Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 8842496, 10 pages
https://doi.org/10.1155/2020/8842496

Research Article

Exact Single Traveling Wave Solutions for Generalized Fractional Gardner Equations

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Received 26 August 2020; Revised 9 November 2020; Accepted 25 November 2020; Published 7 December 2020

Academic Editor: Maria Patrizia Pera

In this paper, the classification of all single traveling wave solutions to generalized fractional Gardner equations is presented by utilizing the complete discrimination system method. Under the fractional traveling wave transformation, generalized fractional Gardner equations can be reduced to an ordinary differential equations. All possible exact traveling wave solutions are given through the complete discrimination system of the fourth-order polynomial. Moreover, graphical representations of different kinds of the exact solutions reveal that the method is of significance for searching the exact solutions to generalized fractional Gardner equations.

1. Introduction

It is common knowledge that fractional partial differential equations (FPDEs) [1–9] have gained great attention because they have been widely used to model various complex physical phenomena in the domain of science and engineering. Therefore, it is of great significance to search the exact traveling wave solutions [10–24] of FPDEs in the research of nonlinear science, which can accurately reflect the propagation of nonlinear waves and better understand nonlinear physical phenomena. So far, many powerful methods have been established and developed to analyze the exact solutions to the FPDE, which include the (G'/G)-expansion method [25, 26], the integral bifurcations [27, 28], the Lie symmetry analysis method [29, 30], first integral method [31], modified trial equation method [32], the exp-function method [33], F-expansion method [34], and the Kudryashov method [35].

In this paper, we shall consider the following generalized fractional Gardner equation [36]:

$$D_t^{\alpha}u + pu_x + \left(qu^n + ru^{2n}\right)u_x + u_{xxx} = 0, \quad n \geq 0, r < 0, 0 < \alpha \leq 1, \tag{1}$$

where $D_t^{\alpha}u$ is the conformable derivative of u depending on the variable t. $u(t,x)$ represents the amplitude of the wave mode, and variables t and x represent the time and spatial variable, respectively. The coefficients p, q, and r are constants. Equation (1) can be usually used to describe the nonlinear propagation of ion-acoustic waves at an unmagnetized plasma. As we all know, equation (1) is a kind of very important FPDE. When the parameters of equation (1) are changed, it can be simplified to the following famous nonlinear FPDE. For example, when $n = 1$, $q \neq 0$, and $r = 0$, equation (1) becomes the fractional KdV equation; when $n = 1$, $q = 0$, and $r \neq 0$, equation (1) becomes the fractional mKdV equation; and when $n = 1$, $q \neq 0$, and $r \neq 0$, equation (1) becomes a fractional KdV-mKdV equation. In [36], Reazadeh et al. obtained the hyperbolic and trigonometric function solutions to the generalized fractional Gardner equation by using modified Kudryashov method and hyperbolic function method, respectively. But, their research only focused on acquiring the hyperbolic and trigonometric function solutions. Motivated by the aforementioned discussion, in the paper, we will construct new exact traveling wave solution to the generalized fractional Gardner equations via the polynomial method.

The complete discrimination system for the polynomial method was first proposed by Liu [37]. It is one of the most powerful methods to find the single traveling wave solutions to partial differential equations (PDEs). With the development of fractional calculus, the study of exact solution to FPDEs has been gaining more and more attention by many experts and scholars. Because of the complexity of fractional derivative, the exact solution of FPDE develops very slowly than PDE of integers. Many scholars [38, 39] have been trying to find new methods to construct the exact solutions of FPDEs. Recently, Khalil et al. [40] introduced the conformable fractional derivative. FPDEs can be reduced into nonlinear ordinary differential equations by the fractional traveling wave transformation. In the paper, we will find the exact solutions to the generalized fractional Gardner equation by the complete discrimination system of the polynomial method.

The main objective of the paper is to draw support from the complete discrimination system to construct exact traveling wave solutions to the generalized fractional Gardner equation. In Section 2, we review the definition of conformable derivative and introduce the complete discrimination system for constructing the exact traveling wave solutions of FPDE. Then, in Section 3, we discuss the exact solutions to the generalized fractional Gardner equation by using the complete discrimination system. Finally, we give a brief conclusion in Section 4.

2. Mathematical Preliminaries

2.1. The Conformable Derivative. The definition and properties of the conformable derivative are defined as

Definition 1. Let $f\colon [0,\infty) \longrightarrow \mathbf{R}$. Then, the conformable derivative of f of order α is defined as

$$D_t^{\alpha} f(t) = \lim_{\varepsilon \longrightarrow 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \quad \forall t \in (0,+\infty), \alpha \in (0,1], \tag{2}$$

and the function f is α-conformable differentiable at a point t if the limit in equation (2) exists.

Remark 1. The conformable derivative possesses many important properties (See [41, 42] and references therein). In recent years, several scholars [43–47] have developed and constructed exact solutions of FPDE in the sense of conformable derivative.

2.2. Description of the Method. Consider the following conformable FPDE:

$$P\left(u, D_t^{\alpha} u, u_x, D_t^{2\alpha} u, u_{xx}, u_{xxx}, \ldots\right) = 0, \quad 0 < \alpha \leq 1, \tag{3}$$

where $u = u(t,x)$ is an unknown function.

Introducing the following fractional traveling wave transformation,

$$u(t,x) = u(\xi), \quad \xi = k\left(x - \frac{\omega t^{\alpha}}{\alpha}\right), \tag{4}$$

where k and ω are nonzero constants.

Substituting equation (4) into equation (3), then equation (3) can be converted into the following integer-order ordinary differential equation:

$$Q\left(u, u', u'', \ldots\right) = 0, \tag{5}$$

where Q is a polynomial in u and its derivatives and notation $(')$ is the derivative with respect to ξ.

Equation (5) is usually reduced to

$$\left(u'\right)^2 = F(u), \tag{6}$$

where $F(u)$ is a polynomial.

Then, integrating equation (6) once, we can obtain

$$\pm\left(\xi - \xi_0\right) = \int \frac{\mathrm{d}u}{\sqrt{F(u)}}, \tag{7}$$

where ξ_0 is an integral constant.

According to the above procedures, recent results have been reported via the complete discrimination system [48–50].

3. Applications

When $n = 1$, $q \neq 0$, and $r \neq 0$, equation (1) becomes the following fractional KdV-mKdv equation:

$$D_t^{\alpha} u + p u_x + \left(qu + ru^2\right) u_x + u_{xxx} = 0. \tag{8}$$

Substituting equation (4) into equation (8), equation (8) can be converted to an ordinary differential equation as follows:

$$-k\omega u' + pku' + \left(qu + ru^2\right) ku' + k^3 u''' = 0. \tag{9}$$

Integrating the above once with respect to ξ, we obtain

$$k^3 u'' + (p - \omega) ku + \frac{kq}{2} u^2 + \frac{kr}{3} u^3 = c_0. \tag{10}$$

Multiplying equation (9) on both sides by u' and then integrating it with respect to ξ again,

$$k^3 \left(u'\right)^2 + \frac{(p - \omega)k}{2} u^2 + \frac{kq}{6} u^3 + \frac{kr}{12} u^4 = c_0 u + c_1, \tag{11}$$

where c_0 and c_1 are integral constants. Then, we obtain

$$\left(u'\right)^2 = -\frac{r}{12k^2} u^4 - \frac{q}{6k^2} u^3 - \frac{(p - \omega)}{2k^2} u^2 + \frac{c_0}{k^3} u + \frac{c_1}{k^3}. \tag{12}$$

Suppose that $a_4 = -r/12k^2$, $a_3 = -q/6k^2$, $a_2 = -p-\omega/2k^2$, $a_1 = c_0/k^3$, and $a_0 = c_1/k^3$; then,

$$\left(u'\right)^2 = a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0. \tag{13}$$

Making the transformation, we obtain

$$\Phi = (a_4)^{1/4}\left(u + \frac{a_3}{4a_4}\right),$$
$$\xi_1 = (a_4)^{1/4}\xi. \tag{14}$$

Substituting transformation (14) into equation (13), it will be changed into

$$\Phi_{\xi_1}^2 = \Phi^4 + b_1\Phi^2 + b_2\Phi + b_3, \tag{15}$$

where $b_1 = a_2/\sqrt{a_4}$, $b_2 = (a_3^3/8a_4^2 - a_2a_3/2a_4 + a_1)\ (a_4)^{-1/4}$, and $b_3 = -3a_3^4/256a_4^3 + a_2a_3^2/16a_4^2 - a_1a_3/4a_4 + a_0$.

Integrating equation (15) once, we obtain

$$\pm(\xi_1 - \xi_0) = \int \frac{d\Phi}{\sqrt{\Phi^4 + b_1\Phi^2 + b_2\Phi + b_3}}, \tag{16}$$

where ξ_0 is the integration constant.

Let $G(\Phi) = \Phi^4 + b_1\Phi^2 + b_2\Phi + b_3$; then, we can obtain its complete discrimination system:

$$D_1 = 4,$$
$$D_2 = -b_1,$$
$$D_3 = -2b_1^3 + 8b_1b_3 - 9b_2^2,$$
$$E_2 = 9b_1^2 - 32b_1b_3,$$
$$D_4 = -b_1^3b_2^2 + 4b_1^4b_3 + 36b_1b_2^2b_3 - 32b_1^2b_3^2 - \frac{27}{4}b_2^4 + 64b_3^3. \tag{17}$$

Integrating formula (16), we will obtain the exact traveling wave solutions of equation (8) under nine cases.

Case 1. $D_2 < 0$, $D_3 = 0$, and $D_4 = 0$. $G(\Phi)$ has a pair of conjugate complex roots of multiplicities two, i.e.,

$$G(\Phi) = \left[(\Phi - \beta)^2 + \gamma^2\right]^2, \tag{18}$$

where $\gamma > 0$. By using equation (16), we attain

$$\xi_1 - \xi_0 = \int \frac{d\Phi}{(\Phi - \beta)^2 + \gamma^2} = \frac{1}{\gamma}\arctan\frac{\Phi - \beta}{\gamma}. \tag{19}$$

Then, equation (19) is simplified to

$$\Phi(\xi_1) = \gamma\tan(\gamma(\xi_1 - \xi_0)) + \beta, \tag{20}$$

which is a trigonometric function solution. Namely, when $b_1 > 0, b_2 = 0$, and $b_3 = b_1^2/4$; therefore, $\gamma = b_1/2$ and then the solution of equation (15) is expressed as follows:

$$u_1(\xi) = \pm a_4^{-1/4}\gamma\tan\left(\gamma\left(a_4^{1/4}\xi - \xi_0\right)\right) - \frac{a_3}{4a_4}. \tag{21}$$

For instance, when $k = 1$, $p = 2$, $\omega = 4$, $q = -12$, $r = -12$, $c_0 = 0$, $c_1 = 11/16$, and $\xi_0 = 0$, we can obtain a trigonometric solution of equation (8) as follows. Under the given parameters, we draw the traveling wave 3D solution surfaces and corresponding 2D solution graphs for the obtained solution $u_1(t, x)$ in Figures 1 and 2:

$$u_1(t, x) = \frac{1}{2}\tan\left(\frac{1}{2}x - \frac{2t^\alpha}{\alpha}\right) - \frac{1}{2}. \tag{22}$$

Case 2. $D_2 = 0$, $D_3 = 0$, and $D_4 = 0$. $G(\Phi)$ has real roots of multiplicities four, namely,

$$G(\Phi) = \Phi^4. \tag{23}$$

By using equation (16), we can obtain

$$\xi_1 - \xi_0 = \int \frac{d\Phi}{\Phi^2} = -\frac{1}{\Phi}. \tag{24}$$

Then, we can obtain a rational function solution:

$$\Phi(\xi_1) = -\frac{1}{\xi_1 - \xi_0}. \tag{25}$$

Therefore, the solutions of equation (16) can be shown as

$$u_2(\xi) = \mp a^{-1/4}\frac{1}{a^{-1/4}\xi - \xi_0} - \frac{a_3}{4a_4}. \tag{26}$$

For example, when $k = 1$, $p = 2$, $\omega = 2$, $q = 0$, $r = -12$, $c_0 = 0$, $c_1 = 0$, and $\xi_0 = 0$, we can obtain a rational function solution of equation (8) as

$$u_2(t, x) = \frac{1}{x - (2t^\alpha/\alpha)}. \tag{27}$$

Case 3. $D_2 > 0$, $D_3 = 0$, $D_4 = 0$, and $E_2 > 0$. $G(\Phi)$ has two real roots of multiplicities two, namely,

$$G(\Phi) = (\Phi - \beta)^2(\Phi - \gamma)^2, \tag{28}$$

where $\beta > \gamma$. By using equation (16), we can obtain

$$\pm(\xi_1 - \xi_0) = \int \frac{d\Phi}{(\Phi - \beta)(\Phi - \gamma)} = \frac{1}{\beta - \gamma}\ln\left|\frac{\Phi - \beta}{\Phi - \gamma}\right|. \tag{29}$$

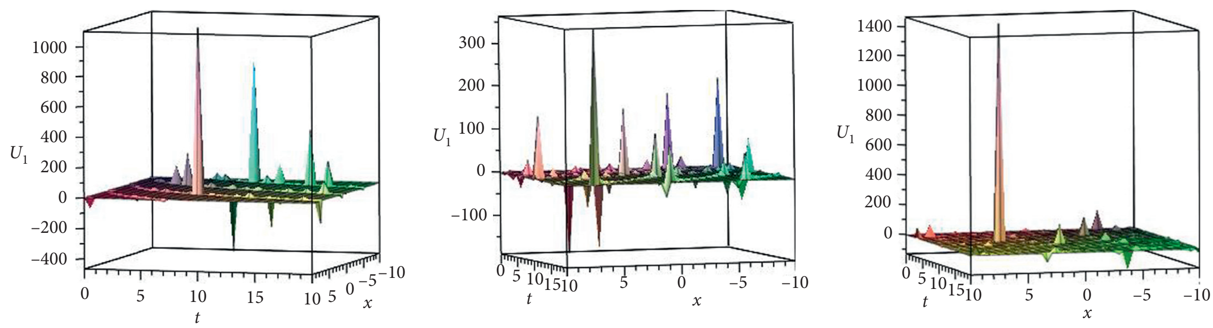
When $\Phi > \beta$ or $\Phi < \gamma$, we can obtain the solution of equation (16) as follows:

$$\Phi(\xi_1) = \frac{\gamma - \beta}{e^{(\beta-\gamma)(\xi_1-\xi_0)} - 1} + \gamma$$
$$= \frac{\gamma - \beta}{2}\left[\coth\frac{(\beta - \gamma)(\xi_1 - \xi_0)}{2} - 1\right] + \gamma. \tag{30}$$

Then, we have

$$u_3(\xi) = \frac{(\gamma - \beta)a_4^{-1/4}}{2}\left[\coth\frac{(\beta - \gamma)\left(a_4^{1/4}\xi - \xi_0\right)}{2} - 1\right]$$
$$+ \gamma - \frac{a_3}{4a_4}. \tag{31}$$

When $\gamma < \Phi < \beta$, we can gain the solution of equation (16) as follows:



(a) (b) (c)

FIGURE 1: The graphics of solutions of $u_1(t,x)$ for differential values of fractional parameter α.

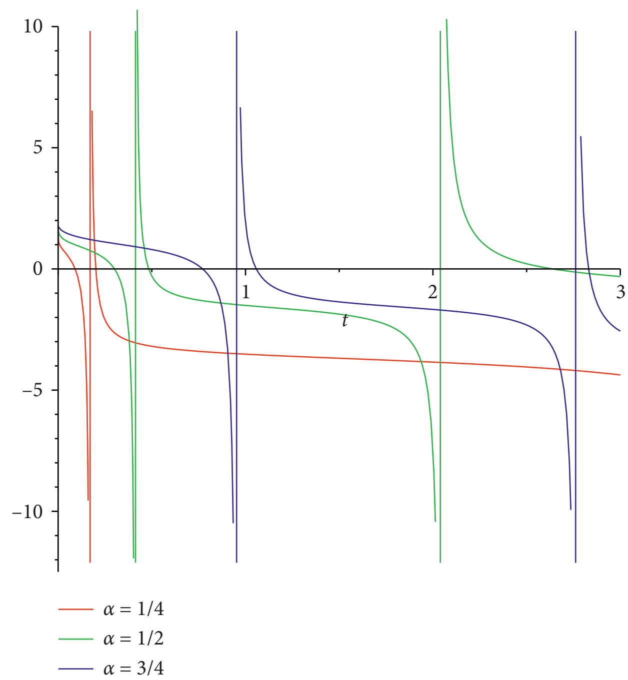


FIGURE 2: Two-dimensional graphic of solution of $u_1(t,x)$ for differential values of fractional parameter α.

$$\Phi(\xi_1) = \frac{\gamma - \beta}{-e^{(\beta-\gamma)(\xi_1-\xi_0)} - 1} + \gamma = \frac{\gamma-\beta}{2}\left[\tanh\frac{(\beta-\gamma)(\xi_1-\xi_0)}{2} - 1\right] + \gamma, \tag{32}$$

Similarly, we can have

$$u_4(\xi) = \frac{(\gamma-\beta)a_4^{-1/4}}{2}\left[\tanh\frac{(\beta-\gamma)\left(a_4^{1/4}\xi - \xi_0\right)}{2} - 1\right] + \gamma - \frac{a_3}{4a_4}. \tag{33}$$

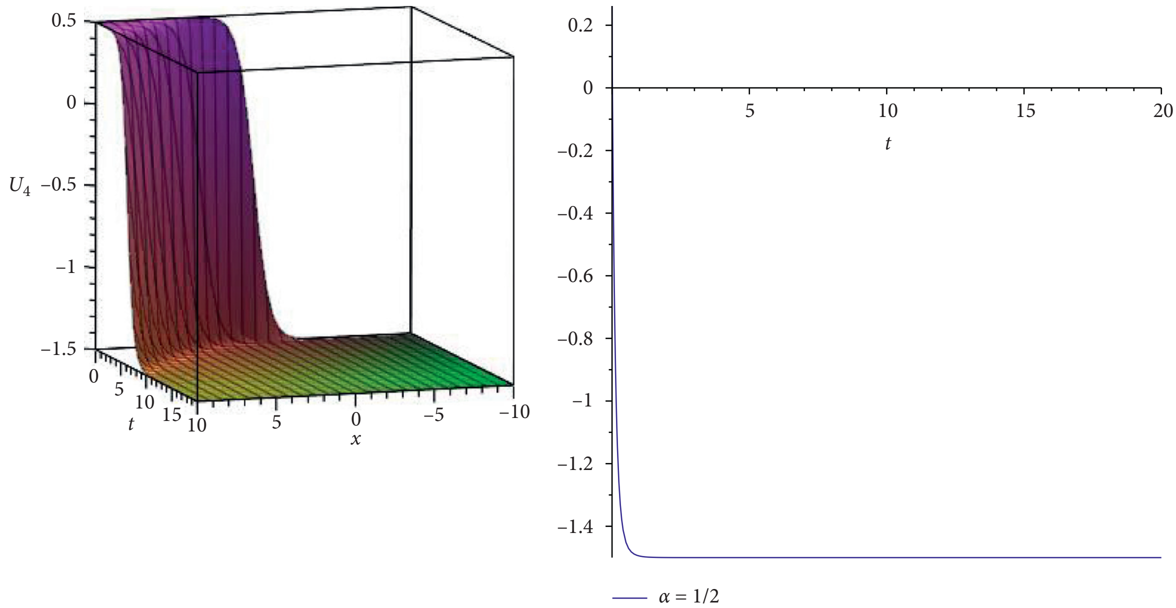
We can see that equations (31) and (33) are two solitary wave solutions. Especially, when $k = 1$, $p = 4$, $\omega = 2$, $q = -12$, $r = -12$, $c_0 = -2$, $c_1 = 11/16$, and $\xi_0 = 0$, we have $b_1 = -1$, $b_2 = 0$, $b_3 = 1/4$, and $-1 < \Phi < 1$, and then we can obtain a hyperbolic solution of equation (8) as follows. Under the given parameters, we draw the traveling wave 3D solution surface and corresponding 2D solution graphs for the obtained solution $u_4(t,x)$ in Figure 3:

$$u_4(t,x) = \tanh\left(x - \frac{2t^{\alpha}}{\alpha}\right) - \frac{1}{2}. \tag{34}$$

Case 4. $D_2 > 0$, $D_3 > 0$, and $D_4 = 0$. $G(\Phi)$ has two real roots and real roots with multiplicities two, namely,

$$G(\Phi) = (\Phi - \beta_1)^2(\Phi - \beta_2)(\Phi - \beta_3), \tag{35}$$

where $\beta_i (i = 1, 2, 3)$ are real numbers and $\beta_2 > \beta_3$.



(a) (b)

Figure 3: The graphics of solutions of $u_4(t, x)$.

When $\beta_1 > \beta_2$ and $\Phi < \beta_2$ or when $\beta_1 < \beta_3$ and $\Phi < \beta_3$, the following formula can be obtained:

$$\pm(\xi_1 - \xi_0) = \int \frac{d\Phi}{\sqrt{(\Phi - \beta_1)^2 (\Phi - \beta_2)(\Phi - \beta_3)}} = \frac{1}{(\beta_1 - \beta_2)(\beta_1 - \beta_3)} \ln \frac{\left[\sqrt{(\Phi - \beta_2)(\beta_1 - \beta_3)} - \sqrt{(\beta_1 - \beta_2)(\Phi - \beta_3)}\right]^2}{|\Phi - \beta_1|}. \quad (36)$$

When $\beta_1 > \beta_2$ and $\Phi < \beta_3$ or when $\beta_1 < \beta_3$ and $\Phi < \beta_2$, we can obtain the solution of equation (28):

$$\pm(\xi_1 - \xi_0) = \int \frac{d\Phi}{\sqrt{(\Phi - \beta_1)^2 (\Phi - \beta_2)(\Phi - \beta_3)}} = \frac{1}{(\beta_1 - \beta_2)(\beta_1 - \beta_3)} \ln \frac{\left[\sqrt{(\Phi - \beta_2)(\beta_3 - \beta_1)} - \sqrt{(\beta_2 - \beta_1)(\Phi - \beta_3)}\right]^2}{|\Phi - \beta_1|}. \quad (37)$$

When $\beta_2 > \beta_1 > \beta_3$, we can have the solution of equation (16):

$$\pm(\xi_1 - \xi_0) = \int \frac{d\Phi}{\sqrt{(\Phi - \beta_1)^2 (\Phi - \beta_2)(\Phi - \beta_3)}} = \frac{1}{(\beta_2 - \beta_1)(\beta_1 - \beta_3)} \arcsin \frac{(\Phi - \beta_2)(\beta_1 - \beta_3) + (\beta_1 - \beta_2)(\Phi - \beta_3)}{|(\Phi - \beta_1)(\beta_2 - \beta_3)|}. \quad (38)$$

Especially, when $k = 1$, $p = 4$, $\omega = 2$, $q = -12$, $r = -12$, $c_0 = -2$, $c_1 = -9/16$, and $\xi_0 = 0$, we have $b_1 = -1$, $b_2 = 0$, and $b_3 = 0$. By taking them into (38), then we can obtain

$$u_5(t, x) = \sin\left(x - \frac{2t^{\alpha}}{\alpha}\right) - \frac{1}{2}. \tag{39}$$

Case 5. $D_2 > 0$, $D_3 = 0$, $D_4 = 0$, and $E_2 = 0$. $G(\Phi)$ has real roots of multiplicities three and real roots with multiplicities one, namely,

$$G(\Phi) = (\Phi - \beta)^3 (\Phi - \gamma), \tag{40}$$

where β and γ are real numbers. Using formula (16) can yield

$$\pm(\xi_1 - \xi_0) = \int \frac{d\Phi}{\sqrt{(\Phi - \beta)^3 (\Phi - \gamma)}} = \frac{2}{\gamma - \beta}\sqrt{\frac{\Phi - \gamma}{\Phi - \beta}}. \tag{41}$$

When $\Phi > \beta$ and $\Phi > \gamma$ or when $\Phi < \beta$ and $\Phi < \gamma$, the solution of equation (16) is given by

$$\Phi(\xi_1) = \frac{4(\beta - \gamma)}{(\gamma - \beta)^2 (\xi_1 - \xi_0)^2 - 4} + \beta. \tag{42}$$

Then, we can obtain a rational solution of equation (8) as

$$u_6(\xi) = \frac{4a^{-1/4}(\beta - \gamma)}{\left(a^{1/4}\xi - \xi_0\right)^2 (\gamma - \beta)^2 - 4} + \beta - \frac{a_3}{4a_4}. \tag{43}$$

Case 6. $D_2 D_3 < 0$ and $D_4 = 0$. $G(\Phi)$ has real roots of multiplicities two and a pair of conjugate complex roots, namely,

$$G(\Phi) = (\Phi - \beta_1)^2 \left[(\Phi - \beta_2)^2 + \beta_3^2\right], \tag{44}$$

where $\beta_i (i = 1, 2, 3)$ are real numbers. By using equation (16), we can obtain

$$\begin{aligned}\pm(\xi_1 - \xi_0) &= \int \frac{d\Phi}{(\Phi - \beta_1)\sqrt{(\Phi - \beta_2)^2 + \beta_3^2}} \\ &= \frac{1}{\sqrt{(\beta_1 - \beta_2)^2 + \beta_3^2}} \ln\left|\frac{c_1\Phi + c_2 - \sqrt{(\Phi - \beta_2)^2 + \beta_3^2}}{\Phi - \beta_1}\right|,\end{aligned} \tag{45}$$

where $c_1 = \beta_1 - 2\beta_2/\sqrt{(\beta_1 - \beta_2)^2 + \beta_3^2}$ and $c_2 = \sqrt{(\beta_1 - \beta_2)^2 + \beta_3^2} - (\beta_1(\beta_1 - 2\beta_2)/\sqrt{(\beta_1 - \beta_2)^2 + \beta_3^2})$. Thus, we obtain the solution of equation (16):

$$\Phi(\xi_1) = \frac{\left(e^{\pm\sqrt{(\beta_1-\beta_2)^2+\beta_3^2}\,(\xi_1-\xi_0)} - c_1\right) + \sqrt{(\beta_1 - \beta_2)^2 + \beta_3^2}\,(2 - c_1)}{\left(e^{\pm\sqrt{(\beta_1-\beta_2)^2+\beta_3^2}\,(\xi_1-\xi_0)} - c_1\right)^2 - 1}. \tag{46}$$

Hence,

$$u_7(\xi) = \frac{\left[e^{\pm\sqrt{(\beta_1-\beta_2)^2+\beta_3^2}\,(a_4^{1/4}\xi-\xi_0)} - c_1\right] + \sqrt{(\beta_1 - \beta_2)^2 + \beta_3^2}\,(2 - c_1)}{a_4^{1/4}\left[\left(e^{\pm\sqrt{(\beta_1-\beta_2)^2+\beta_3^2}\,(a_4^{1/4}\xi-\xi_0)} - c_1\right)^2 - 1\right]} - \frac{a_3}{4a_4}, \tag{47}$$

which is a solitary wave solution.

Case 7. $D_2 > 0$, $D_3 > 0$, and $D_4 > 0$. $G(\Phi)$ has four distinct real roots, namely,

$$G(\Phi) = (\Phi - \beta_1)(\Phi - \beta_2)(\Phi - \beta_3)(\Phi - \beta_4), \tag{48}$$

where β_1, β_2, β_3, and β_4 are real numbers and $\beta_1 > \beta_2 > \beta_3 > \beta_4$.

When $\Phi > \beta_1$ or $\Phi < \beta_4$, we make the following transformation:

$$\Phi = \frac{\beta_2(\beta_1 - \beta_4)\sin^2\theta - \beta_1(\beta_2 - \beta_4)}{(\beta_1 - \beta_4)\sin^2\theta - (\beta_2 - \beta_4)}. \tag{49}$$

When $\beta_3 < \Phi < \beta_2$, similarly

$$\Phi = \frac{\beta_4(\beta_2 - \beta_3)\sin^2\theta - \beta_3(\beta_2 - \beta_4)}{(\beta_2 - \beta_3)\sin^2\theta - (\beta_2 - \beta_4)}. \tag{50}$$

By using equation (16), we obtain

$$\begin{aligned}\pm(\xi_1 - \xi_0) &= \int \frac{d\Phi}{\sqrt{(\Phi - \beta_1)(\Phi - \beta_2)(\Phi - \beta_3)(\Phi - \beta_4)}} \\ &= \frac{2}{\sqrt{(\beta_1 - \beta_3)(\beta_2 - \beta_4)}} \int \frac{d\theta}{\sqrt{1 - m^2\sin^2\theta}},\end{aligned} \tag{51}$$

where $m^2 = (\beta_1 - \beta_4)(\beta_2 - \beta_3)/(\beta_1 - \beta_3)(\beta_2 - \beta_4)$.

From equation (50) and the definition of Jacobian elliptic sine function, we obtain

$$\sin\theta = sn\left(\frac{\sqrt{(\beta_1-\beta_3)(\beta_2-\beta_4)}}{2}(\xi_1-\xi_0), m\right). \quad (52)$$

Combining equation (51) with expression (48), we can gain the solutions of equation (16):

$$\Phi(\xi_1) = \frac{\beta_2(\beta_1-\beta_4)sn^2\left(\sqrt{(\beta_1-\beta_3)(\beta_2-\beta_4)}/2(\xi_1-\xi_0), m\right) - \beta_1(\beta_2-\beta_4)}{(\beta_1-\beta_4)sn^2\left(\sqrt{(\beta_1-\beta_3)(\beta_2-\beta_4)}/2(\xi_1-\xi_0), m\right) - (\beta_2-\beta_4)}, \quad (53)$$

and then we can give the solution of equation (8):

$$u_8(\xi) = \frac{\beta_2(\beta_1-\beta_4)a_4^{-1/4}sn^2\left(\sqrt{(\beta_1-\beta_3)(\beta_2-\beta_4)}/2\left(a_4^{1/4}\xi-\xi_0\right), m\right) - \beta_1(\beta_2-\beta_4)}{(\beta_1-\beta_4)sn^2\left(\sqrt{(\beta_1-\beta_3)(\beta_2-\beta_4)}/2\left(a_4^{1/4}\xi-\xi_0\right), m\right) - (\beta_2-\beta_4)} - \frac{a_3}{4a_4}. \quad (54)$$

Similarly, combining equation (51) with expression (49), we can obtain the solution of equation (8):

$$u_9(\xi) = \frac{\beta_4(\beta_2-\beta_3)a_4^{-1/4}sn^2\left(\sqrt{(\beta_1-\beta_3)(\beta_2-\beta_4)}/2\left(a_4^{1/4}\xi-\xi_0\right), m\right) - \beta_3(\beta_2-\beta_4)}{(\beta_2-\beta_3)sn^2\left(\sqrt{(\beta_1-\beta_3)(\beta_2-\beta_4)}/2\left(a_4^{1/4}\xi-\xi_0\right), m\right) - (\beta_2-\beta_4)} - \frac{a_3}{4a_4}. \quad (55)$$

Case 8. $D_2D_3 \geq 0$ and $D_4 < 0$. $G(\Phi)$ has two different real roots and a pair of conjugate complex roots, namely,

$$G(\Phi) = (\Phi-\beta_1)(\Phi-\beta_2)\left[(\Phi-\beta_3)^2+\beta_4^2\right], \quad (56)$$

where $\beta_1, \beta_2, \beta_3$, and β_4 are real constants, $\beta_1 > \beta_2$, and $\beta_4 > 0$.

Making the following transformation, we obtain

$$\Phi = \frac{c_1\cos\theta + c_2}{c_2\cos\theta + c_4}, \quad (57)$$

where $c_1 = 1/2(\beta_1+\beta_2)c_3 - 1/2(\beta_1-\beta_2)c_4$, $c_2 = 1/2(\beta_1+\beta_2)c_4 - 1/2(\beta_1-\beta_2)c_3$, $c_3 = \beta_1-\beta_3-\beta_4/m_1$, $c_4 = \beta_1-\beta_3-\beta_4m_1$, $E = \beta_4^2 + (\beta_1-\beta_3)(\beta_2-\beta_3)/\beta_4(\beta_1-\beta_2)$, and $m_1 = E \pm \sqrt{E^2+1}$.

By using equation (16), we obtain:

$$\begin{aligned}\xi_1-\xi_0 &= \int \frac{d\Phi}{\sqrt{\pm(\Phi-\beta_1)(\Phi-\beta_2)\left((\Phi-\beta_3)^2+\beta_4^2\right)}}\\ &= \frac{2m_1m_2}{\sqrt{(\mp 2\beta_4m_1(\beta_1-\beta_2))}}\int\frac{d\theta}{\sqrt{1-m_2^2\sin^2\theta}},\end{aligned} \quad (58)$$

where $m_2^2 = 2/1 + m_1^2$.

From equation (57) and the definition of Jacobian elliptic function, we obtain

$$\cos\theta = cn\left(\frac{\sqrt{\mp 2\beta_4m_1(\beta_1-\beta_2)}}{2m_1m_2}(\xi_1-\xi_0), m_2\right). \quad (59)$$

Combining equation (58) with expression (56), we can gain the solutions of equation (16):

$$\Phi(\xi_1) = \frac{c_1cn\left(\sqrt{\mp 2\beta_4m_1(\beta_1-\beta_2)}/2m_1m_2(\xi_1-\xi_0), m_2\right) + c_2}{c_3cn\left(\sqrt{\mp 2\beta_4m_1(\beta_1-\beta_2)}/2m_1m_2(\xi_1-\xi_0), m_2\right) + c_3}, \quad (60)$$

and then we can give the solution of equation (8):

$$u_{10}(\xi)=\frac{a_4^{-1/4}\left[c_1 cn\left(\sqrt{\mp 2\beta_4 m_1(\beta_1-\beta_2)}/2m_1m_2\left(a_4^{1/4}\xi-\xi_0\right),m_2\right)+c_2\right]}{c_3 cn\left(\sqrt{\mp 2\beta_4 m_1(\beta_1-\beta_2)}/2m_1m_2\left(a_4^{1/4}\xi-\xi_0\right),m_2\right)+c_3}, \quad (61)$$

which is an elliptic double periodic function solution.

Case 9. $D_2D_3 \le 0$ and $D_4 > 0$. $G(\Phi)$ has two pairs of conjugate complex roots, namely,

$$G(\Phi)=\left[(\Phi-\beta_1)^2+l_2^2\right]\left[(\Phi-\beta_2)^2+l_2^2\right], \quad (62)$$

where β_1, β_2, l_1, and l_2 are real constants and $l_1 \ge l_2 > 0$.

Making the following transformation, we obtain

$$\Phi=\frac{c_1\tan\theta+c_2}{c_2\tan\theta+c_4}, \quad (63)$$

where $c_1=\beta_1c_3+l_1c_4$, $c_2=\beta_1c_4-l_1c_3$, $c_3=-l_1-l_2/m_1$, $c_4=\beta_1-\beta_2$, $E=(\beta_1-\beta_2)^2+l_1^2+l_2^2/2l_1l_2$, and $m_1=E+\sqrt{E^2-1}$.

By using equation (16), we obtain

$$\xi_1-\xi_0=\int\frac{d\Phi}{\sqrt{\left[(\Phi-\beta_1)^2+l_2^2\right]\left[(\Phi-\beta_2)^2+l_2^2\right]}}=\frac{c_3^2+c_4^2}{l_2\sqrt{\left(c_3^2+c_4^2\right)\left(m_1^2c_3^2+c_4^2\right)}}\int\frac{d\theta}{\sqrt{1-m_2^2\sin^2\theta}}, \quad (64)$$

where $m_2^2=m_1^2-1/m_1^2$.

From equation (62) and the definition of Jacobian elliptic function, we obtain

$$\sin\theta=sn\left(\frac{l_2\sqrt{\left(c_3^2+c_4^2\right)\left(m_1^2c_3^2+c_4^2\right)}}{c_3^2+c_4^2}(\xi_1-\xi_0),m_2\right), \quad (65)$$

$$\cos\theta=cn\left(\frac{l_2\sqrt{\left(c_3^2+c_4^2\right)\left(m_1^2c_3^2+c_4^2\right)}}{c_3^2+c_4^2}(\xi_1-\xi_0),m_2\right). \quad (66)$$

Combining equations (64) and (65) with expression (62), we can gain the solutions of equation (16):

$$\Phi(\xi_1)=\frac{c_1 sn(\mu(\xi_1-\xi_0),m_2)+c_2 cn(\mu(\xi_1-\xi_0),m_2)}{c_3 sn(\mu(\xi_1-\xi_0),m_2)+c_4 cn(\mu(\xi_1-\xi_0),m_2)}, \quad (67)$$

where $\mu=l_2\sqrt{(c_3^2+c_4^2)(m_1^2c_3^2+c_4^2)}/c_3^2+c_4^2$. Then, we can give the solution of equation (8):

$$u_{11}(\xi)=\frac{a_4^{-1/4}c_1 sn\left(\mu\left(a_4^{1/4}\xi-\xi_0\right),m_2\right)+a_4^{-1/4}c_2 cn\left(\mu\left(a_4^{1/4}\xi-\xi_0\right),m_2\right)}{c_3 sn\left(\mu\left(a_4^{1/4}\xi-\xi_0\right),m_2\right)+c_4 cn\left(\mu\left(a_4^{1/4}\xi-\xi_0\right),m_2\right)}-\frac{a_3}{4a_4}, \quad (68)$$

which is an elliptic double periodic function solution.

4. Conclusion

By using the complete discrimination system method, we obtain exact traveling wave solutions to generalized fractional Gardner equations under the given parameter conditions. Many exact solutions have been obtained, which include hyperbolic function solutions, Jacobi elliptic function solutions, trigonometric function solutions, and rational function solutions. Compared with the previous work, the solution obtained in the paper has not been reported. Furthermore, the method we employ here can be used to analyze the exact solutions to other FPDEs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors read and approved the final manuscript.

Acknowledgments

This work was supported by Science Research Fund of Education Department of Sichuan Province of China under grant no. 18ZB0537 and Scientific Research Funds of Chengdu University under grant no. 2081920034.

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