

## Research Article

# New Vertically Planed Pendulum Motion

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This article is concerned about the planed rigid body pendulum motion suspended with a spring which is suspended to move on a vertical plane moving uniformly about a horizontal  $X$ -axis. This model depends on a system containing three generalized coordinates. The three nonlinear differential equations of motion of the second order are obtained to the elastic string length and the oscillation angles  $\varphi_1$  and  $\varphi_2$  which represent the freedom degrees for the pendulum motions. It is assumed that the body moves in a rotating vertical plane uniformly with an arbitrary angular velocity  $\omega$ . The relative periodic motions of this model are considered. The governing equations of motion are obtained using Lagrange's equations and represent a nonlinear system of second-order differential equations that can be solved in terms of generalized coordinates. The numerical solutions are investigated using the approximated fourth-order Runge–Kutta method through programming packages. These solutions are represented graphically to describe and discuss the behavior of the body at any instant for different values of the different physical parameters of the body. The obtained results have been discussed and compared with some previously published works. Some concluding remarks have been presented at the end of this work. The value of this study comes from its wide applications in both civil and military life. The main findings and objectives of the current study are obtaining periodic solutions for the problem and satisfying their accuracy and stabilities through the numerical procedure.

## 1. Introduction

The pendulum motion is studied by many outstanding scientists in the last century due to the wide application of this problem in applied mathematics, physics, and engineering. In [1], El-Barki and others studied the rotary motion of a pendulum model about an elliptic path. They described the problem dynamically and then deduced the equations of motion for this model using Lagrange's equation. The authors defined a small parameter that depends on the different parameters of the moving model. They solved the problem analytically using the small parameter technique and numerically using the Runge–Kutta method to make a comparison between the two sets of solutions. This comparison proved the validity of both obtained solutions.

Ismail in [2] presented a case of relative periodicity motion of a rigidity pendulum model in presence of

multidegrees of freedom. He described the motion dynamically and used the Lagrangian function to describe the motion. The system of equations of motion is obtained. He defined a small parameter and used the small parameter technique to find the approximated periodic solutions of the obtained motion. He achieved computer programs through numerical consideration for proving the validity of the obtained solutions. In [3], the author studied the periodic solutions of a pendulum in a relative case. This case is considered as an especial one from the problem in [2]. The author used Poincare's method to find the approximated solutions. In [4], the author studied the oscillated motion of a simple pendulum model. He used the Lagrangian function for deriving the equations of motion. The processing method of analysis is used to find approximated solutions of the second order. In [5], the elastic pendulum oscillations are given by Vitt and Gorelik in 1933. They give an example of oscillated linear systems with two parametrical couples.

In [6], Lynch presented the three dimensions of elastic pendulum motions in the resonant case. He used the Lagrangian function for describing the motion. In [7], Holm and others studied a resonant elastic pendulum in the case of stepwise precession. In [8], the authors studied the motion of a harmonically excited elastic pendulum in the chaos response case. They derived the equations of motion of the pendulum model using Lagrange's equations. In [9], Amer described the dynamical oscillations of an elastic rigid pendulum in a plane to the equilibrium position. The author considered the plane rotates about the downward vertical fixed axis with uniform velocity. He used the Lagrangian function to deduce the equations of motion of the model. The numerical considerations [10] are considered using one of the numerical methods for searching the accuracy of the solutions.

The phase diagram procedure is used for studying the stability of the solutions [11]. In [12], Brearley studied a simple pendulum model when its string length is changing uniformly. In [13], Pinsky and others studied the oscillated pendulum model for swing with a length which varies periodically.

Nayfeh in [14] presented many perturbation techniques for solving a lot of problems in mathematics, physics, and engineering. Such techniques are named the multiple scales, small parameter, KBM, processing analysis, and finite element method which are used in solving most of the previous problems. None of the authors thought about the use of the large parameter technique which gives accurate results for the required solutions. In [15], the two freedom degrees motion of a dynamic nonlinear model for an elastic damped pendulum in the inviscid flow of fluid was considered. The system for equations of motion was considered applying the Lagrangian function. The multiple scales technique is used for solving such equations to obtain the approximated solutions. The cases of resonance and steady state were investigated. The graphical representations of the motion were considered to show the behavior of the motion. The stabilities of the motion were studied. In [16, 17], the restricted motion for the harmonically damped elastic pendulum motion of a rigid body in the elliptic path was investigated when the damped coefficients are linear. In [18], the near resonance pendulum motions in the presence of a tuned absorber dynamical model system were considered. The authors in [19] studied the pendulum motion of a rigid body which moves in a plane with a constant angular velocity  $\omega$  attached to a damped spring. The obtained solutions are analyzed numerically through computerized programs for showing motion behavior.

In this paper, a new problem is given for the elastic rigid pendulum motion in a vertical plane which rotates about a horizontal fixed axis in space by a uniform angular velocity  $\omega$ . The importance of this motion comes from its wide applications in physics, engineering, and other fields.

## 2. Formulation of the Problem

In this section, the motion of an elastic pendulum model is considered which consists of a rigid body suspended with a massless spring at point  $O_2$  which is suspended from the other hand by point  $O_1$ , see Figure 1. Let the coordinate system  $OXY$  rotate about its horizontal axis  $OX$  with a uniform angular velocity  $\omega$  relative to the pendulum motion. Consider  $OO_1 = h \cos \omega t$  at any instant of the time  $t$  such that at  $t = 0$ ,  $OO_1 = h$ . Let the point  $C$  represent a mass center of the body,  $\varphi_1$  represent the angle between  $O_1Y_1$  and  $O_1C$ , and  $\varphi_2$  denote the angle between  $O_2C$  and the vertical. Assuming  $C\xi$ ,  $C\eta$ , and  $C\zeta$  are the principal axes of inertia of the body such that  $C\zeta$  is perpendicular to the plane  $OXY$ .

Thus, the mass center  $(x_C, y_C)$  of the body to the system  $OXY$  becomes

$$\begin{aligned} x_C &= h \cos \omega t + \rho \sin \varphi_1 + a \sin \varphi_2, \\ y_C &= \rho \cos \varphi_1 + a \cos \varphi_2, \quad a = O_2C, \end{aligned} \quad (1)$$

where  $\rho$  is the elastic string length.

The potential and kinetic energies  $V$  and  $T$  are given as

$$\begin{aligned} V &= 0.5k^2(\rho - \ell)^2 - mg(\rho \cos \varphi_1 + a \cos \varphi_2), \\ T &= 0.5m\{(-h\omega \sin \omega t)^2 + \dot{\rho}^2 + (\rho\dot{\varphi}_1)^2 + (a\dot{\varphi}_2)^2 \\ &\quad - 2h\omega[\dot{\rho} \sin \varphi_1 \sin \omega t + \rho\dot{\varphi}_1 \sin \omega t \cos \varphi_1 + a\dot{\varphi}_2 \sin \omega t \cos \varphi_2] \\ &\quad + 2a[\dot{\rho}\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) + \dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)] \\ &\quad + \omega^2(h \cos \omega t + \rho \sin \varphi_1 + a \sin \varphi_2)^2 + m^{-1}I_3\dot{\varphi}_2^2 \\ &\quad + m^{-1}\omega^2(I_1 \sin^2 \varphi_2 + I_2 \cos^2 \varphi_2)\}, \end{aligned} \quad (2)$$

where  $I_1, I_2$ , and  $I_3$  are the principal inertia moments to the axes  $C\xi\eta\zeta$ ,  $k^2$  is the constant of the spring,  $\ell$  denotes the unstretched spring length, and  $g$  is the acceleration of gravity.

The Lagrangian function  $L$  for this model is of the form [15]

$$L = T - V, \quad (3)$$

where  $L$  is a function of  $\rho$ ,  $\varphi_1$ , and  $\varphi_2$  and their derivatives. Applying Lagrange's equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\rho}} \right) = \frac{\partial L}{\partial \rho}, \quad \rho, \varphi_1, \varphi_2. \quad (4)$$

Making use of equations (1) to (4), the second-order differential equations of the motion are obtained as follows:

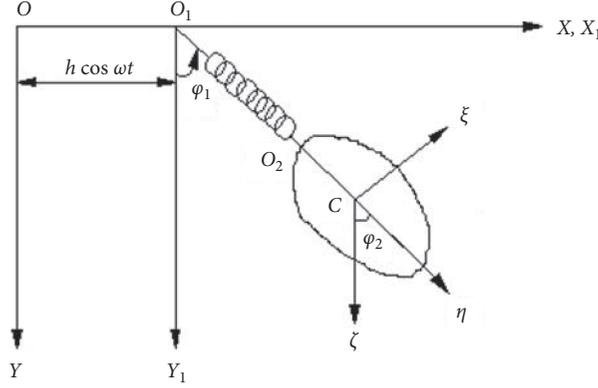


FIGURE 1: The elastic rigid pendulum model.

$$\begin{aligned}
 & \ddot{\rho} + a\ddot{\varphi}_2 \sin(\varphi_1 - \varphi_2) - a\dot{\varphi}_2^2 \cos(\varphi_1 - \varphi_2) - \rho\dot{\varphi}_1^2 \\
 & - \omega^2(2h \cos \omega t + \rho \sin \varphi_1 + a \sin \varphi_2) \sin \varphi_1 - g \cos \varphi_1 \\
 & + k^2 m^{-1}(\rho - \ell) = 0, \\
 & \rho\ddot{\varphi}_1 + 2\dot{\rho}\dot{\varphi}_1 + a\ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + a\dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2) \\
 & - \omega^2(2h \cos \omega t + \rho \sin \varphi_1 + a \sin \varphi_2) \cos \varphi_1 + g \sin \varphi_1 = 0, \\
 & \ell_1 \ddot{\varphi}_2 + (2\dot{\rho}\dot{\varphi}_1 + \rho\ddot{\varphi}_1) \cos(\varphi_1 - \varphi_2) + (\ddot{\rho} - \rho\dot{\varphi}_1^2) \sin(\varphi_1 - \varphi_2) \\
 & - \omega^2(2h \cos \omega t + \rho \sin \varphi_1 + a \sin \varphi_2) \cos(\varphi_1 - \varphi_2) + g \sin \varphi_2 \\
 & + 2m^{-1}\omega^2(I_2 - I_1) \sin 2\varphi_2 = 0, \\
 & \ell_1 = a - I_3 m^{-1} a^{-1}.
 \end{aligned} \tag{5}$$

Equations (5) are the differential equations of motion of second order in the three generalized coordinates.

Let the system oscillate in the closing relative equilibrium position, and the following is obtained:

$$I_1 = I_2. \tag{6}$$

The relative equilibrium admits the equality of the initial values for the angles  $\varphi_1$  and  $\varphi_2$ , and thus

$$\begin{aligned}
 \rho &= b + \xi(t), \\
 \varphi_1 &= \varphi_0 + \varphi(t), \\
 \varphi_2 &= \varphi_0 + \psi(t),
 \end{aligned} \tag{7}$$

where  $b$  is the relative equilibrium for the length of the pendulum string. The quantities  $b$  and  $\varphi_0$  are determined as follows:

$$\begin{aligned}
 m^{-1}k^2(b - \ell) &= \omega^2(a + b)\sin^2 \varphi_0 + g \cos \varphi_0, \\
 g &= \omega^2(a + b)\cos \varphi_0.
 \end{aligned} \tag{8}$$

Making use of (7) into (5) and then (6) and (8), one obtains

$$\begin{aligned}
 \ddot{\xi} + a_{11}\xi + a_{12}\varphi + a_{13}\psi &= f_1, \\
 b\ddot{\varphi} + a\ddot{\psi} + b_{11}\xi + b_{12}\varphi + b_{13}\psi &= f_2, \\
 b\ddot{\varphi} + \ell_1\ddot{\psi} + c_{11}\xi + c_{12}\varphi + c_{13}\psi &= f_3,
 \end{aligned} \tag{9}$$

where

$$\begin{aligned}
 a_{11} &= m^{-1}k^2 - \omega^2(\sin^2 \varphi_0 + 2h \cos \varphi_0 \cos \omega t), \\
 a_{12} &= bc_{11}, \\
 a_{13} &= ac_{11}, \\
 b_{11} &= c_{11}, \\
 b_{12} &= \frac{k^2(b - \ell)}{m - \omega^2 b \cos^2 \varphi_0}, \\
 b_{13} &= -\omega^2 a \cos^2 \varphi_0, \\
 c_{11} &= -\omega^2 \sin \varphi_0 \cos \varphi_0, \\
 c_{12} &= -\omega^2 b \cos^2 \varphi_0, \\
 c_{13} &= k^2 m^{-1}(b - \ell) + \omega^2(2h \sin \varphi_0 \cos \omega t - a \cos^2 \varphi_0), \\
 f_1 &= (\xi + b)\dot{\varphi}^2 + A\xi\varphi + a\dot{\psi}(\psi - \varphi) + a\dot{\psi}^2 + B\psi\varphi + C_1, \\
 f_2 &= -\xi\ddot{\varphi} - 2\dot{\xi}\dot{\varphi} + (\psi - \varphi)a\dot{\psi} + c_{11}(\xi + b)\varphi^2 \\
 & + (D\xi + 2a_{13}\psi)\varphi + C_2, \\
 f_3 &= \ddot{\xi}(\psi - \varphi) - \dot{\varphi}^2(b + \xi)(\psi - \varphi) - \xi\ddot{\varphi} - 2\dot{\xi}\dot{\varphi} \\
 & + 2\omega^2 h \cos \omega t \cos \varphi_0 \\
 & - [b + (1 - \psi^2) - (b + \xi)\varphi\psi]c_{11} - g \sin \varphi_0 \\
 & + \omega^2 \xi(\varphi \cos^2 \varphi_0 - \psi \sin^2 \varphi_0),
 \end{aligned} \tag{10}$$

$$\begin{aligned}
 A &= -2c_{11}, \\
 B &= a\omega^2 \cos^2 \varphi_0, \\
 D &= \frac{B}{a}, \\
 C_1 &= -(k^2 - \omega^2 \sin^2 \varphi_0) + \omega^2[2h \sin \varphi_0 \cos \omega t \\
 & + a(\cos^2 \omega t + \sin^2 \varphi_0)] + g \cos \varphi_0 + k^2 \ell, \\
 C_2 &= 2h\omega^2 \cos \omega t \\
 & - (b + a)c_{11} - g \sin \varphi_0.
 \end{aligned} \tag{11}$$

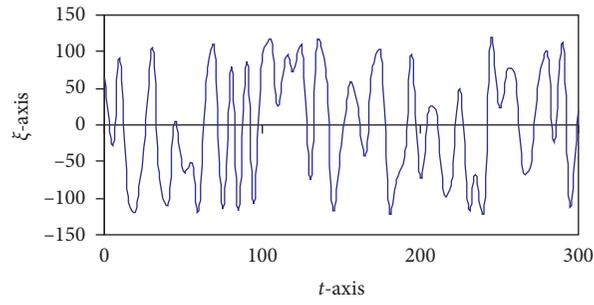


FIGURE 2: The solution  $\xi$  against the time  $t$ .

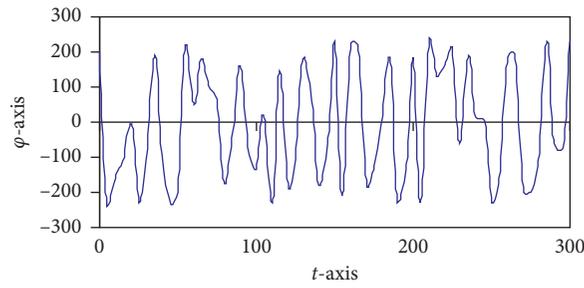


FIGURE 3: The solution  $\varphi$  against the time  $t$ .

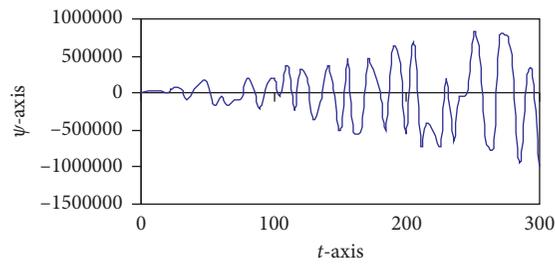


FIGURE 4: The solution  $\psi$  against the time  $t$ .

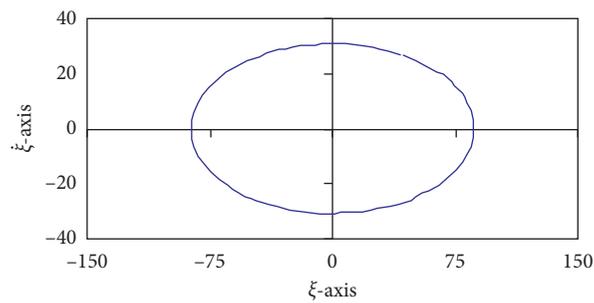


FIGURE 5: The stability  $\dot{\xi}$  diagram against  $\xi$ .

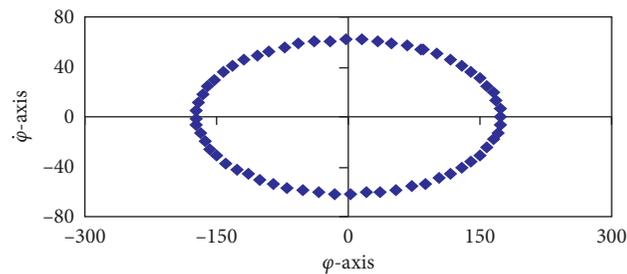


FIGURE 6: The stability  $\dot{\varphi}$  diagram against  $\varphi$ .

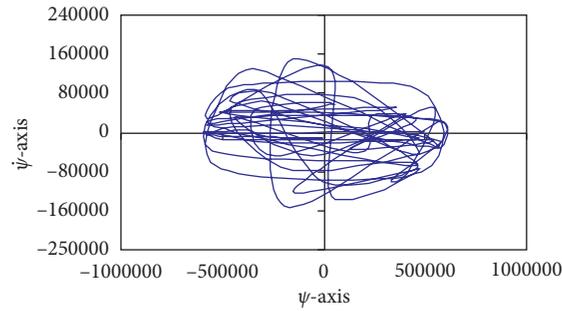


FIGURE 7: The instability  $\dot{\psi}$  diagram against  $\psi$ .

Next, numerical considerations for solving system (9) in three degrees of freedom  $\xi$ ,  $\varphi$ , and  $\psi$  are presented. The fourth-order Runge–Kutta method [10] is used for satisfying the numerical solutions for this system.

### 3. Numerical Investigations

In this section, the fourth-order Runge–Kutta method is used for solving the problem in the previous sections through computerized data. These solutions are investigated to illustrate and describe the oscillations of this system at different values of the time.

Making use of (10), (11), and (12), system (9) is reformulated in the form:

$$\ddot{\xi} = \ddot{\xi}(\xi, \varphi, \psi, \dot{\xi}, \dot{\varphi}, \dot{\psi}), (\xi\varphi\psi), \quad (13)$$

where the symbol  $(\xi\varphi\psi)$  refers to the equations which are neglected. These functions are determined accordingly to equations (9)–(12). Introducing the following data:

$$\begin{aligned} m &= 10\text{kg}, \\ g &= 9.8\text{m} \cdot \text{s}^{-2}, \\ I_1 &= 3\text{kg} \cdot \text{m}^2, \\ \ell &= 0.9\text{m}, \\ \omega &= 2\text{rad} \cdot \text{s}^{-1}, \\ a &= 0.7\text{m}, \\ b &= 0.5\text{m}, \\ \varphi_0 &= 0.2\text{rad}, \\ h &= 4, \\ t &= 0 \longrightarrow 300\text{s}. \end{aligned} \quad (14)$$

The graphical representations for the solutions  $\xi$ ,  $\varphi$ ,  $\psi$ ,  $\dot{\xi}$ ,  $\dot{\varphi}$ , and  $\dot{\psi}$  are given in Figures 2–7.

### 4. Conclusion

It is concluded that the model of relativistic elastic rigid pendulum motions is considered a wide application problem in many scientific fields. The Lagrangian function was used, and Lagrange's equations were applied for deriving the

system of equations of motion for this problem. Computer programs were achieved applying the fourth-order Runge–Kutta method for obtaining the numerical solutions for the considered system. The obtained solutions are sketched at different values of rigid body parameters. From the figures, it is deduced that the approximated solutions are seemed to be periodic in a big interval of the motion.  $\xi$  and  $\varphi$  solutions represented in Figures 2, 3, 5, and 6 have uniform motion and stable solutions, but  $\psi$  is a chaotic and excited solution, see Figures 4 and 7. In all figures, the positive vibration waves come from the motion of the vertical plane above the horizontal axis and vice versa. The solution  $\psi$  is not stable since it moves with fast oscillations with small amplitudes in the beginning time and then goes slowly with larger amplitudes. The changing of the values of  $h$ ,  $b$ , and  $\varphi_0$  affects the behavior of the motion and vice versa.

### Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

### Conflicts of Interest

The author declares that there are no conflicts of interest.

### References

- [1] F. A. El-Barki, A. I. Ismail, M. O. Shaker, and T. S. Amer, "On the motion of the pendulum on an ellipse," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 79, no. 1, pp. 65–72, 1999.
- [2] A. I. Ismail, "Relative periodic motion of a rigid body pendulum on an ellipse," *Journal of Aerospace Engineering*, vol. 22, no. 1, pp. 67–77, 2009.
- [3] N. V. Stoianov, "On the relative periodic motions of a pendulum," *Journal of Applied Mathematics and Mechanics*, vol. 28, no. 1, pp. 188–193, 1964.
- [4] S. J. Liao, "A second-order approximate analytical solution of a simple pendulum by the process analysis method," *Journal of Applied Mechanics*, vol. 59, no. 4, pp. 970–975, 1992.
- [5] A. Vitt and G. Gorelik, "Oscillations of an elastic pendulum as an example of the oscillations of two parametrically coupled linear systems," in *Historical Note No. 3Met Éireann*, Dublin, Ireland, Translated by Lisa Shields, with an Introduction by Peter Lynch, 1999.

- [6] P. Lynch, "Resonant motions of the three-dimensional elastic pendulum," *International Journal of Non-linear Mechanics*, vol. 37, no. 2, pp. 345–367, 2002.
- [7] D. D. Holm and P. Lynch, "Stepwise precession of the resonant swinging spring," *SIAM Journal on Applied Dynamical Systems*, vol. 1, no. 1, pp. 44–64, 2002.
- [8] T. S. Amer and M. A. Bek, "Chaotic responses of a harmonically excited spring pendulum moving in circular path," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 5, pp. 3196–3202, 2009.
- [9] T. S. Amer, "The dynamical behavior of a rigid body relative equilibrium position," *Advances in Mathematical Physics*, vol. 2017, Article ID 8070525, 13 pages, 2017.
- [10] A. Gilat, *Numerical Methods for Engineers and Scientists*, Wiley, New York, NY, USA, 2013.
- [11] R. Starosta, G. Sypniewska-Kamińska, and J. Awrejcewicz, "Asymptotic analysis of kinematically excited dynamical systems near resonances," *Nonlinear Dynamics*, vol. 68, no. 4, pp. 459–469, 2012.
- [12] M. N. Brearley, "The simple pendulum with uniformly changing string length," *Proceedings of the Edinburgh Mathematical Society*, vol. 15, no. 1, pp. 61–66, 1966.
- [13] M. A. Pinsky and A. A. Zevin, "Oscillations of a pendulum with a periodically varying length and a model of swing," *International Journal of Non-Linear Mechanics*, vol. 34, no. 1, pp. 105–109, 1999.
- [14] A. H. Nayfeh, "A perturbation method for treating nonlinear oscillation problems," *Journal of Mathematics and Physics*, vol. 44, no. 1–4, p. 368, 1965.
- [15] M. A. Bek, T. S. Amer, M. A. Sirwah, J. Awrejcewicz, and A. A. Araba, "The vibrational motion of a spring pendulum in a fluid flow," *Results in Physics*, vol. 19, Article ID 103465, 2020.
- [16] T. S. Amer, M. A. Bek, and M. K. Abouhmr, "On the vibrational analysis for the motion of a harmonically damped rigid body pendulum," *Nonlinear Dynamics*, vol. 91, no. 4, pp. 2485–2502, 2018.
- [17] T. S. Amer, M. A. Bek, and M. K. Abohamer, "On the motion of a harmonically excited damped spring pendulum in an elliptic path," *Mechanics Research Communications*, vol. 95, pp. 23–34, 2019.
- [18] W. S. Amer, M. A. Bek, and M. K. Abohamer, "On the motion of a pendulum attached with tuned absorber near resonances," *Results in Physics*, vol. 11, pp. 291–301, 2018.
- [19] F. M. El-Sabaa, T. S. Amer, H. M. Gad, and M. A. Bek, "On the motion of a damped rigid body near resonances under the influence of harmonically external force and moments," *Results in Physics*, vol. 19, Article ID 103352, 2020.