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Research Article

Nonparametric Estimation of Fractional Option Pricing Model

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The establishment of the fractional Black–Scholes option pricing model is under a major condition with the normal distribution for the state price density (SPD) function. However, the fractional Brownian motion is deemed to not be martingale with a long memory effect of the underlying asset, so that the estimation of the state price density (SPD) function is far from simple. This paper proposes a convenient approach to get the fractional option pricing model by changing variables. Further, the option price is transformed as the integral function of the cumulative density function (CDF), so it is not necessary to estimate the distribution function individually by complex approaches. Finally, it encourages to estimate the fractional option pricing model by the way of nonparametric regression and makes empirical analysis with the traded 50 ETF option data in Shanghai Stock Exchange (SSE).

1. Introduction

In the financial market, the memory effect of asset price has been described by the fractional Brownian motion (FBM). The first finding of long memory effects in stock returns was reported by Mandelbrot and Van Ness who also defined the fractional Brownian motion [1]. The memory effect between 0 and 1 is measured by Hurst index (H). Specifically speaking, the asset price has long memory effects if the Hurst index is between 1/2 and 1 whereas the asset price has short memory effects if the Hurst index is between 0 and 1/2. However, there is no memory effect when the Hurst index H is equal to 1/2.

According to the stochastic differential equation driven by the fractional Brownian motion, a large number of literature studies have studied the option pricing models of improving the classical Black–Scholes option pricing model (see Black and Scholes [2]). For instance, the study was reported by Necula [3], Rostek [4], and Hu and Øksendal [5] that fractional Black–Scholes pricing model (FBS) is obtained on the condition that the underlying asset price process obeys the fractional Brownian motion (FBM). Some results reflect the study reported by Ren et al. [6] who found that the option pricing model is linking with the Hurst index between 0.5 and 1. One study done by Wang et al. [7] examined the fractional option pricing formula is carried out when the Hurst index is between 1/3 and 1/2. One study by Chen et al. [8] offers another empirical analysis of the mixed fractional-fractional version of the Black–Scholes model with the Hurst index between 0 and 1.

There are two defects for the existing fractional Black–Scholes option pricing models. Firstly, the existing fractional Black–Scholes option pricing models corroborate the condition of the lognormal distribution of SPD. In practice, it is hard to undertake the estimation of the state price density (SPD) function when the underlying asset process is not a martingale; in addition, the state price density function (SPD) is unknown.

This paper is designed to relax the assumption in the fractional Black–Scholes pricing model (FBS) so that the returns of the underlying asset obey the lognormal distribution, and the option price will be transformed to the integral function of the cumulative density function (CDF). As a result, it is not necessary to estimate the distribution function individually via complex approaches. This idea of variable transformation is inspired by the research found by Ait-Sahalia [9], Xiu [10], and Vogt [11]. The option price can be transformed to a regression equation with the changing

variables, which can be estimated by the local polynomial model proposed by Fan and Gijbels [12] and Li and Racine [13].

Nonparametric pricing option has been present among researchers Ait-Sahalia and Lo [14, 15]. In order to overcome model errors, the semiparametric Black–Scholes model (SBS) has been proposed by Ait-Sahalia and Lo [14] with the implied volatility in the Black–Scholes option pricing model. The research done by Dumas et al. [16] carried out the so-called ad hoc Black–Scholes model in that implied volatility is the parabolic function of moneyness. Inspired by Ait-Sahalia and Lo [14], Fan and Mancini [17] proposed the semiparametric Black-Sholes model in that the implied volatility was the nonparametric estimator of moneyness.

The outline of the article is illustrated as follows. In Section 2, the analysis of the fractional Black–Scholes option pricing model and nonparametric fractional option pricing model established when a variable happens to change along with nonparametric fractional option pricing models is by the local polynomial regression. In Section 3, with the use of the traded 50 ETF option prices in Shanghai Stock Exchange (SSE), the experimental work compares the analysis of the effectiveness among classical Black–Scholes (BS) option pricing model, semiparametric Black–Scholes pricing model (SBS), semiparametric fractional Black–Scholes (SFBS) option pricing model, and nonparametric fractional (NF) option pricing model. In Section 4, several conclusions are given about the different option pricing models.

2. Pricing European Option by Changing Variables

Although the fractional Black–Scholes has improved the pricing performance, the application of the model is still under the condition of lognormal distribution and the framework of parametric Black–Scholes. The importance of the study is that it explores a new achievement in an orthogonal way instead of improving the pricing model to a more flexible level. The nonparametric fractional Black–Scholes model is established to improve the pricing performance by relaxing the lognormal distribution of the returns of the underlying asset (or random variable) to be nonparametric.

2.1. Black–Scholes Option Pricing Model by Changing Variables. This section will propose the following changing variables to obtain closed-form expressions of the Black–Scholes option pricing model. Let $P(f_0^{Q_1}, U_1)$ be the European put option price, and S_T is considered as the underlying asset price at time T and K is the strike price. Then, $\tau = T - t$ is regarded as the time to maturity and $f_0^{Q_1}(S_T|\tau)$ means the state price probability density function, while r is the riskless interest rate, and the price of European put option refers to the discounted expressed payoff in the risk-neutral world:

$$\begin{aligned} P\left(f_0^{Q_1}, U_1\right) &= e^{-r\tau} E\left[\max\left(K - S_T, 0\right)\right] \\ &= e^{-r\tau} \int_0^K (K - S) f_0^{Q_1}(S| \end{aligned} \tag{1}$$

The underlying asset price S_t follows the Brownian motion:

$$\mathrm{d}S_t = rS_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \tag{2}$$

where r is the riskless rate, σ is the diffusion coefficient, and B_t is the standard Brownian motion.

According to Ito's lemma, the price process is as follows:

$$\ln\left(\frac{S_T}{S_0}\right) = \mu_1(U_1) + \sigma_1(U_1) Z_1, \tag{3}$$

where $\mu_1(U_1)$ and $\sigma_1(U_1)$ are the known functions of the characteristics of option parameters $U_1 = (S, K, \tau, r, \sigma)$, and $\mu_1(U_1) = (r - 1/2\sigma^2)\tau$ and $\sigma_1(U_1) = \sigma\sqrt{\tau}$. $Z_1 \sim f_0(\cdot|\tau)$, in which $f_0(\cdot|\tau)$ is the unknown state price density function to be nonparametrically estimated by the market data.

From equation (3), Brownian motion is concretely described by the underlying asset as follows:

$$Z_1 = \frac{\left[\ln\left(S_T/S_0\right) - \mu_1(U_1)\right]}{\sigma_1(U_1)}. \tag{4}$$

By changing variables, the option valuation equation (1) becomes

$$\begin{aligned} P\left(f_0^{Q_1}, U_1\right) &= e^{-r\tau} E\left[\max\left(K - S_T, 0\right)\right] = e^{-r\tau} \int_0^K (K - S) f_0^{Q_1}(S| \\ &= e^{-r\tau} \int_{+\infty}^{d_1(U_1)} \left(K - S_0 e^{\mu_1(U_1) + \sigma_1(U_1) U_1}\right) f_0(Z_1|\tau) \mathrm{d}Z_1 = P_{U_1}(f_0, U_1), \end{aligned} \tag{5}$$

where

$$d_1(U_1) = \frac{\left[\ln\left(K/S_0\right) - \mu_1(U_1)\right]}{\sigma_1(U_1)}. \tag{6}$$

The relationship between $f_0^{Q_1}(S_T|\tau)$ and $f_0(Z_1|\tau)$ is as follows:

$$f_0^{Q_1}(S_T|\tau) = \left[S_0 \sigma_1(U_1) e^{\mu_1(U_1) + \sigma_1(U_1) Z_1}\right]^{-1} f_0(Z_1|\tau). \tag{7}$$

The state price density function $f_0(Z_1|\tau)$ is the normal distribution as follows:

$$f_0(Z_1|\tau) = \frac{1}{\sqrt{2\pi}} e^{-Z_1^2/2}. \tag{8}$$

The systematic analysis of option valuation is the Black–Scholes option pricing model [2] as follows:

$$
\begin{aligned}
P\left(f_0^{Q_1}, U_1\right) &= e^{-r\tau}\int_0^{d_1(U_1)}\left(K - S_0 e^{\mu_1(U_1)+\sigma_1(U_1)Z_1}\right) f_0(Z_1|\tau)\mathrm{d}Z_1 \\
&= e^{-r\tau}K\int_0^{d_1(U_1)} f_0(Z_1|\tau)\mathrm{d}Z_1 - e^{-r\tau}\int_0^{d_1(U_1)} S_0 e^{\mu_1(U_1)+\sigma_1(U_1)Z_1} f_0(Z_1|\tau)\mathrm{d}Z_1 \\
&= e^{-r\tau}K\int_0^{d_1(U_1)} \frac{1}{\sqrt{2\pi}}e^{-Z_1^2/2}\mathrm{d}Z_1 - e^{-r\tau}\int_0^{d_1(U_1)} S_0 e^{\mu_1(U_1)+\sigma_1(U_1)Z_1}\frac{1}{\sqrt{2\pi}}e^{-Z_1^2/2}\mathrm{d}Z_1 \\
&= e^{-r\tau}KN\left(d_1(U_1)\right) - S_0 e^{\mu_1(U_1)+1/2\sigma_1^2(U_1)}\int_0^{d_1(U_1)} \frac{1}{\sqrt{2\pi}}e^{-1/2\left[Z_1-\sigma_1(U_1)\right]^2}\mathrm{d}Z_1 \\
&= e^{-r\tau}KN\left(d_1(U_1)\right) - S_0 e^{\mu_1(U_1)+1/2\sigma_1^2(U_1)}\int_0^{d_1(U_1)-\sigma_1(U_1)} \frac{1}{\sqrt{2\pi}}e^{-1/2x^2}\mathrm{d}x \\
&= e^{-r\tau}KN\left(d_1(U_1)\right) - S_0 N\left(d_1(U_1) - \sigma_1(U_1)\right) \\
&= e^{-r\tau}KN\left(\frac{\ln(K/S_0) - \left(r - \left(\sigma^2/2\right)\right)\tau}{\sigma\sqrt{\tau}}\right) - S_0 N\left(\frac{\ln(K/S_0) - \left(r - \left(\sigma^2/2\right)\right)\tau}{\sigma\sqrt{\tau}} - \sigma\sqrt{\tau}\right) \\
&= e^{-r\tau}KN\left(-\frac{\ln(S_0/K) + \left(r - \left(\sigma^2/2\right)\right)\tau}{\sigma\sqrt{\tau}}\right) - S_0 N\left(-\frac{\ln(S_0/K) + \left(r + \left(\sigma^2/2\right)\right)\tau}{\sigma\sqrt{\tau}}\right) \\
&= P_{BS}(f_0, Z_1),
\end{aligned}
\tag{9}
$$

where let $x = Z_1 - \sigma_1(Z_1)$. That is the classical Black–Scholes option pricing model when volatility turns to the history volatility:

$$P_{\mathrm{BS}} = e^{-r\tau}KN(-d_{12}) - S_0 N(-d_{11}), \tag{10}$$

where $d_{11} = \ln(S_0/K) + (r + (\sigma^2/2))\tau/\sigma\sqrt{\tau}$ and $d_{12} = d_{11} - \sigma\sqrt{\tau} = \ln(S_0/K) + (r - (\sigma^2/2))\tau/\sigma\sqrt{\tau}$.

Furthermore, model (10) has a fine description about semiparametric Black–Scholes model (SBS) proposed by Ait-Sahalia and Lo [14] and Fan and Mancini [17] with implied volatility. Fan and Mancini [17] proposed a nonparametric approach to fit the implied volatility function:

$$\sigma_{t,i}^{IV} = G\left(m_{t,i}\right) + \varepsilon_{t,i}, \quad i = 1, 2, \ldots, n, \tag{11}$$

where $m_{t,i} = K/F_{t,\tau}$ is the moneyness and $F_{t,\tau} = (C_t - P_t)e^{-r_{t,\tau}\tau} + K = S_t e^{(r_{t,\tau}-\delta_{t,\tau})\tau}$ means the forward price, the forward price is obtained from the put-call parity $C_t + Ke^{-r_{t,\tau}\tau} = P_t + F_{t,\tau}e^{-r_{t,\tau}\tau}$, P denotes the put price, and C denotes the call price.

However, the random variable Z_1 does not obey the lognormal distribution, which is unknown. By changing variables, option price can be illustrated by the integral function about random variable Z_1 depending on function $d_1(U_1)$. When the state price cumulative density function $F_0(Z_1|\tau)$ is unknown,

$$
\begin{aligned}
P\left(f_0^{Q_1}, U_1\right) &= e^{-r\tau}\int_0^{K} f_0^{Q_1}(S|\tau)\mathrm{d}S = e^{-r\tau}\int_0^{d_1(U_1)} \\
&\quad \cdot \left(K - S_0 e^{\mu_1(U_1)+\sigma_1(U_1)Z_1}\right) f_0(Z_1|\tau)\mathrm{d}Z_1 \\
&= e^{-r\tau}\int_0^{d_1(U_1)} K\sigma_1(U_1) f_0(Z_1|\tau)\mathrm{d}Z_1 \\
&= e^{-r\tau}K\sigma_1(U_1)\int_0^{d_1(U_1)} f_0(Z_1|\tau)\mathrm{d}Z_1,
\end{aligned}
\tag{12}
$$

where $f_0(Z_1|\tau)$ is the cumulative density function (CDF) of random variable Z_1 and $f_0(Z_1|\tau)$ is unknown function.

Because the function $f_0(Z_1|\tau)$ is unknown, and let $\int_0^{d_1(U_1)} f_0(Z_1|\tau)\mathrm{d}Z_1 = G(d_1(U_1))$, equation (12) will be the form as follows:

$$
\begin{aligned}
P\left(f_0^{Q_1}, U_1\right) &= e^{-r\tau}K\sigma_1(U_1)\int_0^{d_1(U_1)} f_0(Z_1|\tau)\mathrm{d}Z_1 \\
&= e^{-r\tau}K\sigma_1(U_1)G\left(d_1(U_1)\right).
\end{aligned}
\tag{13}
$$

It can be found that the option price is the function of one-dimensional variable $d_1(U_1)$ and distribution function $F_0(B|\tau)$.

From equation (13), the nonparametric estimation equation has been established between put option price function $P(f_0^{Q_1}, Z_1)$ and variable $d_1(U_1)$ as follows:

$$Y_i = G\left(X_i\right) + \varepsilon_i, \quad i = 1, 2, \ldots, n, \tag{14}$$

where $G(\cdot)$ is the unknown function to be estimated, $Y_i = e^{r\tau}/K\sigma_1(U_1)P(f_0^{Q_1}, Z_1)$, $X_i = \ln(K_i/F_{t,\tau}) + (\sigma^2/2)\tau/\sigma\sqrt{\tau}$, and ε_i features i.i.d with zero mean and common variance σ^2.

2.2. Fractional Option Pricing Model by Changing Variables. The correlational analysis of stock price is set out by a fractional Brownian motion when the stock price process has memory effects. In this section, the fractional option pricing model and nonparametric fractional option pricing model have been established on the condition that the stock price is subject to the fractional Brownian motion by changing variables.

Assume that the underlying asset price S_t follows the fractional Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dB_H(t), \tag{15}$$

where $B_H(t)$ is subject to fractional Brownian motion and H means the Hurst index and H can be estimated by R/S analysis approach.

The fractional Brownian motion $B_H(t)$ can be denoted by the standard Brownian motion $B(t)$ as follows:

$$B_H(t) = C_H\left[\int_{-\infty}^{0}\left((t-s)^{H-1/2} - (-s)^{H-1/2}\right)dB(s) + \int_0^t (t-s)^{H-1/2}dB(s)\right]. \tag{16}$$

The increment of the fractional Brownian motion $\Delta B_H(t)$ obeys the standard normal distribution:

$$\Delta B_H(t) \sim N\left(0, (\Delta t)^{2H}\right). \tag{17}$$

The autocovariance function of between $\Delta B_H(t)$ and $\Delta B_H(t+s)$ is as follows:

$$\mathrm{Cov}\left(\Delta B_H(t), \Delta B_H(t+s)\right) = \frac{1}{2}(\Delta t)\left(|s+1|^{2H} + |s-1|^{2H} - 2|s|^{2H}\right). \tag{18}$$

From equation (15), the stock price process is as follows:

$$\ln\left(\frac{S_T}{S_0}\right) = \mu_2(U_2) + \sigma_2(U_2)B_H(t), \tag{19}$$

where $U_2 = (S, K, \tau, r, \sigma, H)$ and $\mu_2(U_2) = \mu\tau - 1/2\sigma^2(\tau)^{2H}, \sigma_2(U_2) = \sigma(\tau)^H$.

In order to make the variable transformation, let Z_2 be the random variable with memory:

$$Z_2 = \frac{\ln(S_T/S_0) - \mu_2(U_2)}{\sigma_2(U_2)}. \tag{20}$$

Then, equation (1) will be

$$\begin{aligned} P\left(f_0^{Q_2}, U_2\right) &= e^{-r\tau}E\left[\max(K - S_T, 0)\right] \\ &= e^{-r\tau}\int_0^K (K-S)f_0^{Q_2}(S|\tau)dS \\ &= e^{-r(T-\tau)}\int_{+\infty}^{d_2(U_2)}\left(K - S_0 e^{\mu_2(U_2)+\sigma_2(U_2)U_2}\right) \\ &\quad \cdot f_0(Z_2|\tau)dZ_2 \\ &= P_{U_2}(f_0, U_2), \end{aligned} \tag{21}$$

where

$$d_2(U_2) = \frac{\ln(K/S_0) - \mu_2(U_2)}{\sigma_2(U_2)}. \tag{22}$$

The density function $f_0(Z_2|\tau)$ is given as normal distribution as follows:

$$f_0(Z_2|\tau) = \frac{1}{\sqrt{2\pi}}e^{-Z_2^2/2}. \tag{23}$$

The option valuation is discussed as the fractional Black–Scholes (FBSM) option pricing model (see Necula [3]):

$$\begin{aligned} P\left(f_0^{Q_2}, U_2\right) &= e^{-r\tau}\int_0^{d_2(U_2)}\left(K - S_0 e^{\mu_2(U_2)+\sigma_2(U_2)Z_2}\right)f_0(Z_2)dZ_2 \\ &= e^{-r\tau}K\int_0^{d_2(U_2)} f_0(Z_2)dZ_2 - e^{-r\tau}\int_0^{d_2(U_2)} S_0 e^{\mu_2(U_2)+\sigma_2(U_2)Z_2}f_0(Z_2)dZ_2 \\ &= e^{-r\tau}K\int_0^{d_2(U_2)}\frac{1}{\sqrt{2\pi}}e^{-Z_2^2/2}dZ_2 - e^{-r\tau}\int_0^{d_2(U_2)} S_0 e^{\mu_2(U_2)+\sigma_2(U_2)Z_2}\frac{1}{\sqrt{2\pi}}e^{-Z_2^2/2}dZ_2 \\ &= e^{-r\tau}KN\left(d_2(U_2)\right) - S_0 e^{\mu_2(U_2)+1/2\sigma_2^2(U_2)}\int_0^{d_2(U_2)}\frac{1}{\sqrt{2\pi}}e^{-1/2\left[Z_2-\sigma_2(U_2)\right]^2}dZ_2 \\ &= e^{-r\tau}KN\left(d_2(U_2)\right) - S_0 e^{\mu_2(U_2)+1/2\sigma_2^2(U_2)}\int_0^{d_2(U_2)-\sigma_2(U_2)}\frac{1}{\sqrt{2\pi}}e^{-1/2x^2}dx \\ &= e^{-r\tau}KN\left(d_2(U_2)\right) - S_0 N\left(d_2(U_2) - \sigma_2(U_2)\right) \\ &= e^{-r\tau}KN\left(\frac{\ln(K/S_0) - \left(r\tau - \left(\sigma^2/2\right)\tau^{2H}\right)}{\sigma\tau^H}\right) - S_0 N\left(\frac{\ln(K/S_0) - \left(r\tau - \left(\sigma^2/2\right)\tau^{2H}\right)}{\sigma\tau^H} - \sigma\tau^H\right) \\ &= e^{-r\tau}KN\left(-\frac{\ln(S_0/K) + \left(r\tau - \left(\sigma^2/2\right)\tau^{2H}\right)}{\sigma\tau^H}\right) - S_0 N\left(-\frac{\ln(S_0/K) + \left(r\tau + \left(\sigma^2/2\right)\tau^{2H}\right)}{\sigma\tau^H}\right) \\ &= P_{\mathrm{FBS}}(f_0, Z_2), \end{aligned} \tag{24}$$

where let $x = Z_2 - \sigma_2(Z_2)$. Generally, the fractional Black–Scholes option pricing model is given by

$$P_{\mathrm{FBS}} = e^{-r\tau}KN(-d_{22}) - S_0N(-d_{21}), \tag{25}$$

where $d_{21} = \ln(S_0/K) + (r\tau + (\sigma^2/2)\tau^{2H})/\sigma\tau^H$ and $d_{22} = d_{21} - \sigma\tau^H = \ln(S_0/K) + (r\tau - (\sigma^2/2)\tau^{2H})/\sigma\tau^H$.

However, the state price density function is unknown in practice. What makes it more complicated is that the fractional Brownian motion is neither martingale nor semimartingale. Therefore, the estimation of the density function is difficult to estimate due to the existing memory effects of the underlying asset:

$$\begin{aligned} P\left(f_0^{Q_2}, U_2\right) &= e^{-r\tau}\int_0^K (K-S)f_0^{Q_2}(S|\tau)\mathrm{d}S = e^{-r\tau}\int_0^{d_2(U_2)} \\ &\quad \cdot \left(K - S_0e^{\mu_2(U_2)+\sigma_2(U_2)Z_2}\right)f_0(Z_2|\tau)\mathrm{d}Z_2 \\ &= e^{-r\tau}\int_0^{d_2(U_2)} K\sigma_2(U_2)f_0(Z_2|\tau)\mathrm{d}Z_2 \\ &= e^{-r\tau}K\sigma_2(U_2)\int_0^{d_2(U_2)} f_0(Z_2|\tau)\mathrm{d}Z_2. \end{aligned} \tag{26}$$

In fact, the density function $f_0(Z_2|\tau)$ is hard to estimate for two reasons: $f_0(Z_2|\tau)$ is unknown and $f_0(Z_2|\tau)$ has memory effects. Therefore, a new idea is put forward not to estimate the function $f_0(Z_2|\tau)$ directly. Let $\int_0^{d_2(U_2)} f_0(Z_2|\tau)\mathrm{d}Z_2 = G(d_2(U_2))$ and the nonparametric regression equation is proposed as follows:

$$\begin{aligned} P\left(f_0^{Q_2}, U_2\right) &= e^{r\tau}K\sigma_2(U_2)\int_0^{d_2(U_2)} f_0(Z_2|\tau)\mathrm{d}Z_2 \\ &= e^{r\tau}K\sigma_2(U_2)G(d_2(U_2)). \end{aligned} \tag{27}$$

According to equation (27), the nonparametric regression equation is given by

$$Y_i = G(X_i) + \varepsilon_i, \quad i = 1, 2, \ldots, n, \tag{28}$$

where $Y_i = e^{r\tau}/K\sigma_2(Z_2)p(f_0^{Q_2}, U_2) = e^{r\tau}/K\sigma\tau^H P(f_0^{Q_2}, U_2)$, $X_i = \mathrm{d}(U_2) = \ln(K_i/S_0) - \mu_2(U_2)/\sigma_2(U_2) = \ln(K_i/F_{t,\tau}) + (\sigma^2/2)\tau^{2H}/\sigma\tau^H$, and ε_i features i.i.d. with zero mean and common variance σ^2.

2.3. Nonparametric Regression Estimation of Option Prices. We can estimate the nonparametric regression model (28) by local polynomial approach in Fan and Gijbels [12]:

$$Y_i = G(X_i) + \varepsilon_i, \quad i = 1, 2, \ldots, n, \tag{29}$$

where $Y = e^{r\tau}/K\sigma\tau^H P(f_0^{Q_2}, U_2)$, $X = \ln(K_i/F_{t,\tau}) + (\sigma^2/2)\tau^{2H}/\sigma\tau^H$, and ε_i features i.i.d with zero mean and common variance σ^2.

We approximate the unknown regression function $G(X)$ locally by a polynomial of order m, and the Taylor expansion of $G(X)$ in the neighborhood of x is given by

$$G(X) = \sum_{k=0}^{m} \frac{m^{(k)}(x)}{k!}(X-x)^k. \tag{30}$$

The nonparametric regression equation (29) will be estimated by a weighted least squares regression problem [12]:

$$\min \sum_{i=0}^{n}\left\{Y_i - \sum_{k=0}^{m}\beta_k(x)(X-x)^k\right\}^2 K_h\left(\frac{X_i - x}{h}\right), \tag{31}$$

where $K(\cdot)$ is the kernel function, $K(z) = 0.75(1 - z^2)I(|z| < 1)$ (Epanechnikov kernel), h is the bandwidth, and $h = 3.45\sigma n^{-1/5}$ from the experience of cross-validation (CV) approach [15], σ is the std. dev of the regressors, and n is the number of samples.

Generally, the majority of recent studies involve the nonparametric equation by applying a local quadratic polynomial approximation with $m = 2$. It is more convenient to write the weighted least squares problems (31) as matrix notation:

$$\mathrm{minimize}_\beta (Y - X\beta)^T W (Y - X\beta), \tag{32}$$

where

$$\begin{aligned} X &= \begin{pmatrix} 1 & X_1 - x & (X_1 - x)^2 \\ 1 & X_2 - x & (X_2 - x)^2 \\ 3 & & \\ 1 & X_n - x & (X_n - x)^2 \end{pmatrix}_{n\times 3}, \\ Y &= \begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_n \end{pmatrix}_{n\times 1}, \\ W &= \mathrm{diag}\{K_h(X - x)\}, \end{aligned} \tag{33}$$

where W is the weight matrix. And the coefficient $\beta_k(x)$ can be denoted by

$$\beta_{(x)} = \begin{pmatrix} \beta_0(x) \\ \beta_1(x) \\ \beta_2(x) \end{pmatrix}_{3\times 1} = \begin{pmatrix} \beta_0(x_1) & \beta_0(x_2) & \ldots & \beta_0(x_n) \\ \beta_1(x_1) & \beta_1(x_2) & \ldots & \beta_1(x_n) \\ \beta_2(x_1) & \beta_2(x_2) & \ldots & \beta_2(x_n) \end{pmatrix}_{3\times n}. \tag{34}$$

The solution vector of (32) is given as

$$\widehat{\beta} = \left(X^T W X\right)^{-1} X^T W Y. \tag{35}$$

3. Empirical Analysis

3.1. Data and Option Contacts. This section will make an empirical analysis by the option market data in China. The analysis is sourcing from the closing prices of European put option on the 50ETF in China from February 9, 2015, to August 21, 2015, and the option contacts contain from March 2015 to September 2015. To retain only liquid options, it is encouraged to discard the options with implied volatility larger than 70% and price smaller than 0.05, ending up with 3529. As a conclusion, the riskless rate is 2.25% in the year of 2015, and the history volatility is 20.59%.

3.2. Empirical Results. The Hurst index of 50 ETF is $H = 0.4526$, which is estimated by R/S analysis approach. Table 1 summarizes the pricing errors of different option

Table 1: Empirical result.

Model	Minimum	Maximum	Mean	Std. dev.	RMSE	MAE
BS	−0.218300	−0.000200	−0.072371	0.040202	0.082785	0.459366
SBS	−0.114000	0.144400	0.006069	0.041827	0.042259	0.233587
SFBS	−0.164617	0.356268	0.006773	0.0.045880	0.046373	0.227103
NF	−0.342859	0.235327	0.001111	0.041441	0.041449	0.146550

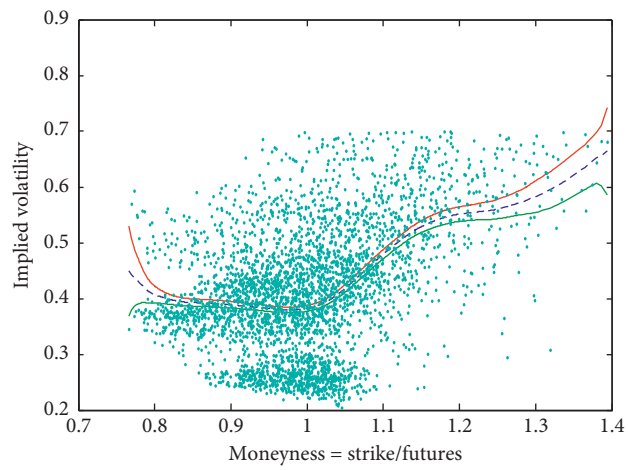

Figure 1: Nonparametric estimation of (11).

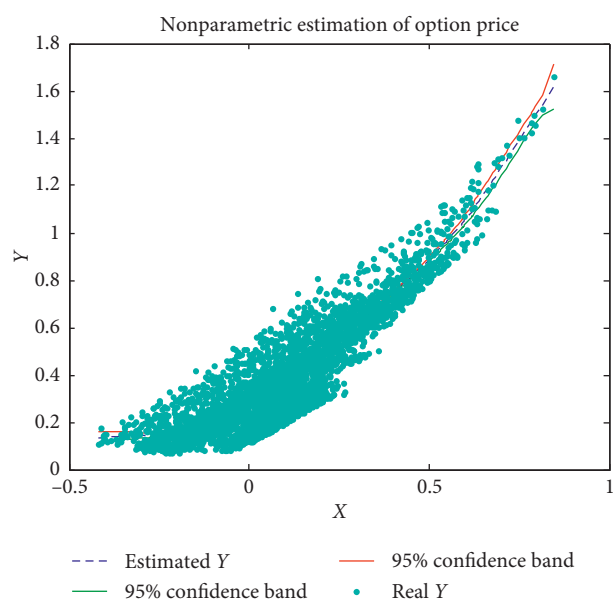

Figure 2: Nonparametric estimation of (28).

pricing models. From Table 1, the result of the MAE and RMSE of NF model is found to be lower than the BS, SBS, and SFBS models. To conclude, the nonparametric fractional option pricing model (NF) is superior to Black–Scholes model (BS), Semiparametric Black–Scholes model (SBS), and semiparametric fractional Black–Scholes pricing model (SFBS).

BS is the classical Black–Scholes option pricing model, and whole SBS is the semiparametric Black–Scholes option pricing model in that implied volatility is the local linear estimator of moneyness; SFBS is the semiparametric fractional Black–Scholes option pricing model, and NF is the nonparametric regression fractional option pricing model. The items are shown as the minimum, maximum, mean, std. dev, RMSE, and MAE of the price error (model price-market price $\text{RMSE} = \sqrt{1/2 \sum_{i=1}^{n} |P_{\text{model}} - P_{\text{market}}|^2}$, $\text{MAE} = 1/2 \sum_{i=1}^{n} |P_{\text{model}} - P_{\text{market}}/P_{\text{marhet}}|$).

Figure 1 presents the expression of the regression of implied volatility smile about moneyness; Figure 2 describes the results of the local quadratic polynomial estimation of equation (28).

Figures 3–6 demonstrate the price error histogram of several models, which is concentrated on zero. From the

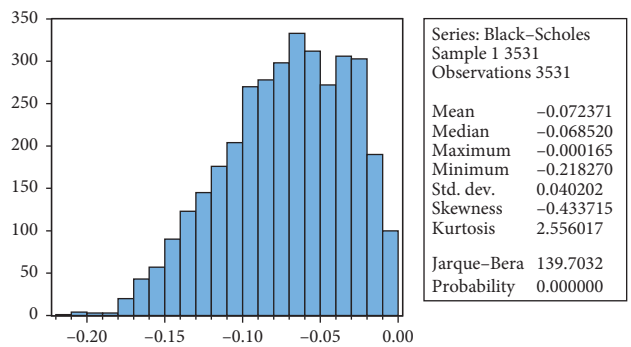

Figure 3: Price error of BS model.

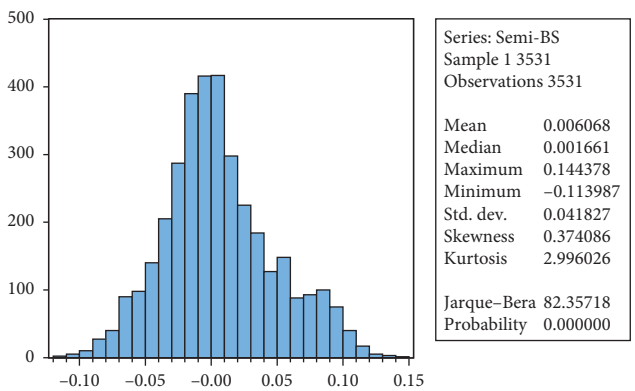

Figure 4: Price error of SBS model.

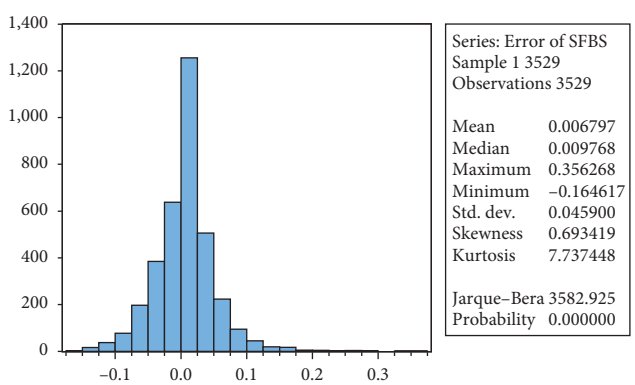

Figure 5: Price error of SFBS model.

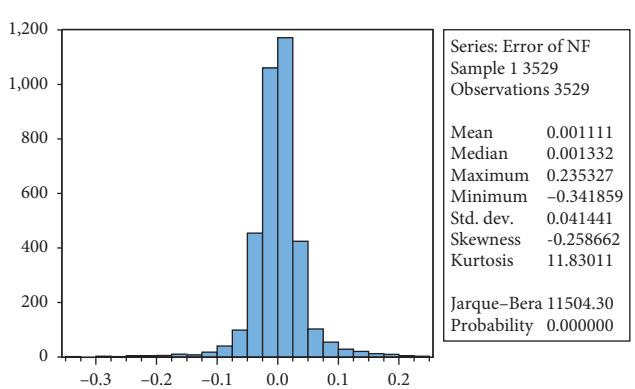

Figure 6: Price error of NF model.

results, it is expected to found that the NF model outperforms the other models.

4. Conclusions

A lot of efforts being spent on proposing the nonparametric fractional option pricing model (NF), which is better than Black–Scholes model (BS), semiparametric Black–Scholes model (SBS), and semiparametric fractional Black–Scholes option pricing model (SFBS). Comparing the pricing error histogram of semiparametric fractional Black–Scholes pricing model (SFBS) to nonparametric fractional option pricing model (NF), the experimental results have revealed that the error of NF is close to zero.

Data Availability

The datasets used and analysed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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