Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 8868200, 10 pages
https://doi.org/10.1155/2020/8868200

Research Article

Existence of Solutions for Fractional Evolution Equations with Infinite Delay and Almost Sectorial Operator

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Received 8 September 2020; Revised 27 September 2020; Accepted 8 October 2020; Published 31 October 2020

Academic Editor: Yong Hong Wu

This paper discusses a class of semilinear fractional evolution equations with infinite delay and almost sectorial operator on infinite interval in Banach space. By using the properties of analytic semigroups and Schauder's fixed-point theorem, this paper obtains the existence of mild solutions of the fractional evolution equation. Moreover, this paper also discusses the existence of mild solution when the analytic semigroup lacks compactness by Kuratowski measures of noncompactness and Darbo–Sadovskii fixed-point theorem.

1. Introduction

Fractional differential models play a very important role in describing many complex phenomena such as chaotic system [1], fluid flow [2, 3], anomalous diffusion [4–7], and so on. Compared with the classical partial differential models such as [8–19], the biggest advantage of models with fractional derivatives is their global property and history memory. Delay is short for time delay, which exists widely in the objective world. In the differential equation model with delay, the function depends not only on the current state but also on the past time state, so it is more suitable to describe the process with time memory. This property of delay is very similar to that of fractional derivatives. So many researchers introduced fractional derivatives into differential equations with delay [20–24]. Evolution equation, which is a general appellation for some partial differential equations with time variable, is mainly used to describe the time-dependent state and process. Common evolution equations include the wave equation, the heat equation, Schrodinger equation, KdV equation, Navier–Stokes equation, and so on. By using the operator semigroup theory, some partial differential evolution equations can be represented to some abstract ordinary differential equations (ODEs) in some special functional spaces. At present, the research on integer-order evolution equations has been relatively perfect [25, 26], but the research on fractional-order evolution equations is still in the preliminary stage. The existence of solutions for fractional evolution equations is also the basis of the following study. The mild solution of integer-order evolution equations is defined by the constant variation method, which cannot be directly extended to fractional-order evolution equations.

Li [20] studied the following fractional evolution equations with almost sectorial operator on finite interval:

$$\left\{ {}^{c}D_{t}^{q} x(t) = Ax(t) + f(t, x, x_t), \quad 0 < q < 1, t \in (0, T], x_0 = \phi(t) \in B, \quad t \in (-\infty, 0], \right. \tag{1}$$

where ${}^{c}D_t^q$ is the Caputo fractional derivative operator, the evolution operator A is an almost sectorial operator, and B is a phase space. x_t is the element of B defined by $x_t(\theta) = x(t+\theta)$, $\theta \in (-\infty, 0]$. Here, $x_t(\cdot)$ represents the history of state up to the present time.

Baliki et al. [22] discussed a second-order evolution equation with infinite delay and obtained the existence and attractivity of mild solutions by Schauder's fixed point as follows:

$$\begin{cases} x''(t) - A(t)x(t) = f(t, x_t), & 0 < q < 1, t \in (0, \infty), \\ x_0 = \phi(t) \in B, \ x'(0) = \widetilde{x}, \end{cases} \tag{2}$$

where $\{A(t)\}_{0 \le t < \infty}$ is a family of linear closed operators, $x_t(\theta) = x(t+\theta)$, $\theta \in (-\infty, 0]$, and B is a phase space. The existence of mild solutions for fractional evolution equations and evolution equations with infinite delay has been discussed in several papers (see [20, 21]). However, we find that most of the previous papers discuss the fractional evolution equations in the conventional spaces of continuous function on finite or infinite interval and in Banach space on finite interval. To our knowledge, no paper is devoted to the existence of mild solutions with infinite delay and almost sectorial operator on infinite interval on Banach space.

In this paper, we consider the following fractional evolution problem with infinite time delay:

$$\left\{ {}^{c}D_{0+}^q x(t) + Ax(t) = f(t, x_t), \quad t \in (0, +\infty), \ 0 < q < 1, x(t) = \phi(t) \in B, \quad t \in (-\infty, 0], \right. \tag{3}$$

where ${}^{c}D_{0+}^q$ is the Caputo fractional derivative operator, the evolution operator A is an almost sectorial operator, f is a given function which will be introduced later, and B is a phase space. For any continuous function x and any $t \ge 0$, x_t is the same as in equation (1) which represents the history of state up to the present time.

The rest of this paper is organized as follows. In Section 2, we recall some definitions, propositions, notations, and lemmas. In Section 3, the main results of this paper are obtained. We consider two cases: the semigroup $Q(t)$ generated by operator A with compactness and without compactness. For the case that $Q(t)$ is compact, we construct a special Banach space B' and obtain the existence of global mild solution by using Schauder's fixed-point theorem. For the case that $Q(t)$ is not compact, we expand the result of Theorem 1.2.4 in Guo et al. [27] from any compact interval to infinite interval (see Lemma 10) and obtain the existence of global mild solution by applying Kuratowski measures of noncompactness theory and Darbo–Sadovskii fixed-point theorem.

2. Preliminaries

In this section, we introduce some notations, definitions, lemmas, and preliminary facts that will be used in the rest of this paper. Let $(E, |\cdot|)$ be a Banach space. Denote $B(E)$ as the space of all bounded linear operators from E to itself with norm $\|\cdot\|_{B(E)}$.

Definition 1 (see [28, 29]). Let $-1 < \gamma < 0$ and $0 < \omega < (\pi/2)$. Denote by $\Theta_\omega^\gamma(E)$ all the linear closed operators A: $D(A) \subset E \longrightarrow E$ which satisfy

(1) $\sigma(A) \subset S_\omega = \{z \in C \setminus \{0\}, \arg|z| \le \omega\} \cup \{0\}$.

(2) For every $\omega < \mu < \pi$, there exists a constant C_μ such that

$$|R(z, A)| \le C_\mu |z|^\gamma \text{ for all } z \in C \backslash S_\mu. \tag{4}$$

A linear operator A will be called an almost sectorial operator on E if $A \in \Theta_\omega^\gamma(E)$.

Define the power of A as

$$A^\beta = \frac{1}{2\pi i} \int_{\Gamma_\theta} z^\beta R(z, A) \mathrm{d}z, \quad \beta > 1 + \gamma, \tag{5}$$

where $\Gamma_\theta = \{R_+ e^{i\theta} \cup R_+ e^{-i\theta}\}$ is an appropriate path oriented counterclockwise and $\omega < \theta < \mu$. Then, the linear power space $X_\beta := D(A^\beta)$ can be defined and X_β is a Banach space with the graph norm $\|x\|_\beta = |A^\beta x|$, $x \in D(A^\beta)$.

Next, let us introduce the semigroup associated with A. If A is an almost sectorial operator, then A generates an analytic semigroup $Q(t)$ of growth order $1 + \gamma$ as follows:

$$Q(t) = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{-tz} R(z, A) \mathrm{d}z, \quad t \in S^0_{(\pi/2)-\omega}, \tag{6}$$

where $\Gamma_\theta = \{R_+ e^{i\theta} \cup R_+ e^{-i\theta}\}$ is oriented counterclockwise and $\omega < \theta < \mu < (\pi/2) - \arg|t|$. $S^0_{(\pi/2)-\omega}$ is the open sector $\{z \in C \setminus \{0\}, |\arg z| < (\pi/2) - \omega\}$. Furthermore, $Q(t)$ satisfies the following properties.

Proposition 1 (see [28, 29]). Let $A \in \Theta_\omega^\gamma(E)$ with $-1 < \gamma < 0$ and $0 < \omega < (\pi/2)$. Then, the following properties remain true:

(1) $Q(t)$ is analytic in $S^0_{(\pi/2)-\omega}$ and $(d^n/dt^n)Q(t) = (-A)^n Q(t)$, $t \in S^0_{(\pi/2)-\omega}$.

(2) The functional equation holds: $Q(s+t) = Q(s)Q(t)$ for all $s, t \in S^0_{(\pi/2)-\omega}$.

(3) *There is a constant* $C_0 = C_0(\gamma) > 0$ *such that* $|Q(t)| \le C_0 t^{-\gamma-1}$, $t > 0$.

(4) *The range* $R(Q(t))$ *of* $Q(t)$ $(t \in S^0_{(\pi/2)-\omega})$ *is contained in* $D(A^\infty)$. *Particularly,* $R(Q(t)) \subset D(A^\beta)$ *for all* $\beta \in C$ *with* $\operatorname{Re}\beta > 0$:

$$A^\beta Q(t)x = \frac{1}{2\pi i}\int_{\Gamma_\theta} z^\beta e^{-tz} R(z, A)x\mathrm{d}z, \quad t \in S^0_{(\pi/2)-\omega}, x \in E, \tag{7}$$

and there exists a constant $C' = C'(\gamma, \beta) > 0$ *such that for all* $t > 0$,

$$\left|A^\beta Q(t)\right| \le C' t^{-\gamma+\operatorname{Re}\beta-1}. \tag{8}$$

(5) *If* $\beta > 1 + \gamma$, *then* $D(A^\beta) \subset \sum_Q = \left\{x \in E, \lim_{t\longrightarrow 0} Q(t)x = x\right\}$.

By Theorem 3.13 in Periago [28], if A is an almost sectorial operator, then for every $\lambda \in C$ with $\operatorname{Re}\lambda > 0$,

$$R(\lambda, -A) = \int_0^{+\infty} e^{-\lambda t} Q(t)\mathrm{d}t. \tag{9}$$

Let X *be the following set:*

$$X \coloneqq \left\{x\colon R \longrightarrow X_\beta,\ x_{[0,+\infty)} \in C\left([0, +\infty), X_\beta\right),\ \lim_{t\longrightarrow+\infty} e^{-kt}x(t) = 0,\ x_0 \in B\right\}, \tag{10}$$

where $x_{[0,+\infty)}$ *is the restriction of* x *on* $[0, +\infty)$ *and* k *is a constant.*

In this paper, we use an axiomatic definition of the phase space B. $(B, \|\cdot\|_B)$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into E and satisfies the following axioms which are introduced by Hale and Kato in [30].

(A) If $x\colon (-\infty, b] \longrightarrow E$, $b > 0$ is continuous on $[0, b]$ and $x_0 \in B$, then for any $t \in [0, b]$, the following conditions hold:

  (i) $x_t \in B$.

  (ii) There exists a positive constant H such that $|x| \le H\|x_t\|_B$.

  (iii) There exist positive continuous functions $K(\cdot), M(\cdot)$ independent of $x(\cdot)$ such that

$$\left\|x_t\right\|_B \le K(t) \sup_{0\le s\le t} \|x(s)\|_\beta + M(t)\left\|x_0\right\|_B. \tag{11}$$

(B) For the functions in (A), x_t is a B-value continuous function on $[0, b]$.

(C) The space B is complete.

Definition 2 (see [31, 32]). Let $f \in L^1((0, +\infty), E)$ and $q > 0$; then,

$$I^q_{0+} f(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)\mathrm{d}s \tag{12}$$

is called the Riemann–Liouville fractional integral of order q.

Definition 3 (see [31, 32]). The Caputo fractional derivative of order $q > 0$ of the function $f\colon (0, +\infty) \longrightarrow E$ is given by

$${}^cD^q_{0+} f(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1} f^{(n)}(s)\mathrm{d}s, \tag{13}$$

where n is the smallest integer greater than or equal to q, provided that the right side is well defined on $(0, +\infty)$.

Lemma 1 (see [31, 32]). For all $f, g \in L^q((0, +\infty), E)$, $1 \le q < \infty$,

$$I^q_{0+}(f * g) = \left(I^q_{0+} f\right) * g. \tag{14}$$

Next, we will introduce the mild solution of equation (3). Shu et al. [33] define the mild solution of equation (3) as

$$x(t) = S_q(t)\phi(0) + \int_0^t (t-s)^{q-1} P_q(t-s) f(s, x_s)\mathrm{d}s, \tag{15}$$

where $S_q(t)$ and $P_q(t)$ have the following expressions and Γ is an appropriate path in $\rho(-A)$.

$$\begin{aligned} S_q(t) &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{q-1} R(\lambda^q, -A)\mathrm{d}\lambda, \\ P_q(t) &= \frac{t^{1-q}}{2\pi i}\int_\Gamma e^{\lambda t} R(\lambda^q, -A)\mathrm{d}\lambda. \end{aligned} \tag{16}$$

Using the properties of the Mittag-Leffler function (for more details, we refer the readers to [32]),

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i}\int_\Gamma \frac{\lambda^{\alpha-\beta} e^\alpha}{\lambda^\alpha - z}\mathrm{d}\lambda, \tag{17}$$

where Γ is the same path as in (4) (see [32]), the above operators $S_q(t)$ and $P_q(t)$ can be represented as the generalized Mittag-Leffler-type functions:

$$\begin{aligned} S_q(t) &= E_{q,1}(-t^q A) = E_q(-t^q A), \\ P_q(t) &= E_{q,q}(-t^q A). \end{aligned} \tag{18}$$

Moreover, Wang et al. [29] and Zhou et al. [34–36] introduced the function of Wright-type $M_q(z)$:

$$M_q(z) = \sum_{n=1}^{+\infty} \frac{(-z)^{n-1}}{(n-1)!\Gamma(1-nq)}, \quad 0<q<1, z \in C, \tag{19}$$

and obtained another expression of $S_q(t)$, $P_q(t)$:

$$\begin{aligned} S_q(t) &= \int_0^{+\infty} M_q(s)Q(t^q s)\mathrm{d}s, \\ P_q(t) &= \int_0^{+\infty} qsM_q(s)Q(t^q s)\mathrm{d}s. \end{aligned} \tag{20}$$

In fact, these three expressions ((16)–(20)) are equivalent in the case that $t>0$ and $A \in \Theta_\omega^\gamma(E)$. Therefore, in this paper, we use the same expression of $S_q(t)$, $P_q(t)$ as Wang et al. in [29] and Zhou et al. in [34–36]. Then, the global mild solution of problem (3) is given in the following definition.

Definition 4. A function $x: R \longrightarrow X$ is called a global mild solution to the problem (3), if $x(t) \in C(R, X)$ and

$$x(t) = \begin{cases} S_q(t)\phi(0) + \int_0^t (t-s)^{q-1}P_q(t-s)f(s,x_s)\mathrm{d}s, & t \in (0,+\infty), \\ \phi(t), & t \in (-\infty, 0]. \end{cases} \tag{21}$$

Lemma 2 (see [29]). For any fixed $t>0$, $S_q(t)$ and $P_q(t)$ are linear and bounded operators and there exist constants C_s and C_p such that for all $x \in E$,

$$\begin{aligned} \left|S_q(t)x\right| &\le C_s t^{-q(1+\gamma)}|x|, \\ \left|P_q(t)x\right| &\le C_p t^{-q(1+\gamma)}|x|. \end{aligned} \tag{22}$$

Lemma 3 (see [29]). For $t>0$, operators $\{S_q(t)\}$ and $\{P_q(t)\}$ are continuous in the uniform operator topology. Moreover, for every $r>0$, the continuity is uniform on $[r,+\infty)$.

Lemma 4 (see [29]). Let $0<\beta<1-\gamma$; then,

(1) For $t>0$, the range $R(P_q(t))$ of $P_q(t)$ is contained in $D(A^\beta)$.

(2) For all $x \in D(A)$ and $t>0$, $|AS_q(t)x| \le Ct^{-q(1+\gamma)}|Ax|$, where C is a constant depending on γ, q.

Remark 1. Moreover, for all $x \in D(A^\beta)(0<\beta<1-\gamma)$ and $t>0$,

$$\begin{aligned} \left|A^\beta S_q(t)x\right| &\le C_s t^{-q(1+\gamma)}\left|A^\beta x\right|, \\ \left|A^\beta P_q(t)x\right| &\le C_p t^{-q(1+\gamma)}\left|A^\beta x\right|, \end{aligned} \tag{23}$$

that is,

$$\begin{aligned} \left\|S_q(t)x\right\|_\beta &\le C_s t^{-q(1+\gamma)}\|x\|_\beta, \\ \left\|P_q(t)x\right\|_\beta &\le C_p t^{-q(1+\gamma)}\|x\|_\beta. \end{aligned} \tag{24}$$

Lemma 5 (see [29]). Let $\beta>1+\gamma$; then, $\lim_{t\longrightarrow 0+}S_q(t)x = x$ for all $x \in D(A^\beta)$.

3. Main Results

In this section, our main purpose is to establish sufficient conditions for the existence of global mild solutions to problem (3) in X. Assume that:

(H) $f: [0,+\infty) \times B \longrightarrow X_\beta$, $(1+\gamma<\beta<1-\gamma)$ is continuous and satisfies

$$\|f(t,x)\|_\beta \le p(t)e^{-kt}\|x\|_B, \tag{25}$$

where $p(t)$ is a nonnegative and continuous function on $[0,+\infty)$ and here exists a big enough $k>0$ such that

(i) For any $t \ge 0$,

$$C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1}p(s)K(s)\mathrm{d}s \le \frac{1}{2}, \tag{26}$$

(ii) $\lim_{t\longrightarrow+\infty} e^{-kt}\int_0^t (t-s)^{-q\gamma-1}p(s)K(s)\mathrm{d}s = 0$, $\lim_{t\longrightarrow+\infty} e^{-kt}\int_0^t (t-s)^{-q\gamma-1}p(s)M(s)\mathrm{d}s = 0$.

In order to obtain the existence of global mild solution of problem (3), we transform it into a fixed-point problem. For any $\phi(0) \in X_\beta$, define the operator $\widehat{T}: X \longrightarrow X$ as

$$\widehat{T}x(t) = \begin{cases} S_q(t)\phi(0) + \int_0^t (t-s)^{q-1}P_q(t-s)f(s,x_s)\mathrm{d}s, & t \in (0,+\infty), \\ \phi(t), & t \in (-\infty, 0]. \end{cases} \tag{27}$$

Let $z(t)$: $R \longrightarrow X$ be the function

$$z(t)=\begin{cases} S_q(t)\phi(0), & t\in(0,+\infty),\\ \phi(t), & t\in(-\infty,0],\end{cases} \tag{28}$$

and $x(t)=y(t)+z(t)$, $t\in R$. It is easy to know that $x(t)$ satisfies (21) if and only if

$$y(t)=\begin{cases} \int_0^t (t-s)^{q-1}P_q(t-s)f(s,y_s+z_s)\mathrm{d}s, & t\in(0,+\infty),\\ y_0=0, & t\in(-\infty,0].\end{cases} \tag{29}$$

Define the set $B':=\{y\in X:\ y_0=0\in B\}$ endowed with seminorm $\|\cdot\|_b$:

$$\|y\|_b=\|y_0\|_B+\sup_{t\ge0}\{e^{-kt}\|y(t)\|_\beta\}=\sup_{t\ge0}\{e^{-kt}\|y(t)\|_\beta\}. \tag{30}$$

Thus, $(B',\|\cdot\|_b)$ is a Banach space. Define the operator T: $B'\longrightarrow B'$ as

$$Ty(t)=\begin{cases} \int_0^t (t-s)^{q-1}P_q(t-s)f(s,y_s+z_s)\mathrm{d}s, & t\in(0,+\infty),\\ y_0=0, & t\in(-\infty,0].\end{cases} \tag{31}$$

Consequently, the operator $\widehat{T}$: $X\longrightarrow X$ having a fixed point in X is equivalent to the operator T: $B'\longrightarrow B'$ having a fixed point in B'.

Lemma 6. *Assume that condition (H) is valid; then, there exists a constant $r>0$ such that*

$$C_pe^{-kt}\int_0^t (t-s)^{-q\gamma-1}e^{-ks}p(s)[K(s)M+M(s)\|\phi\|_B]\mathrm{d}s\le\frac{r}{2}, \tag{32}$$

where M satisfies $\sup_{t>0}\|S_q(t)\phi(0)\|_\beta\le M$. Consider $B_r:=\{y\in B',\|y\|_b\le r\}$; then, for any $\phi(0)\in X_\beta$, the operator T: $B_r\longrightarrow B_r$ is continuous.

Proof. By $\phi(0)\in X_\beta$ and Lemma 5 (1), there exists $0<\delta_1<T$, and for any $t\in(0,\delta_1]$, such that $\|S_q(t)\phi(0)-\phi(0)\|_\beta<\varepsilon$. for any $t\ge\delta_1$, $\|S_q(t)\phi(0)\|_\beta\le C_s\|\phi(0)\|_\beta\delta_1^{-q(1+\gamma)}$. Therefore, there exists a constant $M>0$ such that $\sup_{t>0}\|S_q(t)\phi(0)\|_\beta\le M$.

For any $y(t)\in B_r$, $0<s<t$, note that

$$\begin{aligned}\|y_s+z_s\|_B&\le\|y_s\|_B+\|z_s\|_B,\\ &\le K(s)e^{ks}\|y\|_b+K(s)\sup_{0<\tau\le s}\left\|S_q(\tau)\phi(0)\right\|_\beta\\ &\quad+M(s)\|\phi\|_B,\\ &\le K(s)e^{ks}\|y\|_b+K(s)M+M(s)\|\phi\|_B,\\ &:=\eta(s).\end{aligned} \tag{33}$$

Then, by condition (H) and Remark 1, we have

$$\begin{aligned}e^{-kt}\|Ty(t)\|_\beta&\le e^{-kt}\int_0^t (t-s)^{q-1}\left\|P_q(t-s)f(s,y_s+z_s)\right\|_\beta\mathrm{d}s,\\ &\le C_pe^{-kt}\int_0^t (t-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s,\\ &\le\left(C_pe^{-kt}\int_0^t (t-s)^{-q\gamma-1}e^{-ks}p(s)K(s)\mathrm{d}s\right)\|y\|_b,\\ &\quad+C_pe^{-kt}\int_0^t (t-s)^{-q\gamma-1}e^{-ks}p(s)[K(s)M+M(s)\|\phi\|_B]\mathrm{d}s,\\ &\le\frac{r}{2}+\frac{r}{2}=r,\end{aligned} \tag{34}$$

which implies that $\|Ty\|_b\le r$ and T: $B_r\longrightarrow B_r$.

Next, we will prove the continuity of T. Let $\{y^n(t)\}_{n=1}^{\infty}\in B_r$ and $\|y^n-y\|_b\longrightarrow0$ as $n\longrightarrow\infty$ for any $t\ge0$. Then, for any $t>0$, by the continuity of f,

$$\begin{aligned}e^{-kt}\left\|Ty^n(t)-Ty(t)\right\|_\beta&\le C_pe^{-kt}\int_0^t (t-s)^{-q\gamma-1}\\ &\quad\left\|f(s,y_s^n)-f(s,y_s)\right\|_\beta\mathrm{d}s\longrightarrow0\ (n\longrightarrow\infty),\end{aligned} \tag{35}$$

which implies that $\|Ty^n(t)-Ty(t)\|_b\longrightarrow0$ as $n\longrightarrow\infty$. Therefore, the continuity of T is proved. □

Lemma 7. *Assume that condition (H) is satisfied; then, for any $\phi(0)\in X_\beta$,*

(1) *$\{e^{-kt}Ty(t),\ y\in B'\}$ is equicontinuous on any compact interval of $[0,+\infty)$.*

(2) *For any given $\varepsilon>0$, there exists a constant $T>0$ such that $e^{-kt}\|Ty(t)\|_\beta<\varepsilon$ for any $t\ge T$ and $y\in B'$.*

Proof. (1) Without loss of generality, we take $[0,T) \subset [0,+\infty)$ as the compact interval and $0 \le t_1 < t_2 \le T$. Firstly, for $t_1 = 0$, $t_1 < t_2 \le T$ and any $y \in B'$, according to the continuity of $p(s)$ and $\eta(s)$, we have

$$\left\| e^{-kt_1} Ty(t_1) - e^{-kt_2} Ty(t_2) \right\|_\beta \le C_p e^{-kt_2} \int_0^{t_2} (t_2 - s)^{-q\gamma-1} e^{-ks} p(s)\eta(s)\mathrm{d}s \longrightarrow 0 \; (t_2 \longrightarrow 0). \tag{36}$$

Next, for $0 < t_1 < t_2 \le T$, by Lemma 2 and Remark 1, we have

$$\begin{aligned}
&\left\| e^{-kt_1} Ty(t_1) - e^{-kt_2} Ty(t_2) \right\|_\beta, \\
&\le e^{-kt_2} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \left\| P_q(t_2 - s) f(s, y_s + z_s) \right\|_\beta \mathrm{d}s, \\
&\quad + \left(e^{-kt_1} - e^{-kt_2}\right) \int_0^{t_1} (t_2 - s)^{q-1} \left\| P_q(t_2 - s) f(s, y_s + z_s) \right\|_\beta ds, \\
&\quad + e^{-kt_1} \int_0^{t_1} \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right] \left\| P_q(t_2 - s) f(s, y_s + z_s) \right\|_\beta \mathrm{d}s, \\
&\quad + e^{-kt_1} \int_0^{t_1} (t_1 - s)^{q-1} \left\| \left(P_q(t_2 - s) - P_q(t_1 - s) \right) f(s, y_s + z_s) \right\|_\beta \mathrm{d}s, \\
&\le C_p e^{-kt_2} \int_{t_1}^{t_2} (t_2 - s)^{-q\gamma-1} e^{-ks} p(s)\eta(s)\mathrm{d}s, \\
&\quad + C_p\left(e^{-kt_1} - e^{-kt_2}\right) \int_0^{t_1} (t_2 - s)^{-q\gamma-1} e^{-ks} p(s)\eta(s)\mathrm{d}s, \\
&\quad + C_p e^{-kt_1} \int_0^{t_1} \left[(t_1 - s)^{q-1} - (t_2 - s)^{-q\gamma-1} \right] e^{-ks} p(s)\eta(s)\mathrm{d}s, \\
&\quad + \sup_{s \in [0, t_1 - \delta]} \left\| P_q(t_2 - s) - P_q(t_1 - s) \right\|_{B(E)} e^{-kt_1} \int_0^{t_1-\delta} (t_1 - s)^{-q\gamma-1} e^{-ks} p(s)\eta(s)\mathrm{d}s, \\
&\quad + e^{-kt_1} \int_{t_1-\delta}^{t_1} (t_1 - s)^{q-1} \left\| \left(P_q(t_2 - s) - P_q(t_1 - s) \right) f(s, y_s + z_s) \right\|_\beta \mathrm{d}s, \\
&\coloneqq I_{11}(t) + I_{12}(t) + I_{13}(t) + I_{14}(t) + I_{15}(t).
\end{aligned} \tag{37}$$

For $I_{11}(t)$, $I_{12}(t)$, and $I_{14}(t)$ by the continuity of $p(s)$, $\eta(s)$, e^{-ks}, and $P_q(s)$, we have $I_{11}(t)$, $I_{12}(t) I_{14}(t) \longrightarrow 0$ as $t_2 \longrightarrow t_1$, $\delta \longrightarrow 0$. For $I_{13}(t)$ and $I_{15}(t)$, note that

$$I_{1i}(t) \le 2C_p e^{-kt_1} \int_0^{t_1} (t_1 - s)^{-q\gamma-1} e^{-ks} p(s)\eta(s)\mathrm{d}s, \quad i = 3, 5. \tag{38}$$

Then, by using Lebesgue's dominated convergence theorem, we have $I_{13}(t), I_{15}(t) \longrightarrow 0$ as $t_2 \longrightarrow t_1$, $\delta \longrightarrow 0$. Therefore, for any $0 \le t_1 < t_2 \le T$ and $y \in B'$, $\|Ty(t_1) - Ty(t_2)\|_b \longrightarrow 0$ as $t_2 \longrightarrow t_1$, $\delta \longrightarrow 0$.

(2) By condition (H), for big enough $T > 0$,

$$e^{-kt} \int_0^t (t - s)^{-q\gamma-1} e^{-ks} p(s)\eta(s)\mathrm{d}s < \frac{1}{C_p}\varepsilon. \tag{39}$$

Then, for any $t \ge T$, $y \in B'$, we have

$$e^{-kt} \|Ty(t)\|_\beta \le C_p e^{-kt} \int_0^t (t - s)^{-q\gamma-1} e^{-ks} p(s)\eta(s)\mathrm{d}s < \varepsilon. \tag{40}$$

□

3.1. The Case That $Q(t)$ Is Compact. In this section, we assume that $Q(t)$ is compact for $t > 0$, i.e., $Q(t)$ is a compact operator for every $t > 0$.

Lemma 8. *Let $Z \subseteq B'$ be a bounded set; then, Z is relatively compact in B' if the following conditions hold:*

(1) The set $\{y(t), y \in Z\}$ is equicontinuous on any compact interval of $[0,+\infty)$ and for any $t \ge 0$, $\{y(t), y \in Z\}$ is relatively compact in X.

(2) For any given $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that $e^{-kt}\|y(t)\|_\beta < \varepsilon$ for any $t \ge T$ and $y(t) \in Z$.

Proof. It is sufficient to prove that Z is totally bounded. We consider the compact interval $[0,T]$ of $[0,+\infty)$. Define

$$Z_{[0,T]} := \{y(t):\ y(t) \in Z, \quad t \in [0,T]\}, \tag{41}$$

with norm $\|y\|_{b_1} := \sup_{0\le t\le T}\{e^{-kt}\|y(t)\|_\beta\}$; then, condition (1) combined with Arzelà–Ascoli theorem in Banach space indicates that $Z_{[0,T]}$ is relatively compact. Therefore, for any $\varepsilon > 0$, there exist finitely many balls $B_\varepsilon(y^i)$ such that $Z_{[0,T]} \subset \cup_{i=1}^{n} B_\varepsilon(y^i)$, where $y^i \in B'$.

$$B_\varepsilon(y^i) = \left\{y(t) \in Z_{[0,T]}, \|y - y^i\|_{b_1} = \sup_{0\le t\le T}\left\{e^{-kt}\|y(t) - y^i(t)\|_\beta\right\} \le \varepsilon\right\}. \tag{42}$$

Hence, for any $y(t) \in Z$, there exists an $i \in \{1, 2, \ldots, n\}$ such that $y_{[0,T]} \in B_\varepsilon(y^i)$, i.e., for $t \in [0,T]$,

$$e^{-kt}\|y(t) - y^i(t)\|_\beta \le \varepsilon. \tag{43}$$

Moreover, for $t \in [T,+\infty]$, with conditions (3) and (43),

$$\begin{aligned}
&e^{-kt}\|y(t) - y^i(t)\|_\beta, \\
&\le \left\|e^{-kt}y(t) - e^{-kT}y(T)\right\|_\beta + \left\|e^{-kT}y(T) - e^{-kT}y^i(T)\right\|_\beta \\
&\quad + \left\|e^{-kT}y^i(T) - e^{-kt}y^i(t)\right\|_\beta, \\
&\le 5\varepsilon.
\end{aligned} \tag{44}$$

Therefore, by (43) and (44), we have $\|y(t) - y^i(t)\|_b \le 5\varepsilon$ for any $t \ge 0$. Then, Z can be covered by balls $B_{5\varepsilon}(y^i) = \{y(t) \in Z, |y - y^i|_b \le 5\varepsilon\}$. Consequently, Z is totally bounded and the process is complete. □

Theorem 1. *Assume that condition (H) holds; then, for $\phi(0) \in X_\beta$, problem (3) has at least one global mild solution in B_r.*

Proof. We aim to prove this theorem by using Schauder's fixed-point theorem. In view of Lemma 6, $T\colon B_r \longrightarrow B_r$ and T is continuous, so we just need to prove that for any bounded subset $V \subset B_r$, TV is relatively compact in X. Then, it is easy to prove that TV satisfies all conditions in Lemma 8.

Consider Lemma 6; we have proved that $\|Ty\|_b = \sup\{e^{-kt}\|Ty(t)\|_\beta\} \le r$ for any $y \in B_r$ which implies $\{Ty,\ y \in {}^{t}B_r^0\}$ is uniformly bounded. By Lemma 7, $\{Ty, y \in B'\}$ is equicontinuous on any compact interval $[0,T]$ of $[0,+\infty)$ and $e^{-kt}\|Ty(t)\|_\beta < \varepsilon$ for any $t \ge T$ and $y \in B'$. Then, it remains to show that $V(t) = \{(Ty)(t),\ y(t) \in V\}$ is relatively compact in X for any $t \in [0,T]$.

It is easy to know that $V(0) = \{0\}$ is compact in X. Let $t \in [0,T)$ be fixed and for any $\varepsilon \in (0,t)$, $\delta > 0$, we define an operator T_ε^δ on V by the formula

$$\begin{aligned}
\left(T_\varepsilon^\delta y\right)(t) &= \int_0^{t-\varepsilon}\int_\delta^{+\infty} q\theta(t-s)^{q-1}M_q(\theta)\, Q\left((t-s)^q\theta\right)f(s, y_s + z_s)\mathrm{d}\theta\mathrm{d}s, \\
&= Q(\varepsilon^q\delta)\int_0^{t-\varepsilon}\int_\delta^{+\infty} q\theta(t-s)^{q-1}M_q(\theta)\, Q\left((t-s)^q\theta - \varepsilon^q\delta\right)f(s, y_s + z_s)\mathrm{d}\theta\mathrm{d}s,
\end{aligned} \tag{45}$$

where $y \in V$. Under the compactness of $Q(\varepsilon^q\delta)\,(\varepsilon^q\delta > 0)$ and the boundedness of

$$\begin{aligned}
&\int_0^{t-\varepsilon}\int_\delta^{+\infty} q\theta(t-s)^{q-1}M_q(\theta)Q\left((t-s)^q\theta - \varepsilon^q\delta\right) \\
&f(s, y_s + z_s)\mathrm{d}\theta\mathrm{d}s,
\end{aligned} \tag{46}$$

we obtain that the set $V_\varepsilon^\delta(t) = \{(T_\varepsilon^\delta y)(t),\ y \in V\}$ is relatively compact in X for any $\varepsilon \in (0,t)$ and $\delta > 0$. Moreover, for any $y \in V$, $t > 0$, we have

$$\begin{aligned}
&e^{-kt}\left\|(Ty)(t) - \left(T_\varepsilon^\delta y\right)(t)\right\|_\beta, \\
&\le qe^{-kt}\left\|\int_0^t\int_0^\delta \theta(t-s)^{q-1}M_q(\theta)\left((t-s)^q\theta\right)f(s, y_s + z_s)\mathrm{d}\theta\mathrm{d}s\right\|_\beta, \\
&\quad + e^{-kt}\left\|\int_{t-\varepsilon}^t\int_\delta^{+\infty} q\theta(t-s)^{q-1}M_q(\theta)\, Q\left((t-s)^q\theta\right)f(s, y_s + z_s)\mathrm{d}\theta\mathrm{d}s\right\|_\beta, \\
&\le qC_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s\int_0^\delta \theta^{-\gamma}M_q(\theta)\mathrm{d}\theta, \\
&\quad + C_p e^{-kt}\int_{t-\varepsilon}^t (t-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s.
\end{aligned} \tag{47}$$

According to $\int_0^{+\infty} \theta^r M_q(\theta)d\theta = (\Gamma(1+r)/\Gamma(1+qr))$ and condition (H), we have

$$\int_0^{\delta} \theta^{-\gamma} M_q(\theta)\mathrm{d}\theta \longrightarrow 0,$$
$$\int_{t-\varepsilon}^{t} (t-s)^{-q\gamma-1} p(s)\eta(s)\mathrm{d}s \longrightarrow 0, \quad \text{as}\ \varepsilon \longrightarrow 0, \delta \longrightarrow 0, \tag{48}$$

which implies $\|(Ty)(t) - (T_\varepsilon^\delta y)(t)\|_b \longrightarrow 0$ as $\varepsilon \longrightarrow 0, \delta \longrightarrow 0$.

Therefore, the relatively compact set $V_\varepsilon^\delta(t)$ is arbitrarily close to the set $V(t)$. Hence, for any $t \in [0,T]$, the set $V(t)$, $t \in [0,T]$ is also relatively compact in X.

Hence, T: $B_r \longrightarrow B_r$ is a completely continuous operator. So, by Schauder's fixed-point theorem, T has at least one fixed point in B_r which implies that problem (3) has at least one global mild solution in B_r. □

3.2. The Case That $Q(t)$ Is Not Compact. In this section, we assume that $Q(t)$ is not compact. In the following, α and $\alpha_{B'}$ denote the Kuratowski measures of noncompactness of bounded sets in X_β and in B'. For more details about Kuratowski measures of noncompactness, we refer the readers to [27]. Assume that:

(H^*) There exists $m(t) \in L([0,+\infty), [0,+\infty))$ such that $I_{0+}^q m$ exists and for any bounded set $V \subset B$,

$$\alpha(f(t,V)) \le m(t)\, e^{-kt} \sup\nolimits_{-\infty<\tau\le 0} \alpha(V(\tau)), \tag{49}$$

and for any $t \ge 0$,

$$C_p e^{-kt} \int_0^t (t-s)^{-q\gamma-1} m(s)\mathrm{d}s < 1. \tag{50}$$

Lemma 9 (see [27]). If $V \subset C(J,E)$ is bounded and equicontinuous, then $\alpha(V(t))$ is continuous and

$$\alpha\left(\left\{\int_J y(t)\mathrm{d}t, \quad y \in V\right\}\right) \le \int_J \alpha(V(t))\mathrm{d}t, \tag{51}$$

where J is any compact interval of $[0,+\infty)$.

Lemma 10. *Let V be a bounded set in B'. Suppose that $V(t)$ is equicontinuous on any compact interval $[0,T]$ of $[0,+\infty)$ and for any $t \ge T$, $\varepsilon > 0$, and $y \in V$,*

$$e^{-kt}\|y(t)\|_\beta < \varepsilon. \tag{52}$$

Then, for each $V(t) = \{y(t),\ y \in V\}$,

$$\alpha_{B'}(V) = \sup_{t\ge 0}\left\{e^{-kt}\alpha(V(t))\right\}. \tag{53}$$

Proof. First, we prove that $\alpha_{B'}(V) \ge \sup_{t\ge 0}\{e^{-kt}\alpha(V(t))\}$. For the above given $\varepsilon > 0$, $t \ge 0$, there exists a partition $V = \cup_{j=1}^n V_j$ such that

$$\mathrm{diam}\left(V_j\right) < \alpha_{B'}(V) + \varepsilon, \quad \text{for any}\ j = 1,2,\ldots,n. \tag{54}$$

Then, $V(t) = \cup_{j=1}^n V_j(t)$. For any $u, v \in V_j$, $t \ge 0$,

$$e^{-kt}\|u(t) - v(t)\|_\beta \le \mathrm{diam}\left(V_j\right) < \alpha_{B'}(V) + \varepsilon. \tag{55}$$

Therefore, $\mathrm{diam}(V_j(t)) \le e^{kt}(\alpha_{B'}(V) + \varepsilon)$ which implies

$$\sup_{t\ge 0}\left\{e^{-kt}\alpha(V(t))\right\} \le \alpha_{B'}(V), \tag{56}$$

by the arbitrariness of ε.

Next, we show that $\alpha_{B'}(V) \le \sup_{t\ge 0}\{e^{-kt}\alpha(V(t))\}$. By the equicontinuity of $V(t)$ on $[0,T]$, there exists a partition $0 = t_0 < t_1 < \cdots < t_m = T$ such that

$$\left\|e^{-kt_i'} y(t_i') - e^{-kt_i''} y(t_i'')\right\|_\beta < \varepsilon, \tag{57}$$

for any $t_i', t_i'' \in [t_i, t_{i+1}]$, $y \in V$, $i = 0,1,\ldots,m-1$. Let $I_i = [t_i, t_{i+1}]$, $i = 0,1,\ldots,m-1$ and $I_m = [t_m, +\infty)$; then, by (51) and (57),

$$\left\|e^{-kt_i'} y(t_i') - e^{-kt_i''} y(t_i'')\right\|_\beta < 2\varepsilon, \quad \text{for any}\ y \in V,\ t_i', t_i'' \in I_i, \quad i = 0,1,\ldots,m. \tag{58}$$

For each $i \in \{0,1,\ldots,m\}$, there exists a division $V = \cup_{j=1}^n V_j^i$ such that $V(t_i') = \cup_{j=1}^n V_j^i(t_i')$ and

$$\mathrm{diam}\left(V_j^i(t_i')\right) < \alpha(V(t_i')) + 2\varepsilon, \quad j = 1,2,\ldots,n. \tag{59}$$

Let Y be the finite set of all maps $i \longrightarrow \gamma(i)$ of $\{0,1,\ldots,m\}$ into $\{1,2,\ldots,n\}$. For $\gamma \in Y$,

$$Z_\gamma := \left\{y \in V,\ y(t_i') \in V_{\gamma(i)}^i(t_i'), \quad i = 0,1,\ldots,m\right\}, \tag{60}$$

so $V = \{y(t),\ y \in Z_\gamma, \gamma \in Y\}$. For any $u, v \in Z_\gamma$ and $t \geq 0$, there exists $i \in \{0, 1, \ldots, m\}$ such that $t \in I_i$; then,

$$\begin{aligned} &e^{-kt}\|u(t) - v(t)\|_\beta, \\ &\leq \left\|e^{-kt}u(t) - e^{-kt_i'}u(t_i')\right\|_\beta + \left\|e^{-kt_i'}u(t_i') - e^{-kt_i'}v(t_i')\right\|_\beta + \left\|e^{-kt}v(t) - e^{-kt_i'}v(t_i')\right\|_\beta, \\ &< \alpha(V(t_i')) + 6\varepsilon. \end{aligned} \tag{61}$$

Therefore, $\operatorname{diam}(Z_\gamma) \leq \alpha(V(t_i')) + 6\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have

$$\alpha_{B'}(V) \leq \sup_{t\geq 0}\left\{e^{-kt}\alpha(V(t))\right\}. \tag{62}$$

□

Lemma 11 (see [27]). Let D be a bounded, closed, and convex subset of Banach space E. If the operator $T\colon D \longrightarrow D$ is a strict set contraction, then T has a fixed point in D.

Remark 2. A bounded and continuous operator $T\colon D \longrightarrow E$ is called a strict set contraction if there is a constant $0 \leq \lambda < 1$ such that $\alpha(TV) \leq \lambda\alpha(V)$ for any bounded set $V \subset D$.

Theorem 2. *Assume that conditions (H), (H^*) are satisfied; then, for $\phi(0) \in X_\beta$, problem (3) has a global mild solution in B_r.*

Proof. Let V be an arbitrary bounded set in B_r. According to Lemmas 6 and 7, we know that $T\colon B_r \longrightarrow B_r$ is bounded and continuous and $\{Ty(\cdot), y \in V\}$ is equicontinuous on $[0, T]$ and $e^{-kt}\|Ty(t)\|_\beta < \varepsilon$ for any $t \geq T$, $y \in V$, $\varepsilon > 0$. Then, by Lemma 3.6, it follows that

$$\alpha_{B'}(TV) = \sup_{t\geq 0}\left\{e^{-kt}\alpha(TV(t))\right\}. \tag{63}$$

Consider Lemma 9 and condition (H^*); let any $t \geq 0$ be fixed, and for the above $\varepsilon > 0$, we have

$$\begin{aligned} e^{-kt}\alpha(TV(t)) &= e^{-kt}\alpha\left(\left\{\int_0^t (t-s)^{q-1}P_q(t-s)f(s, y_s + z_s)\mathrm{d}s, \quad y \in V\right\}\right), \\ &\leq e^{-kt}\int_0^t \alpha\left(\left\{(t-s)^{q-1}P_q(t-s)f(s, y_s + z_s), \quad y \in V\right\}\right)\mathrm{d}s, \\ &\leq C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1}\alpha(\{f(s, y_s + z_s), \quad y \in V\})\mathrm{d}s, \\ &\leq C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1}m(s)e^{-ks}\sup_{0\leq\tau\leq s}\alpha(V(\tau))\mathrm{d}s, \\ &\leq \left(C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1}m(s)\mathrm{d}s\right)\alpha_{B'}(V), \end{aligned} \tag{64}$$

which implies that $\alpha_{B'}(TV) \leq \lambda\alpha_{B'}(V)$ where $\lambda := C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1}m(s)\mathrm{d}s < 1$. Then, T is a strict set contraction.

Consequently, by Lemma 11, T has a fixed point in B_r which implies that problem (3) has a global mild solution in B_r. The proof process is completed. □

4. Conclusions

In this paper, we investigated a class of fractional evolution equations with infinite delay and almost sectorial operator on unbounded domains in Banach space. We considered the case of compact semigroups and noncompact semigroups and obtained sufficient conditions of the existence of global mild solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported by the National Natural Science Foundation of China (no. 11671181).

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