Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 8826338, 8 pages
https://doi.org/10.1155/2020/8826338

Research Article

Convergence Analysis of Implicit Euler Method for a Class of Nonlinear Impulsive Fractional Differential Equations

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Received 9 September 2020; Revised 10 November 2020; Accepted 17 November 2020; Published 9 December 2020

Academic Editor: Luciano Mescia

For a class of nonlinear impulsive fractional differential equations, we first transform them into equivalent integral equations, and then the implicit Euler method is adapted for solving the problem. The convergence analysis of the method shows that the method is convergent of the first order. The numerical results verify the correctness of the theoretical results.

1. Introduction

In recent years, fractional differential equations have become a research hotspot due to their wide application in many fields. We refer the readers to the research papers [1–5] and the monographs by Podlubny [6], Diethelm [7], Kilbas et al. [8], Zhou [9], and the references cited therein. When the fractional differential equations are affected by instantaneous mutation, the impulsive fractional differential equations are obtained. The research of impulsive fractional differential equations can be found in literatures [10–20] and monograph [21]. However, there are few literatures on numerical methods for impulsive fractional differential equations.

In this paper, implicit Euler method is constructed for solving a class of nonlinear impulsive fractional differential equations. It is proved that the method is convergent of the first order. The numerical results also verify the correctness of the theoretical results.

2. Construction of Numerical Scheme

Consider the following impulsive fractional differential equations:

$$\begin{cases} {}_0^C D_t^{\alpha} u(t) = f(t, u(t)), & t \in J' := J \backslash \{t_1, t_2, \ldots, t_m\}, J := [0, T], \\ \triangle u(t_k) = I_k(u(t_k)), & \triangle u'(t_k) = \overline{I}_k(u(t_k)), k = 1, 2, \ldots, m, \\ u(0) = \beta, & u'(0) = \gamma, \end{cases} \quad (1)$$

where $\alpha \in (1, 2)$, $\beta, \gamma \in \mathbb{R}$ are constants, and ${}_0^C D_t^{\alpha} u(t)$ is the α-order Caputo derivative of solution $u(t)$ defined by (see [6–8])

$${}_0^C D_t^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha+1-n}} \mathrm{d}s, \quad 0 \le n-1 < \alpha < n, \quad (2)$$

$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, $\triangle u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $u(t_k^-) = \lim_{\varepsilon \longrightarrow 0^-} u(t_k + \varepsilon)$, and $u(t_k^+) = \lim_{\varepsilon \longrightarrow 0^+} u(t_k + \varepsilon)$ represent the left and right limits of $u(t)$ at $t = t_k$, and $f\colon J \times \mathbb{R} \longrightarrow \mathbb{R}$ and $I_k, \overline{I}_k, \colon \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and satisfy the following conditions:

$$\begin{aligned} |f(t,u) - f(t,v)| &\le L_1 |u - v|, t \in J,\ u, v \in \mathbb{R}, \\ \left|I_k(u) - I_k(v)\right| &\le L_2 |u - v|, u, v \in \mathbb{R},\ k = 1, 2, \ldots, m, \\ \left|\overline{I}_k(u) - \overline{I}_k(v)\right| &\le L_3 |u - v|, u, v \in \mathbb{R},\ k = 1, 2, \ldots, m, \end{aligned} \tag{3}$$

where L_1, L_2, L_3 are nonnegative constants with moderate size.

Throughout this paper, let $C(J, \mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm $\|u\|_c \coloneqq \sup\{|u(t)|\colon t \in J\}$ for $u \in C(J, \mathbb{R})$. We also define $PC(J, \mathbb{R}) \coloneqq \{u\colon J \longrightarrow \mathbb{R}, u \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, 1, \ldots, m,$ and $u(t_k^+)$ exists, $k = 1, 2, \ldots m\}$. The space $PC(J, \mathbb{R})$ is a Banach space equipped with the norm $\|u\|_{pc} \coloneqq \sup\{|u(t)|\colon t \in J\}$.

In addition, due to the need of convergence analysis, for function $u(t) \in PC(J, \mathbb{R})$, there are constants L_4 and L_5 such that

$$\left|u'(t)\right| \le L_4, \left|\frac{\partial f}{\partial t}(t, u)\right| \le L_5, \quad t \in \left(t_k, t_{k+1}\right], k = 0, 1, \ldots, m. \tag{4}$$

In order to obtain the numerical scheme for solving problem (1), according to Lemma 3.1 in reference [12], we can express equation (1) as the following equivalent integral equation:

$$u(t) = u(0) + u'(0)t + \sum_{0<t_k<t} I_k(u(t_k)) + \sum_{0<t_k<t} \overline{I}_k(u(t_k))(t - t_k) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\mathrm{d}s, \quad t \in J. \tag{5}$$

Let $h_k = (t_{k+1} - t_k)/N$, N be a given positive integer, the grid points $t_{k,i} = t_k + ih_k = t_{0,0} + (\sum_{j=0}^{k-1} Nh_j) + ih_k$, $k = 0, 1, \ldots, m$, $i = 1, 2, \ldots, N$, $h = \max\{h_0, h_1, \ldots, h_m\}$, and $u(t_{k,i})$ express the true solution of equation (1) at $t_{k,i}$. Then, an approximation to the integral equation can be attained by right rectangle formula:

$$\begin{aligned} u\left(t_{k,i}\right) &= u(0) + u'(0)t_{k,i} + \sum_{l=1}^{k} I_l\left(u\left(t_{l-1,N}\right)\right) + \sum_{l=1}^{k} \overline{I}_l\left(u\left(t_{l-1,N}\right)\right)\left(t_{k,i} - t_{l-1,N}\right) \\ &\quad + \sum_{l=1}^{k}\sum_{r=1}^{N} \frac{1}{\Gamma(\alpha)} \int_{t_{l-1,r-1}}^{t_{l-1,r}} \left(t_{k,i} - s\right)^{\alpha-1} f(s, u(s))\mathrm{d}s \\ &\quad + \sum_{r=1}^{i} \frac{1}{\Gamma(\alpha)} \int_{t_{k,r-1}}^{t_{k,r}} \left(t_{k,i} - s\right)^{\alpha-1} f(s, u(s))\mathrm{d}s \\ &\approx u(0) + u'(0)t_{k,i} + \sum_{l=1}^{k} I_l\left(u\left(t_{l-1,N}\right)\right) + \sum_{l=1}^{k} \overline{I}_l\left(u\left(t_{l-1,N}\right)\right)\left(t_{k,i} - t_{l-1,N}\right) \\ &\quad + \sum_{l=1}^{k}\sum_{r=1}^{N} \frac{1}{\Gamma(\alpha)} f\left(t_{l-1,r}, u\left(t_{l-1,r}\right)\right) \int_{t_{l-1,r-1}}^{t_{l-1,r}} \left(t_{k,i} - s\right)^{\alpha-1} \mathrm{d}s \\ &\quad + \sum_{r=1}^{i} \frac{1}{\Gamma(\alpha)} f\left(t_{k,r}, u\left(t_{k,r}\right)\right) \int_{t_{k,r-1}}^{t_{k,r}} \left(t_{k,i} - s\right)^{\alpha-1} \mathrm{d}s \\ &= u(0) + u'(0)t_{k,i} + \sum_{l=1}^{k} I_l\left(u\left(t_{l-1,N}\right)\right) + \sum_{l=1}^{k} \overline{I}_l\left(u\left(t_{l-1,N}\right)\right)\left(t_{k,i} - t_{l-1,N}\right) \\ &\quad + \sum_{l=1}^{k}\sum_{r=1}^{N} \frac{1}{\Gamma(\alpha+1)} f\left(t_{l-1,r}, u\left(t_{l-1,r}\right)\right)\left[\left(t_{k,i} - t_{l-1,r-1}\right)^{\alpha} - \left(t_{k,i} - t_{l-1,r}\right)^{\alpha}\right] \\ &\quad + \sum_{r=1}^{i} \frac{1}{\Gamma(\alpha+1)} f\left(t_{k,r}, u\left(t_{k,r}\right)\right)\left[\left(t_{k,i} - t_{k,r-1}\right)^{\alpha} - \left(t_{k,i} - t_{k,r}\right)^{\alpha}\right]. \end{aligned} \tag{6}$$

By using the numerical solution $u_{k,i}$ instead of the true solution $u(t_{k,i})$ in equation (6), we obtain the implicit Euler method for solving impulsive fractional differential equation (1):

$$\begin{aligned} u_{k,i} &= u_{0,0} + u_{0,0}' t_{k,i} + \sum_{l=1}^{k} I_l\left(u_{l-1,N}\right) + \sum_{l=1}^{k} \overline{I}_l\left(u_{l-1,N}\right)\left(t_{k,i} - t_{l-1,N}\right) \\ &+ \sum_{l=1}^{k}\sum_{r=1}^{N} \frac{1}{\Gamma(\alpha+1)} f\left(t_{l-1,r}, u_{l-1,r}\right)\left[\left(t_{k,i} - t_{l-1,r-1}\right)^{\alpha} - \left(t_{k,i} - t_{l-1,r}\right)^{\alpha}\right] \\ &+ \sum_{r=1}^{i} \frac{1}{\Gamma(\alpha+1)} f\left(t_{k,r}, u_{k,r}\right)\left[\left(t_{k,i} - t_{k,r-1}\right)^{\alpha} - \left(t_{k,i} - t_{k,r}\right)^{\alpha}\right], k = 0, 1, \ldots, m, i = 1, 2, \ldots, N. \end{aligned} \tag{7}$$

Remark 1. It is well known that there are various forms of definition of fractional calculus in the literature, such as Riemann–Liouville fractional calculus, Grü nwald–Letnikov fractional calculus, and Caputo fractional calculus (see [6–8]). The α-order Riemann–Liouville derivative of function $u(t)$ is defined by

$${}_0^R D_t^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{u(s)}{(t-s)^{\alpha+1-n}} ds, \quad 0 \le n-1 < \alpha < n. \tag{8}$$

We have the following relationship between the Riemann–Liouville derivative and Caputo derivative:

$${}_0^R D_t^{\alpha} u(t) = {}_0^C D_t^{\alpha} u(t) + \sum_{j=0}^{n-1} \frac{t^{j-\alpha} y^{(j)}(0)}{\Gamma(j-\alpha+1)}, \quad 0 \le n-1 < \alpha < n. \tag{9}$$

Therefore, the fractional differential equations studied in the present paper only consider Caputo derivative.

3. Convergence Analysis

Let $z_{k,i} = u_{k,i} - u(t_{k,i})$, where $u_{k,i}$ and $u(t_{k,i})$ denote the numerical solution and true solution of problem (1) at grid point $t_{k,i}$, respectively. Then, for the error $z_{k,i}$, we have

$$\begin{aligned} z_{k,i} &= \sum_{l=1}^{k}\left(I_l\left(u_{l-1,N}\right) - I_l\left(u\left(t_{l-1,N}\right)\right)\right) + \sum_{l=1}^{k}\left(\overline{I}_l\left(u_{l-1,N}\right) - \overline{I}_l\left(u\left(t_{l-1,N}\right)\right)\right)\left(t_{k,i} - t_{l-1,N}\right) \\ &+ \sum_{l=1}^{k}\sum_{r=1}^{N} \frac{1}{\Gamma(\alpha+1)}\left(f\left(t_{l-1,r}, u_{l-1,r}\right) - f\left(t_{l-1,r}, u\left(t_{l-1,r}\right)\right)\right) \times \left[\left(t_{k,i} - t_{l-1,r-1}\right)^{\alpha} - \left(t_{k,i} - t_{l-1,r}\right)^{\alpha}\right] \\ &+ \sum_{r=1}^{i} \frac{1}{\Gamma(\alpha+1)}\left(f\left(t_{k,r}, u_{k,r}\right) - f\left(t_{k,r}, u\left(t_{k,r}\right)\right)\right)\left[\left(t_{k,i} - t_{k,r-1}\right)^{\alpha} - \left(t_{k,i} - t_{k,r}\right)^{\alpha}\right] - R_{k,i}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} R_{k,i} &= \sum_{l=1}^{k}\sum_{r=1}^{N} \frac{1}{\Gamma(\alpha)} \int_{t_{l-1,r-1}}^{t_{l-1,r}} \left(t_{k,i} - s\right)^{\alpha-1} f(s, u(s)) ds \\ &+ \sum_{r=1}^{i} \frac{1}{\Gamma(\alpha)} \int_{t_{k,r-1}}^{t_{k,r}} \left(t_{k,i} - s\right)^{\alpha-1} f(s, u(s)) ds \\ &- \sum_{l=1}^{k}\sum_{r=1}^{N} \frac{1}{\Gamma(\alpha)} f\left(t_{l-1,r}, u\left(t_{l-1,r}\right)\right) \int_{t_{l-1,r-1}}^{t_{l-1,r}} \left(t_{k,i} - s\right)^{\alpha-1} ds \\ &- \sum_{r=1}^{i} \frac{1}{\Gamma(\alpha)} f\left(t_{k,r}, u\left(t_{k,r}\right)\right) \int_{t_{k,r-1}}^{t_{k,r}} \left(t_{k,i} - s\right)^{\alpha-1} ds. \end{aligned} \tag{11}$$

In order to obtain the convergence result of numerical method (7), we first prove the following lemma.

Lemma 1. *Assume the functions* $f: J \times \mathbb{R} \longrightarrow \mathbb{R}$ *and* $I_k, \overline{I}_k: \mathbb{R} \longrightarrow \mathbb{R}, k = 1, 2, \ldots, m$ *are continuous and satisfy* *conditions (3) and (4); then, the truncation error of the discrete scheme (7) satisfies*

$$\left|R_{k,i}\right| \le Ch, \quad k = 0, 1, \ldots, m, i = 1, 2, \ldots, N. \tag{12}$$

Throughout the paper, C will denote a positive constant not necessarily the same at different places, which may depend on L_j, $j = 1, 2, \ldots, 5$, but is independent of h and N.

Proof. From (11), we have

$$\begin{aligned}\left|R_{k,i}\right| &= \left|\sum_{l=1}^{k}\sum_{r=1}^{N}\frac{1}{\Gamma(\alpha)}\int_{t_{l-1,r-1}}^{t_{l-1,r}}\left(f(s,u(s)) - f\left(t_{l-1,r}, u\left(t_{l-1,r}\right)\right)\right)\left(t_{k,i}-s\right)^{\alpha-1}ds + \sum_{r=1}^{i}\frac{1}{\Gamma(\alpha)}\int_{t_{k,r-1}}^{t_{k,r}}\left(f(s,u(s)) - f\left(t_{k,r}, u\left(t_{k,r}\right)\right)\right)\left(t_{k,i}-s\right)^{\alpha-1}ds\right| \\ &\le \left|\sum_{l=1}^{k}\sum_{r=1}^{N}\frac{1}{\Gamma(\alpha)}\int_{t_{l-1,r-1}}^{t_{l-1,r}}\left(f(s,u(s)) - f\left(t_{l-1,r}, u\left(t_{l-1,r}\right)\right)\right)\left(t_{k,i}-s\right)^{\alpha-1}ds\right| + \left|\sum_{r=1}^{i}\frac{1}{\Gamma(\alpha)}\int_{t_{k,r-1}}^{t_{k,r}}\left(f(s,u(s)) - f\left(t_{k,r}, u\left(t_{k,r}\right)\right)\right)\left(t_{k,i}-s\right)^{\alpha-1}ds\right|.\end{aligned} \tag{13}$$

Applying the integral mean value theorem, we know that there exists $\xi_{l,r} \in (t_{l,r-1}, t_{l,r})$ such that

$$\begin{aligned}\left|R_{k,i}\right| \le& \sum_{l=1}^{k}\sum_{r=1}^{N}\frac{1}{\Gamma(\alpha+1)}\left|f\left(\xi_{l-1,r}, u\left(\xi_{l-1,r}\right)\right) - f\left(t_{l-1,r}, u\left(t_{l-1,r}\right)\right)\right| \times \left[\left(t_{k,i}-t_{l-1,r-1}\right)^{\alpha} - \left(t_{k,i}-t_{l-1,r}\right)^{\alpha}\right] \\ &+ \sum_{r=1}^{i}\frac{1}{\Gamma(\alpha+1)}\left|f\left(\xi_{k,r}, u\left(\xi_{k,r}\right)\right) - f\left(t_{k,r}, u\left(t_{k,r}\right)\right)\right|\left[\left(t_{k,i}-t_{k,r-1}\right)^{\alpha} - \left(t_{k,i}-t_{k,r}\right)^{\alpha}\right] \\ \le& \sum_{l=1}^{k}\sum_{r=1}^{N}\frac{1}{\Gamma(\alpha+1)}\left\{f\left(\xi_{l-1,r}, u\left(\xi_{l-1,r}\right)\right) - f\left(\xi_{l-1,r}, u\left(t_{l-1,r}\right)\right) + \left|f\left(\xi_{l-1,r}, u\left(t_{l-1,r}\right)\right) - f\left(t_{l-1,r}, u\left(t_{l-1,r}\right)\right)\right|\right\}\left[\left(t_{k,i}-t_{l-1,r-1}\right)^{\alpha} - \left(t_{k,i}-t_{l-1,r}\right)^{\alpha}\right] \\ &+ \sum_{r=1}^{i}\frac{1}{\Gamma(\alpha+1)}\left\{f\left(\xi_{k,r}, u\left(\xi_{k,r}\right)\right) - f\left(\xi_{k,r}, u\left(t_{k,r}\right)\right) + \left|f\left(\xi_{k,r}, u\left(t_{k,r}\right)\right) - f\left(t_{k,r}, u\left(t_{k,r}\right)\right)\right|\right\}\left[\left(t_{k,i}-t_{k,r-1}\right)^{\alpha} - \left(t_{k,i}-t_{k,r}\right)^{\alpha}\right].\end{aligned} \tag{14}$$

Using conditions (2) and (3) and differential mean value theorem, we know that there exists $\eta_{l,r}, \zeta_{l,r} \in (t_{l,r-1}, t_{l,r})$ such that

$$\begin{aligned}\left|R_{k,i}\right| \le& \sum_{l=1}^{k}\sum_{r=1}^{N}\frac{1}{\Gamma(\alpha+1)}\left\{L_1\left|u\left(\xi_{l-1,r}\right) - u\left(t_{l-1,r}\right)\right| + h_{l-1}\left|\frac{\partial f}{\partial t}\left(\eta_{l-1,r}, u\left(t_{l-1,r}\right)\right)\right|\right\} \times \left[\left(t_{k,i}-t_{l-1,r-1}\right)^{\alpha} - \left(t_{k,i}-t_{l-1,r}\right)^{\alpha}\right] \\ &+ \sum_{r=1}^{i}\frac{1}{\Gamma(\alpha+1)}\left\{L_1\left(u\left(\xi_{k,r}\right) - u\left(t_{k,r}\right)\right) + h_k\left|\frac{\partial f}{\partial t}\left(\eta_{k,r}, u\left(t_{k,r}\right)\right)\right|\right\} \times \left[\left(t_{k,i}-t_{k,r-1}\right)^{\alpha} - \left(t_{k,i}-t_{k,r}\right)^{\alpha}\right] \\ \le& C\sum_{l=1}^{k}\sum_{r=1}^{N}h_{l-1}\left(\left|u'\left(\zeta_{l-1,r}\right)\right| + \left|\frac{\partial f}{\partial t}\left(\eta_{l-1,r}, u\left(t_{l-1,r}\right)\right)\right|\right)\left[\left(t_{k,i}-t_{l-1,r-1}\right)^{\alpha} - \left(t_{k,i}-t_{l-1,r}\right)^{\alpha}\right] \\ &+ C\sum_{r=1}^{i}h_k\left(\left|u'\left(\zeta_{k,r}\right)\right| + \left|\frac{\partial f}{\partial t}\left(\eta_{k,r}, u\left(t_{k,r}\right)\right)\right|\right)\left[\left(t_{k,i}-t_{k,r-1}\right)^{\alpha} - \left(t_{k,i}-t_{k,r}\right)^{\alpha}\right] \\ \le& C\sum_{l=1}^{k}\sum_{r=1}^{N}h_{l-1}\left[\left(t_{k,i}-t_{l-1,r-1}\right)^{\alpha} - \left(t_{k,i}-t_{l-1,r}\right)^{\alpha}\right] + C\sum_{r=1}^{i}h_k\left[\left(t_{k,i}-t_{k,r-1}\right)^{\alpha} - \left(t_{k,i}-t_{k,r}\right)^{\alpha}\right] \\ \le& Ch\left[\left(t_{k,i}^{\alpha} - \left(t_{k,i}-t_{k-1,N}\right)^{\alpha}\right) + \left(t_{k,i}-t_{k,0}\right)^{\alpha}\right] \le Ch.\end{aligned} \tag{15}$$

This means the proof of Lemma 1 is completed. □

Theorem 1. *Let $u_{k,i}$ and $u(t_{k,i})$ denote the numerical solution and true solution of problem (1) at grid point $t_{k,i}$, respectively. Then, the convergence inequality*

$$\left|u_{k,i}-u\left(t_{k,i}\right)\right|\leq Ch,\quad k=0,1,\ldots,m, i=1,2,\ldots,N, \tag{16}$$

holds when h is small enough and the conditions of Lemma 1 are satisfied. This means the numerical method (7) is convergent of the first order.

Proof. From (10), we can obtain that

$$\begin{aligned}
&\left(1-\frac{1}{\Gamma(\alpha+1)}L_1 h_k^\alpha\right)\left|z_{k,i}\right| \\
&\leq\left|\sum_{l=1}^{k}\left(I_l\left(u_{l-1,N}\right)-I_l\left(u\left(t_{l-1,N}\right)\right)\right)\right|+\left|\sum_{l=1}^{k}\left(\overline{I}_l\left(u_{l-1,N}\right)-\overline{I}_l\left(u\left(t_{l-1,N}\right)\right)\right)\left(t_{k,i}-t_{l-1,N}\right)\right| \\
&+\left|\sum_{l=1}^{k}\sum_{r=1}^{N}\frac{1}{\Gamma(\alpha+1)}\left(f\left(t_{l-1,r},u_{l-1,r}\right)-f\left(t_{l-1,r},u\left(t_{l-1,r}\right)\right)\right)\times\left[\left(t_{k,i}-t_{l-1,r-1}\right)^\alpha-\left(t_{k,i}-t_{l-1,r}\right)^\alpha\right]\right| \\
&+\left|\sum_{r=1}^{i-1}\frac{1}{\Gamma(\alpha+1)}\left(f\left(t_{k,r},u_{k,r}\right)-f\left(t_{k,r},u\left(t_{k,r}\right)\right)\right)\times\left[\left(t_{k,i}-t_{k,r-1}\right)^\alpha-\left(t_{k,i}-t_{k,r}\right)^\alpha\right]\right| \\
&+\left|R_{k,i}\right|\leq\sum_{l=1}^{k}\left|z_{l-1,N}\right|\left(L_2+L_3\left(t_{k,i}-t_{l-1,N}\right)\right)+\sum_{l=1}^{k}\sum_{r=1}^{N}\frac{1}{\Gamma(\alpha+1)}L_1\left|z_{l-1,r}\right|\left[\left(t_{k,i}-t_{l-1,r-1}\right)^\alpha-\left(t_{k,i}-t_{l-1,r}\right)^\alpha\right] \\
&+\sum_{r=1}^{i-1}\frac{1}{\Gamma(\alpha+1)}L_1\left|z_{k,r}\right|\left[\left(t_{k,i}-t_{k,r-1}\right)^\alpha-\left(t_{k,i}-t_{k,r}\right)^\alpha\right]+\left|R_{k,i}\right|.
\end{aligned} \tag{17}$$

When h is small enough, i.e.,

$$\Gamma(\alpha+1)-L_1 h_k^\alpha>0, \tag{18}$$

we have

$$\left|z_{k,i}\right|\leq p_{k,i}+w_k\sum_{l=1}^{k}\sum_{r=1}^{N}v_{l-1,r}\left|z_{l-1,r}\right|+w_k\sum_{r=1}^{i-1}a_{k,r}\left|z_{k,r}\right|, \tag{19}$$

where

$$\begin{aligned}
p_{k,i}&=\frac{\Gamma(\alpha+1)\left|R_{k,i}\right|}{\Gamma(\alpha+1)-L_1 h_k^\alpha}, \\
w_k&=\frac{1}{\Gamma(\alpha+1)-L_1 h_k^\alpha},\quad k=0,1,\ldots,m, \\
a_{k,r}&=L_1\left[\left(t_{k,i}-t_{k,m-1}\right)^\alpha-\left(t_{k,i}-t_{k,r}\right)^\alpha\right],\quad r=1,2,\ldots,N-1, \\
v_{l-1,r}&=L_1\left[\left(t_{k,i}-t_{l-1,r-1}\right)^\alpha-\left(t_{k,i}-t_{l-1,r}\right)^\alpha\right],\quad l=1,2,\ldots,k, \\
v_{l-1,N}&=\Gamma(\alpha+1)\left(L_2+L_3\left(t_{k,i}-t_{l-1,N}\right)\right)+L_1\left[\left(t_{k,i}-t_{l-1,N-1}\right)^\alpha-\left(t_{k,i}-t_{l-1,N}\right)^\alpha\right].
\end{aligned} \tag{20}$$

By using the discrete analogue of Gronwall's inequality (see Theorem 2 in [22]), it follows that

$$|z_{k,i}| \le p_{k,i} + w_k \prod_{l=1}^{k}\prod_{r=1}^{N}\left(1 + w_{l-1}v_{l-1,r}\right)\prod_{r=1}^{i-1}\left(1 + w_k a_{k,r}\right) \times \left[\sum_{q=1}^{k}\sum_{j=1}^{N}\left(p_{q-1,j}v_{q-1,j}\prod_{l=1}^{q}\prod_{r=1}^{j}\left(1 + w_{l-1}v_{l-1,r}\right)^{-1}\right) + \sum_{g=1}^{i-1} a_{k,g}p_{k,g}\prod_{l=1}^{k}\prod_{r=1}^{N}\left(1 + w_{l-1}v_{l-1,r}\right)^{-1}\prod_{r=1}^{g}\left(1 + w_k a_{k,r}\right)^{-1}\right]. \tag{21}$$

According to Lemma 1, when h is small enough, we have $0 < w_k \le C, k = 0, 1, \ldots, m,$ and

$$0 < p_{k,i} \le Ch. \tag{22}$$

Hence, we obtain

$$\begin{aligned}
&\sum_{q=1}^{k}\sum_{j=1}^{N}\left(p_{q-1,j}v_{q-1,j}\prod_{l=1}^{q}\prod_{r=1}^{j}\left(1 + w_{l-1}v_{l-1,r}\right)^{-1}\right) + \sum_{g=1}^{i-1} a_{k,g}p_{k,g}\prod_{l=1}^{k}\prod_{r=1}^{N}\left(1 + wv_{l-1,r}\right)^{-1}\prod_{r=1}^{g}\left(1 + w_k a_{k,r}\right)^{-1}\\
&\quad\le \sum_{q=1}^{k}\sum_{j=1}^{N} p_{q-1,j}v_{q-1,j} + \sum_{g=1}^{i-1} a_{k,g}p_{k,g}\\
&\quad\le \mathrm{Ch}\left(L_1\left(t_{k,i}^{\alpha} - \left(t_{k,i} - t_{k-1,N}\right)^{\alpha}\right) + \sum_{q=1}^{k}\Gamma(\alpha+1)\left(L_2 + L_3\left(t_{k,i} - t_{q-1,N}\right)\right)\right) + \mathrm{Ch}L_1\left[\left(t_{k,i} - t_{k,0}\right)^{\alpha} - \left(t_{k,i} - t_{k,i-1}\right)^{\alpha}\right]\\
&\quad\le \mathrm{Ch}\left(L_1\left(t_{k,i}^{\alpha} - \left(t_{k,i} - t_{k,i-1}\right)^{\alpha}\right) + \sum_{q=1}^{k}\Gamma(\alpha+1)\left(L_2 + L_3\left(t_{k,i} - t_{q-1,N}\right)\right)\right)\\
&\quad\le \mathrm{Ch},
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\prod_{l=1}^{k}\prod_{r=1}^{N}\left(1 + w_{l-1}v_{l-1,r}\right)\prod_{r=1}^{i-1}\left(1 + w_k a_{k,r}\right) \le \exp\left(\sum_{l=1}^{k}\sum_{r=1}^{N} w_{l-1}v_{l-1,r} + \sum_{r=1}^{i-1} w_k a_{k,r}\right)\\
&\quad\le \exp\left(C\left[L_1\left(t_{k,i}^{\alpha} - \left(t_{k,i} - t_{k-1,N}\right)^{\alpha}\right) + \sum_{l=1}^{k}\Gamma(\alpha+1)\left(L_2 + L_3\left(t_{k,i} - t_{l-1,N}\right)\right)\right] + CL_1\left[\left(t_{k,i} - t_{k,0}\right)^{\alpha} - \left(t_{k,i} - t_{k,i-1}\right)^{\alpha}\right]\right)\\
&\quad\le \exp\left(C\left[L_1\left(t_{k,i}^{\alpha} - \left(t_{k,i} - t_{k,i-1}\right)^{\alpha}\right) + \sum_{l=1}^{k}\Gamma(\alpha+1)\left(L_2 + L_3\left(t_{k,i} - t_{l-1,N}\right)\right)\right]\right) \le C,
\end{aligned} \tag{24}$$

where we have used inequality (22) and $(1 + x) \le e^x$ for $x \ge -1$. Therefore, substituting (22)–(24) into (21) leads to

$$|z_{k,i}| = \left|u_{k,i} - u\left(t_{k,i}\right)\right| \le Ch, \quad k = 0, 1, \ldots, m, i = 1, 2, \ldots, N, \tag{25}$$

which means the method is convergent of the first order. □

4. Numerical Experiments

In the section, we utilize the following example to verify the theoretical results obtained in the previous section. Here, we will give error estimates and convergence rates for the numerical scheme.

Example 1. Consider the following impulsive fractional differential equation:

$$\begin{cases}
{}_0^C D_t^{\alpha} u(t) = -3u(t) + 0.2\sin(u(t)) + 5t^{\alpha} + 2,\ t = [0, 3]\backslash\{0.5, 2\},\ 1 < \alpha < 2,\\
\Delta u(1) = -\dfrac{1}{2}u(0.5^{-}),\ \Delta u(2) = -\dfrac{2}{3}u(2^{-}),\\
\Delta u'(1) = \dfrac{1}{4}u(0.5^{-}),\ \Delta u'(2) = \dfrac{1}{4}u(2^{-}),\\
u(0) = 1,\ u'(0) = 1.
\end{cases} \tag{26}$$

Table 1: Error and convergence order of numerical method (7) with different α.

	$N = 40$	$N = 80$	$N = 160$	$N = 320$	$N = 640$
$\alpha = 1.1$	$4.197e-2$	$2.108e-2$	$1.047e-2$	$5.112e-3$	$2.425e-3$
	0.993	1.010	1.033	1.076	—
$\alpha = 1.3$	$4.849e-2$	$2.430e-2$	$1.205e-2$	$5.881e-3$	$2.789e-3$
	0.997	1.012	1.034	1.076	—
$\alpha = 1.5$	$6.299e-2$	$3.157e-2$	$1.566e-2$	$7.644e-3$	$3.625e-3$
	0.997	1.012	1.034	1.076	—
$\alpha = 1.7$	$8.186e-2$	$4.096e-2$	$2.029e-2$	$9.901e-3$	$4.694e-3$
	0.999	1.013	1.035	1.077	—
$\alpha = 1.9$	$8.612e-2$	$4.283e-2$	$2.115e-2$	$1.031e-2$	$4.882e-3$
	1.008	1.018	1.037	1.078	—

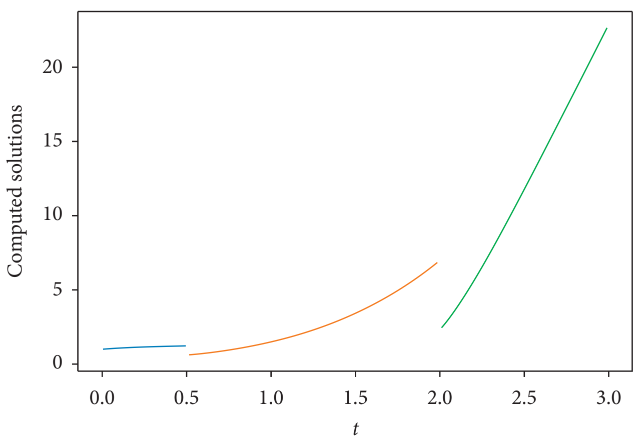

Figure 1: Numerical solution of equation (26) with $\alpha = 1.5$ and $N = 100$.

Because it is difficult to obtain the true solution of the equation, we take the numerical solution $u_{k,i}$ as the true solution $u(t_{k,i})$ at $t = t_{k,i}$ with $N = 6400$. The numerical scheme (7) is solved by using the Newton iteration method with $u_{k,i}^0 = 0$ $(k = 0, 1, \ldots, m, i = 1, 2, \ldots, N)$ as an initial value. We iteratively compute $u_{k,i}^j$ until

$$\max_{0\le k\le m,\,1\le i\le N}\left|u_{k,i}^j - u_{k,i}^{j-1}\right| < 10^{-8}. \tag{27}$$

The L_∞ norm of the global error is denoted as

$$e_{m,N} = \max_{0\le k\le m,\,1\le i\le N}\left|u_{k,i} - u\left(t_{k,i}\right)\right|. \tag{28}$$

Then, the convergence order of the numerical method is

$$r_{m,N} = \log_2\left(\frac{e_{m,N}}{e_{m,2N}}\right). \tag{29}$$

When α takes different values, the error and convergence order of numerical method (7) are shown in Table 1.

Table 1 shows that the numerical method (7) is convergent of the first order, which supports the convergence estimate of Theorem 1.

In Figure 1, we can see that the numerical solution is discontinuous due to the existence of the impulses.

5. Conclusion

In this paper, we focus on the numerical solution of impulsive fractional differential equations. For a class of nonlinear impulsive fractional differential equations, the implicit Euler method is adapted for solving the problem. After careful convergence analysis, we prove that the method is convergent of the first order. For future work, we will study the higher-order methods for solving impulsive fractional differential equations and analyze their convergence.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was supported by the NSF of China (11571291).

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