = u$, then u is a solution of (1).

First, we estimate $(\Omega u)(y)$ from below. By (A3), we have

$$(\Omega u)(y) \geq \int_{y_1}^y \left[\frac{1}{\varsigma(\eta)} \left[\frac{\kappa}{2} + 0 \right] \right]^{1/a} d\eta = \left(\frac{\kappa}{2}\right)^{1/a} Y(y). \quad (29)$$

Now, we estimate $(\Omega u)(y)$ from above. For u in S , we have $u^c(\vartheta(\zeta)) \leq (\kappa^{1/a} Y(\vartheta(\zeta)))^c$. Then, by (26),

$$\begin{aligned} (\Omega u)(y) &\leq \int_{y_1}^y \left[\frac{1}{\varsigma(\eta)} \left[\frac{\kappa}{2} + \int_{\eta}^{\infty} p(\zeta)u^c(\vartheta(\zeta))d\zeta \right] \right]^{1/a} d\eta \\ &\leq \kappa^{1/a} Y(y). \end{aligned} \quad (30)$$

Therefore, Ω maps S to S .

Next, we find a fixed point for Ω in S . Let us define a sequence of functions in S by the recurrence relation

$$\begin{aligned}
v_0(y) &= 0, \quad \text{for } y \geq y_0, \\
v_1(y) &= (\Omega v_0)(y) = \begin{cases} 0, & \text{if } y < y_1, \\ \kappa^{1/a} \Upsilon(y), & \text{if } y \geq y_1, \end{cases} \\
v_{n+1}(y) &= (\Omega v_n)(y), \quad \text{for } n \geq 1, y \geq y_1.
\end{aligned} \quad (31)$$

Note that for each fixed y , we have $v_1(y) \geq v_0(y)$. Using mathematical induction, we can show that $v_{n+1}(y) \geq v_n(y)$. Therefore, the sequence $\{v_n\}$ converges pointwise to a function v . Using the Lebesgue dominated convergence theorem, we can show that v is a fixed point of Ω in S . This shows under assumption (26), there is a nonoscillatory solution that does not converge to zero. This completes the proof. $\square$

Theorem 2. Assume that there exists a constant b_2 , the quotient of two positive odd integers such that $0 < a < b_2 < c$. If (A1)–(A4) hold and $\zeta(y)$ is nondecreasing, then each solution of (1) is oscillatory if and only if

$$\int_{y_1}^{\infty} \left[\frac{1}{\zeta(s)} \int_s^{\infty} p(\zeta) d\zeta \right]^{1/a} d\eta = \infty. \quad (32)$$

Proof. On the contrary, we assume that u is an eventually positive solution that does not converge to zero. Using the same argument as in Lemma 1, there exists $y_1 \geq y_0$ such that $u(\vartheta(y)) > 0$ and $\zeta(y)(u'(y))^a$ is positive and nonincreasing. Since $\zeta(y) > 0$, $u(y)$ is increasing for $y \geq y_1$. Using $u(y) \geq u(y_1)$, we have

$$u^c(y) \geq u^{c-b_2}(y) u^{b_2}(y) \geq u^{c-b_2}(y_1) u^{b_2}(y), \quad (33)$$

and hence

$$u^c(\vartheta(y)) \geq u^{c-b_2}(y_1) u^{b_2}(\vartheta(y)), \quad \text{for } y \geq y_2. \quad (34)$$

Using (34) and $\vartheta(y) \geq \vartheta_0(y)$, from (13), we have

$$\zeta(y)(u'(y))^a \geq u^{c-b_2}(y_1) u^{b_2}(\vartheta_0(y)) \int_y^{\infty} p(\eta) d\eta, \quad (35)$$

for $y \geq y_2$. From $\zeta(y)(u'(y))^a$ being nonincreasing and $\vartheta_0(y) \leq y$, we have

$$\zeta(\vartheta_0(y))(u'(\vartheta_0(y)))^a \geq \zeta(y)(u'(y))^a. \quad (36)$$

We use this in the left-hand side of (35). Then, dividing by $\zeta(\vartheta_0(y)) u^{b_2}(\vartheta_0(y)) > 0$ and raising both sides to the $1/a$ power, we have

$$\frac{u'(\vartheta_0(y))}{u^{b_2/a}(\vartheta_0(y))} \geq \left[\frac{u^{c-b_2}(y_1)}{\zeta(\vartheta_0(y))} \int_y^{\infty} p(\eta) d\eta \right]^{1/a}, \quad (37)$$

for $y \geq y_2$. Multiplying the left-hand side by $\vartheta'_0(y)/\vartheta_0 \geq 1$ and integrating from y_2 to y , we have

$$\begin{aligned}
& \frac{1}{\vartheta_0} \int_{y_2}^y \frac{u'(\vartheta_0(\eta)) \vartheta'_0(\eta)}{u^{b_2/a}(\vartheta_0(\eta))} d\eta \\
& \geq u^{c-b_2}(y_1) \int_{y_2}^y \left[\frac{1}{\zeta(\vartheta_0(\eta))} \int_{\eta}^{\infty} p(\zeta) d\zeta \right]^{1/a} d\eta.
\end{aligned} \quad (38)$$

On the left-hand side, since $a < b_2$, integrating, we have

$$\begin{aligned}
& \frac{1}{\alpha(1-b_2/a)} \left[z^{1-b_2/a}(\vartheta_0(\eta)) \right]_{s=y_2}^y \\
& \leq \frac{1}{\alpha(b_2/a-1)} z^{1-b_2/a}(\vartheta_0(y_2)) < \infty.
\end{aligned} \quad (39)$$

On the right-hand side of (38), we use that $\zeta(\vartheta_0(\eta)) \leq \zeta(\eta)$ to conclude that (32) implies the right-hand side approaching $+\infty$, as $y \rightarrow \infty$, which is a contradiction. Hence, the solution u cannot be eventually positive.

For eventually negative solutions, we use the same change of variables as in Theorem 1 and proceed as above.

To prove the necessity part, we assume that (32) does not hold and obtain an eventually positive solution that does not converge to zero. If (32) does not hold, then for each $\kappa > 0$ there exists $y_1 \geq y_0$ such that

$$\int_{y_1}^{\infty} \left[\frac{1}{\zeta(\eta)} \int_{\eta}^{\infty} p(\zeta) d\zeta \right]^{1/a} d\eta \frac{\kappa^{(1-c/a)}}{2}, \quad \forall y \geq y_1. \quad (40)$$

We define the set of continuous function

$$S = \left\{ u \in C([0, \infty)) : \frac{\kappa}{2} \leq u(y) \leq \kappa \text{ for } y \geq y_1 \right\}. \quad (41)$$

Then, we define the operator

$$(\Omega u)(y) = \begin{cases} 0, & \text{if } y \leq y_1, \\ \frac{\kappa}{2} + \int_{y_1}^y \frac{1}{\zeta(\eta)} \left[\int_{\eta}^{\infty} p(\zeta) u^c(\vartheta(\zeta)) d\zeta \right]^{1/a} d\eta, & \text{if } y > y_1. \end{cases} \quad (42)$$

Note that if u is continuous, Ωu is also continuous at $y = y_1$. Also, note that if $\Omega u = u$, then u is solution of (1).

First, we estimate $(\Omega u)(y)$ from below. Let $u \in M$, we have $(\Omega u)(y) \geq \kappa/2 + 0$, on $[y_1, \infty)$.

Now, we estimate $(\Omega u)(y)$ from above. Let $u \in M$. Then, $u \leq \kappa$ and by (40), we have

$$\begin{aligned}
(\Omega u)(y) & \leq \frac{\kappa}{2} + \kappa^{c/a} \int_{y_1}^y \left[\frac{1}{\zeta(\eta)} \int_{\eta}^{\infty} p(\zeta) d\zeta \right]^{1/a} d\eta \\
& \leq \frac{\kappa}{2} + \frac{\kappa}{2} = \kappa.
\end{aligned} \quad (43)$$

Therefore, Ω maps S to S . To find a fixed point for Ω in S , we define a sequence of functions by the recurrence relation

$$\begin{aligned} v_0(y) &= 0, \quad \text{for } y \geq y_0, \\ v_1(y) &= (\Omega v_0)(y) = 1, \quad \text{for } y \geq y_1, \\ v_{n+1}(y) &= (\Omega v_n)(y), \quad \text{for } n \geq 1, y \geq y_1. \end{aligned} \quad (44)$$

Note that for each fixed y , we have $v_1(y) \geq v_0(y)$. Using mathematical induction, we can prove that $v_{n+1}(y) \geq v_n(y)$. Therefore, $\{v_n\}$ converges pointwise to a function v in S . Then, v is a fixed point of Ω and a positive solution of (1). The proof is completed. $\square$

Example 1. Consider the differential equations

$$(e^{-y}(u'(y))^{11/3})' + \frac{1}{y+1}(u(y-2))^{1/3} = 0. \quad (45)$$

Here, $a = 11/3$, $\zeta(y) = e^{-y}$, $\vartheta_1(y) = y - 2$, $Y(y) = \int_{y_1}^y e^{11\eta/3} d\eta = (3/11)(e^{11y/3} - e^{11y_1/3})$, and $c_1 = 1/3$. For $b = 7/3$, we have $0 < c < b < a$. To check (17), we have

$$\begin{aligned} &\int_0^\infty p(s)Y^c(\vartheta(s))ds \\ &= \int_0^\infty \frac{1}{\eta+1} \left(\frac{3}{11} (e^{11(s-2)/3} - e^{11y_1/3}) \right)^{1/3} ds = \infty. \end{aligned} \quad (46)$$

So, every conditions of Theorem 1 hold true, and therefore, all solutions of (45) are oscillatory or converge to zero.

Example 2. Consider the differential equations

$$((u'(y))^{1/3})' + y(u(y-2))^{7/3} = 0. \quad (47)$$

Here, $a = 1/3$, $\zeta(y) = 1$, $\vartheta_1(y) = y - 2$, and $c_1 = 7/3$. For $b = 5/3$, we have $c > b > a$. To check (32), we have

$$\int_{y_0}^\infty \left[\frac{1}{\zeta(s)} \int_s^\infty p(\zeta) d\zeta \right]^{1/a} ds = \int_2^\infty \left[\int_s^\infty \zeta d\zeta \right]^3 ds = \infty. \quad (48)$$

So, every conditions of Theorem 2 hold true. Thus, all solutions of (47) are oscillatory or converge to zero.

3. Conclusion

The aim of this work is to establish necessary and sufficient conditions for the oscillation of solution to second-order half-linear differential equation. The obtained oscillation theorems complement the well-known oscillation results present in the literature. This work, as well as [31–41], leads us to pose an open problem: Can we find necessary and sufficient conditions for the oscillation of solutions to second-order differential equation

$$\begin{aligned} &[r(t)((y(t) + p(t)y(\tau(t)))')^y]' + \sum_{i=1}^m q_i(t)y^{\alpha_i}(\tau_i(t)) = 0, \\ &\text{for } p \in C(\mathbb{R}_+, \mathbb{R})?. \end{aligned} \quad (49)$$

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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