Research Article

New Simpson Type Integral Inequalities for s -Convex Functions and Their Applications

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First, we consider a new Simpson's identity. This identity investigates our main results that consist of some integral inequalities of Simpson's type for the s -convex functions. From our main results, we obtain some special cases which are discussed in detail. Finally, some applications on the Bessel functions, special means of distinct positive real numbers, and error estimation about Simpson quadrature formula are presented to support our theoretical results.

1. Introduction and Preliminaries

Integral inequalities are generally applicable in many branches of mathematics such as mathematical analysis, fractional calculus, discrete fractional calculus, and abstract spaces; for an overview, the reader should see the literature on integral inequalities, e.g., [1–11] and the references therein.

Nowadays, the study of convexity is considered as an original icon in the investigation of theoretical behavior of mathematical inequalities (e.g., [12–15]). Recently, several works on integral inequalities for convex functions were conducted. In particular, much attention has been given to the theoretical studies of inequalities on different types of convex functions such as s -geometrically convex functions [16], GA-convex functions [17], quasiconvex function [18], MT-convex function [19, 20], (α, m) -convex functions [21], F -convex functions [22], λ_ψ -convex functions [23], and a new class of convex functions [24], and many other types can be found in [25].

Now we recall some useful definitions of convex functions that are essential for our present investigation.

Definition 1 (see [25]). A function $f: \mathcal{T} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex, if

$$f(\nu\omega_1 + (1-\nu)\omega_2) \leq \nu f(\omega_1) + (1-\nu)f(\omega_2) \quad (1)$$

holds for all $\omega_1, \omega_2 \in \mathcal{T}$ and $\nu \in [0, 1]$. Likewise, f is concave if $(-f)$ is convex.

Definition 2 (see [26]). Let $s \in (0, 1]$ be fixed. A function $f: \mathcal{T} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be s -convex (in the second sense), if

$$f(\nu\omega_1 + (1-\nu)\omega_2) \leq \nu^s f(\omega_1) + (1-\nu)^s f(\omega_2) \quad (2)$$

holds for all $\omega_1, \omega_2 \in \mathcal{T}$ and $\nu \in [0, 1]$.

Recently, many well-known inequalities haven't been established for the convex function and its types such as

Hermite–Hadamard type inequalities [27, 28], Hermite–Hadamard–Fejer type inequalities [29], Ostrowski type inequalities [30], Hardy type inequalities [31], Olsen type inequalities [32], Gagliardo–Nirenberg type inequalities [33], Opial type inequalities [34, 35], and Rozanova type inequalities [36]. However, our focus in this paper is on another type of inequality, namely, the Simpson's inequality, which is obtained by Dragomir et al. [37], and their main result is as follows.

Theorem 1 (see [37]). Assume that $f: [\omega_1, \omega_2] \rightarrow \mathbb{R}$ is a four times continuous and differentiable function on (ω_1, ω_2) such that $\|f^{(4)}\|_\infty := \sup_{x \in (\omega_1, \omega_2)} |f^{(4)}(x)| < +\infty$ with $\omega_1 < \omega_2$. Then, the following integral inequality holds:

$$\left| \frac{1}{6} \left[f(\omega_1) + 4f\left(\frac{\omega_1 + \omega_2}{2}\right) + f(\omega_2) \right] - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} f(x) dx \right| \leq \frac{1}{2880} (\omega_2 - \omega_1)^4 \|f^{(4)}\|_\infty. \quad (3)$$

Just like the aforementioned inequalities, several integral inequalities related to the Simpson's integral inequality (3) has been found for the convex functions (see, e.g., [38–45]).

In view of the above indices and literatures, we will prove a new Simpson's integral identity. From this, then we will obtain several Simpson's integral inequalities and some special cases.

The rest of this article is organized in the following way: Section 2 contains the main results of the article which consists of proving several Simpson's integral inequalities and some related results. In Section 3, we give some applications of the Bessel functions, special means, and error estimation about the Simpson quadrature formula to support the main results. Section 4 is for the conclusions.

2. Main Results

We need the following Lemma in order to establish our main results.

Lemma 1. Let $f: \mathcal{T} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be an $\mathcal{L}^1$ function on $\mathcal{T}$ such that $\omega_1, \omega_2 \in \mathcal{T}$ with $\omega_1 < \omega_2$. Also, let $f' \in \mathcal{L}^1$, and then for any positive integer number n , we have

$$\begin{aligned} T_n(f; \omega_1, \omega_2) &:= \sum_{k=0}^{n-1} \frac{1}{6n} \left[f\left(\frac{(n-k)\omega_1 + k\omega_2}{n}\right) + 4f\left(\frac{(2(n-k)-1)\omega_1 + (2k+1)\omega_2}{2n}\right) \right. \\ &\quad \left. + f\left(\frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n}\right) \right] - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} f(x) dx \\ &= \left(\frac{\omega_2 - \omega_1}{n^2} \right) \sum_{k=0}^{n-1} \int_0^1 \varpi(\nu) f' \left((1-\nu) \frac{(n-k)\omega_1 + k\omega_2}{n} + \nu \frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) d\nu, \end{aligned} \quad (4)$$

where

$$\varpi(\nu) = \begin{cases} \nu - \frac{1}{6}, & \nu \in \left[0, \frac{1}{2}\right), \\ \nu - \frac{5}{6}, & \nu \in \left[\frac{1}{2}, 1\right]. \end{cases} \quad (5)$$

Proof. From the definition of $\varpi(\nu)$, we can deduce that

$$\begin{aligned} &\left(\frac{\omega_2 - \omega_1}{n^2} \right) \int_0^1 \varpi(\nu) f' \left((1-\nu) \frac{(n-k)\omega_1 + k\omega_2}{n} + \nu \frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) d\nu \\ &= \left(\frac{\omega_2 - \omega_1}{n^2} \right) \left\{ \int_0^{(1/2)} \left(\nu - \frac{1}{6} \right) f' \left((1-\nu) \frac{(n-k)\omega_1 + k\omega_2}{n} + \nu \frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) d\nu \right. \\ &\quad \left. + \int_{(1/2)}^1 \left(\nu - \frac{5}{6} \right) f' \left((1-\nu) \frac{(n-k)\omega_1 + k\omega_2}{n} + \nu \frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) d\nu \right\}. \end{aligned} \quad (6)$$

By applying the integration by parts to the right side of (6), we have

$$\begin{aligned}
 & \int_0^{(1/2)} \left(\nu - \frac{1}{6} \right) f' \left((1-\nu) \frac{(n-k)\omega_1 + k\omega_2}{n} + \nu \frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) d\nu \\
 &= \left(\frac{n}{\omega_2 - \omega_1} \right) \left[\left(\nu - \frac{1}{6} \right) f \left((1-\nu) \frac{(n-k)\omega_1 + k\omega_2}{n} + \nu \frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) \right]_{t=0}^{(1/2)} \\
 & \quad - \int_0^{(1/2)} f \left((1-\nu) \frac{(n-k)\omega_1 + k\omega_2}{n} + \nu \frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) d\nu \quad (7) \\
 &= \left(\frac{n}{\omega_2 - \omega_1} \right) \left[\frac{1}{3} f \left(\frac{(2(n-k)-1)\omega_1 + (2k+1)\omega_2}{2n} \right) + \frac{1}{6} f \left(\frac{(n-k)\omega_1 + k\omega_2}{n} \right) \right. \\
 & \quad \left. - \int_0^{(1/2)} f \left((1-\nu) \frac{(n-k)\omega_1 + k\omega_2}{n} + \nu \frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) d\nu \right],
 \end{aligned}$$

and similarly,

$$\begin{aligned}
 & \int_{(1/2)}^1 \left(\nu - \frac{5}{6} \right) f' \left((1-\nu) \frac{(n-k)\omega_1 + k\omega_2}{n} + \nu \frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) d\nu \\
 &= \left(\frac{n}{\omega_2 - \omega_1} \right) \left[\frac{1}{3} f \left(\frac{(2(n-k)-1)\omega_1 + (2k+1)\omega_2}{2n} \right) + \frac{1}{6} f \left(\frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) \right. \\
 & \quad \left. - \int_{(1/2)}^1 f \left((1-\nu) \frac{(n-k)\omega_1 + k\omega_2}{n} + \nu \frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) d\nu \right], \quad (8)
 \end{aligned}$$

for all $k = 0, 1, 2, \dots, n-1$. Then, substituting (7) and (8) in the equality (6), we get

$$\begin{aligned}
 & \left(\frac{\omega_2 - \omega_1}{n^2} \right) \int_0^1 \bar{\omega}(\nu) f' \left((1-\nu) \frac{(n-k)\omega_1 + k\omega_2}{n} + \nu \frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) d\nu \\
 &= \frac{1}{6n} \left[f \left(\frac{(n-k)\omega_1 + k\omega_2}{n} \right) + 4f \left(\frac{(2(n-k)-1)\omega_1 + (2k+1)\omega_2}{2n} \right) \right. \\
 & \quad \left. + f \left(\frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right) \right] - \frac{1}{\omega_2 - \omega_1} \int_{((n-k)\omega_1 + k\omega_2/n)}^{((n-k-1)\omega_1 + (k+1)\omega_2/n)} f(x) dx. \quad (9)
 \end{aligned}$$

By summing over k from 0 to $n-1$, we can deduce the desired identity (4). $\square$

Remark 1. Taking $n = 1$ in Lemma 1, we have the following Simpson's rule (see [43]):

$$\frac{1}{6} \left[f(\omega_1) + 4f\left(\frac{\omega_1 + \omega_2}{2}\right) + f(\omega_2) \right] - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} f(x) dx = (\omega_2 - \omega_1) \int_0^1 \bar{\omega}(\nu) f'(\nu\omega_2 + (1-\nu)\omega_1) d\nu, \quad (10)$$

where $\varpi(\nu)$ is defined by (5).

Also, let $f' \in \mathcal{L}^1$ and $|f'|$ be an s -convex function on $[\omega_1, \omega_2]$, and then for any positive integer number n , we have

Theorem 2. Let $f: \mathcal{T} \subseteq [0, +\infty) \longrightarrow \mathbb{R}$ be an $\mathcal{L}^1$ function on $\mathcal{T}$ such that $\omega_1, \omega_2 \in \mathcal{T}$ with $\omega_1 < \omega_2$ and $s \in (0, 1]$ be fixed.

$$|T_n(f; \omega_1, \omega_2)| \leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \sum_{k=0}^{n-1} \left[(A_1(s) + A_3(s)) |f'(\mathbf{e}_{n,k})| + (A_2(s) + A_4(s)) |f'(\mathbf{e}_{n,k+1})| \right], \quad (11)$$

where

$$\begin{aligned} A_1(s) &:= \int_0^{(1/2)} \left| \nu - \frac{1}{6} \right| (1-\nu)^s d\nu = \frac{1}{6} \left[\frac{1 - (5/6)^{s+1}}{s+1} \right] - \frac{1 - (5/6)^{s+1}}{s+1} + \frac{1 - (5/6)^{s+2}}{s+2} \\ &\quad + \frac{(5/6)^{s+1} - (1/2)^{s+1}}{s+1} - \frac{(5/6)^{s+2} - (1/2)^{s+2}}{s+2} - \frac{1}{6} \left[\frac{(5/6)^{s+1} - (1/2)^{s+1}}{s+1} \right], \\ A_2(s) &:= \int_0^{(1/2)} \left| \nu - \frac{1}{6} \right| \nu^s d\nu = \frac{1}{6(s+1)} \left(\frac{2 - 3^{s+1}}{6^{s+1}} \right) + \frac{1}{s+2} \left(\frac{3^{s+2} - 2}{6^{s+2}} \right), \\ A_3(s) &:= \int_{(1/2)}^1 \left| \nu - \frac{5}{6} \right| (1-\nu)^s d\nu = \frac{5}{6} \left[\frac{(1/2)^{s+1} - (1/6)^{s+1}}{s+1} \right] - \frac{(1/2)^{s+1} - (1/6)^{s+1}}{s+1} + \frac{(1/2)^{s+2} - (1/6)^{s+2}}{s+2} \\ &\quad + \frac{1}{6^{s+1}(s+1)} - \frac{1}{6^{s+2}(s+2)} - \frac{5}{6^{s+2}(s+1)}, \\ A_4(s) &:= \int_{(1/2)}^1 \left| \nu - \frac{5}{6} \right| \nu^s d\nu = \frac{5}{6} \left[\frac{(5/6)^{s+1} - (1/2)^{s+1}}{s+1} \right] - \frac{(5/6)^{s+2} - (1/2)^{s+2}}{s+2} + \frac{1 - (5/6)^{s+2}}{s+2} - \frac{5}{6} \left[\frac{1 - (5/6)^{s+1}}{s+1} \right], \\ e_{n,k} &:= \frac{(n-k)\omega_1 + k\omega_2}{n}. \end{aligned} \quad (12)$$

Proof. By using Lemma 1, s -convexity of $|f'|$ and properties of modulus, we can deduce

$$\begin{aligned} |T_n(f; \omega_1, \omega_2)| &\leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \sum_{k=0}^{n-1} \int_0^1 |\varpi(\nu)| |f'((1-\nu)\mathbf{e}_{n,k} + \nu\mathbf{e}_{n,k+1})| d\nu \\ &\leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \sum_{k=0}^{n-1} \int_0^1 |\varpi(\nu)| \left[(1-\nu)^s |f'(\mathbf{e}_{n,k})| + \nu^s |f'(\mathbf{e}_{n,k+1})| \right] d\nu \\ &= \left(\frac{\omega_2 - \omega_1}{n^2} \right) \sum_{k=0}^{n-1} \left[(A_1(s) + A_3(s)) |f'(\mathbf{e}_{n,k})| + (A_2(s) + A_4(s)) |f'(\mathbf{e}_{n,k+1})| \right], \end{aligned} \quad (13)$$

which completes the proof. $\square$

$$|T_n(f; \omega_1, \omega_2)| \leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \sum_{k=0}^{n-1} \left[(A_1 + A_3) |f'(\mathbf{e}_{n,k})| + (A_2 + A_4) |f'(\mathbf{e}_{n,k+1})| \right], \quad (14)$$

Corollary 1. Choosing $s = 1$ in Theorem 2, we get the following inequality for the convex function:

where

$$\begin{aligned}
 A_1 &:= \int_0^{(1/2)} \left| \nu - \frac{1}{6} \right| (1 - \nu) d\nu = \frac{1}{6} \left[\frac{1 - (5/6)^2}{2} \right] - \frac{1 - (5/6)^2}{2} + \frac{1 - (5/6)^3}{3} \\
 &\quad + \frac{(5/6)^2 - (1/2)^2}{2} - \frac{(5/6)^3 - (1/2)^3}{3} - \frac{1}{6} \left[\frac{(5/6)^2 - (1/2)^2}{2} \right], \\
 A_2 &:= \int_0^{(1/2)} \left| \nu - \frac{1}{6} \right| \nu d\nu = \frac{29}{1296}, \\
 A_3 &:= \int_{(1/2)}^1 \left| \nu - \frac{5}{6} \right| (1 - \nu) d\nu = \frac{5}{6} \left[\frac{(1/2)^2 - (1/6)^2}{2} \right] - \frac{(1/2)^2 - (1/6)^2}{2} + \frac{(1/2)^3 - (1/6)^3}{3} + \frac{1}{1296}, \\
 A_4 &:= \int_{(1/2)}^1 \left| \nu - \frac{5}{6} \right| \nu d\nu = \frac{5}{6} \left[\frac{(5/6)^2 - (1/2)^2}{2} \right] - \frac{(5/6)^3 - (1/2)^3}{3} + \frac{1 - (5/6)^3}{3} - \frac{5}{6} \left[\frac{1 - (5/6)^2}{2} \right].
 \end{aligned} \tag{15}$$

Corollary 2. Choosing $n = 1$ in Theorem 2, we get

$$\begin{aligned}
 &\left| \frac{1}{6} \left[f(\omega_1) + 4f\left(\frac{\omega_1 + \omega_2}{2}\right) + f(\omega_2) \right] - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} f(x) dx \right| \leq (\omega_2 - \omega_1) \\
 &\quad \times \left[(A_1(s) + A_3(s)) |f'(\omega_1)| + (A_2(s) + A_4(s)) |f'(\omega_2)| \right].
 \end{aligned} \tag{16}$$

Theorem 3. Let $f: \mathcal{T} \subseteq [0, +\infty) \rightarrow \mathbb{R}$ be an $\mathcal{L}^1$ function on $\mathcal{T}$ such that $\omega_1, \omega_2 \in \mathcal{T}$ with $\omega_1 < \omega_2$ and $s \in (0, 1]$ be fixed.

Also, let $f' \in \mathcal{L}^1$ and $|f'|^q, q > 1$ be an s -convex function on $[\omega_1, \omega_2]$, and then for any positive integer number n , we have

$$|T_n(f; \omega_1, \omega_2)| \leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \left[\frac{2}{p+1} \left(\frac{2^{p+1} + 1}{6^{p+1}} \right) \right]^{(1/p)} \left(\frac{1}{s+1} \right)^{(1/q)} \sum_{k=0}^{n-1} \left[|f'(\mathbf{e}_{n,k})|^q + |f'(\mathbf{e}_{n,k+1})|^q \right]^{(1/q)}, \tag{17}$$

where $\mathbf{e}_{n,k}$ is as defined in Theorem 2 and $(1/p) + (1/q) = 1$.

Proof. By making use of Lemma 1, Hölder's inequality, s -convexity of $|f'|^q$, and properties of modulus, we have

$$\begin{aligned}
 |T_n(f; \omega_1, \omega_2)| &\leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \sum_{k=0}^{n-1} \int_0^1 |\varpi(\nu)| |f'((1-\nu)\mathbf{e}_{n,k} + \nu\mathbf{e}_{n,k+1})| d\nu \\
 &\leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \left(\int_0^1 |\varpi(\nu)|^p d\nu \right)^{(1/p)} \sum_{k=0}^{n-1} \left(\int_0^1 |f'((1-\nu)\mathbf{e}_{n,k} + \nu\mathbf{e}_{n,k+1})|^q d\nu \right)^{(1/q)} \\
 &\leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \left[\frac{2}{p+1} \left(\frac{2^{p+1} + 1}{6^{p+1}} \right) \right]^{(1/p)} \sum_{k=0}^{n-1} \left(\int_0^1 \left[(1-\nu)^s |f'(\mathbf{e}_{n,k})|^q + \nu^s |f'(\mathbf{e}_{n,k+1})|^q \right] d\nu \right)^{(1/q)} \\
 &= \left(\frac{\omega_2 - \omega_1}{n^2} \right) \left[\frac{2}{p+1} \left(\frac{2^{p+1} + 1}{6^{p+1}} \right) \right]^{(1/p)} \left(\frac{1}{s+1} \right)^{(1/q)} \sum_{k=0}^{n-1} \left[|f'(\mathbf{e}_{n,k})|^q + |f'(\mathbf{e}_{n,k+1})|^q \right]^{(1/q)}.
 \end{aligned} \tag{18}$$

This ends our proof. $\square$

Corollary 3. Choosing $s = 1$ in Theorem 3, we get the following inequality for the convex function:

$$|T_n(f; \omega_1, \omega_2)| \leq \left(\frac{\omega_2 - \omega_1}{n^2 \sqrt[q]{q} 2} \right) \left[\frac{2}{p+1} \left(\frac{2^{p+1} + 1}{6^{p+1}} \right) \right]^{(1/p)} \sum_{k=0}^{n-1} \left[|f'(\mathbf{e}_{n,k})|^q + |f'(\mathbf{e}_{n,k+1})|^q \right]^{(1/q)}. \quad (19)$$

Corollary 4. Choosing $n = 1$ in Theorem 3, we get

$$\begin{aligned} & \left| \frac{1}{6} \left[f(\omega_1) + 4f\left(\frac{\omega_1 + \omega_2}{2}\right) + f(\omega_2) \right] - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} f(x) dx \right| \leq (\omega_2 - \omega_1) \\ & \times \left[\frac{2}{p+1} \left(\frac{2^{p+1} + 1}{6^{p+1}} \right) \right]^{(1/p)} \left[\frac{|f'(\omega_1)|^q + |f'(\omega_2)|^q}{s+1} \right]^{(1/q)}. \end{aligned} \quad (20)$$

Theorem 4. Let $f: \mathcal{T} \subseteq [0, +\infty) \rightarrow \mathbb{R}$ be an $\mathcal{L}^1$ function on $\mathcal{T}$ such that $\omega_1, \omega_2 \in \mathcal{T}$ with $\omega_1 < \omega_2$ and $s \in (0, 1]$ be fixed.

Also, let $f' \in \mathcal{L}^1$ and $|f'|^q, q \geq 1$ be s -convex function on $[\omega_1, \omega_2]$, and then for any positive integer number n , we have

$$\begin{aligned} |T_n(f; \omega_1, \omega_2)| & \leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \left(\frac{5}{36} \right)^{1-(1/q)} \\ & \times \sum_{k=0}^{n-1} \left\{ \left[A_1(s) |f'(\mathbf{e}_{n,k})|^q + A_2(s) |f'(\mathbf{e}_{n,k+1})|^q \right]^{(1/q)} + \left[A_3(s) |f'(\mathbf{e}_{n,k})|^q + A_4(s) |f'(\mathbf{e}_{n,k+1})|^q \right]^{(1/q)} \right\}, \end{aligned} \quad (21)$$

where $A_i(s)$ for all $i = 1, 2, 3, 4$ and $\mathbf{e}_{n,k}$ are as defined in Theorem 2.

Proof. By using Lemma 1, the well-known power mean inequality, s -convexity of $|f'|^q$, and properties of modulus, we have

$$\begin{aligned} |T_n(f; \omega_1, \omega_2)| & \leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \sum_{k=0}^{n-1} \int_0^1 |\varpi(\nu)| |f'((1-\nu)\mathbf{e}_{n,k} + \nu\mathbf{e}_{n,k+1})| d\nu \\ & \leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \left(\int_0^1 |\varpi(\nu)| d\nu \right)^{1-(1/q)} \sum_{k=0}^{n-1} \left(\int_0^1 |\varpi(\nu)| |f'((1-\nu)\mathbf{e}_{n,k} + \nu\mathbf{e}_{n,k+1})|^q d\nu \right)^{(1/q)} \\ & \leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \left(\frac{5}{36} \right)^{1-(1/q)} \sum_{k=0}^{n-1} \left(\int_0^1 |\varpi(\nu)| \left[(1-\nu)^s |f'(\mathbf{e}_{n,k})|^q + \nu^s |f'(\mathbf{e}_{n,k+1})|^q \right] d\nu \right)^{(1/q)} \\ & = \left(\frac{\omega_2 - \omega_1}{n^2} \right) \left(\frac{5}{36} \right)^{1-(1/q)} \\ & \times \sum_{k=0}^{n-1} \left\{ \left[A_1(s) |f'(\mathbf{e}_{n,k})|^q + A_2(s) |f'(\mathbf{e}_{n,k+1})|^q \right]^{(1/q)} + \left[A_3(s) |f'(\mathbf{e}_{n,k})|^q + A_4(s) |f'(\mathbf{e}_{n,k+1})|^q \right]^{(1/q)} \right\}, \end{aligned} \quad (22)$$

which completes the proof. $\square$

Remark 2. Choosing $q = 1$ in Theorem 4, we get Theorem 2

Corollary 5. Choosing $s = 1$ in Theorem 4, we get the following inequality for the convex function:

$$|T_n(f; \omega_1, \omega_2)| \leq \left(\frac{\omega_2 - \omega_1}{n^2} \right) \left(\frac{5}{36} \right)^{1-(1/q)} \times \sum_{k=0}^{n-1} \left\{ \left[A_1 |f'(\mathbf{e}_{nk})|^q + A_2 |f'(\mathbf{e}_{n,k+1})|^q \right]^{(1/q)} + \left[A_3 |f'(\mathbf{e}_{nk})|^q + A_4 |f'(\mathbf{e}_{n,k+1})|^q \right]^{(1/q)} \right\}, \quad (23)$$

where A_i for all $i = 1, 2, 3, 4$ are defined as in Corollary 1.

Corollary 6. Choosing $n = 1$ in Theorem 4, we get

$$\left| \frac{1}{6} \left[f(\omega_1) + 4f\left(\frac{\omega_1 + \omega_2}{2}\right) + f(\omega_2) \right] - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} f(x) dx \right| \leq (\omega_2 - \omega_1) \left(\frac{5}{36} \right)^{1-(1/q)} \times \left\{ \left[A_1(s) |f'(\omega_1)|^q + A_2(s) |f'(\omega_2)|^q \right]^{(1/q)} + \left[A_3(s) |f'(\omega_1)|^q + A_4(s) |f'(\omega_2)|^q \right]^{(1/q)} \right\}. \quad (24)$$

3. Examples and Applications

3.1. Bessel Functions. Consider the function $\mathcal{D}_\tau: (0, \infty) \rightarrow [1, \infty)$ given by

$$\mathcal{D}_\tau(x) = 2^\tau \Gamma(\tau + 1) x^{-\tau} \mathcal{M}_\tau(x), \quad (25)$$

where $\mathcal{M}_\tau$ is the modified Bessel function of the first kind defined by (see (2) in [46], p. 77):

$$\mathcal{M}_\tau(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{\tau+2n}}{n! \Gamma(\tau + 1 + n)}, \quad x \in \mathbb{R}. \quad (26)$$

The first order derivative formula of $\mathcal{D}_\tau(x)$ is given by [46]:

$$\mathcal{D}'_\tau(x) = \frac{x}{2(\tau + 1)} \mathcal{D}_{\tau+1}(x), \quad (27)$$

and the second derivative can be easily calculated from (27) to be

$$\mathcal{D}''_\tau(x) = \frac{x^2 \mathcal{D}_{\tau+2}(x)}{4(\tau + 1)(\tau + 2)} + \frac{\mathcal{D}_{\tau+1}(x)}{2(\tau + 1)}. \quad (28)$$

Example 1. Let $0 < \omega_1 < \omega_2$ and $\tau > -1$. Then, by applying Corollary 2 (note that all assumptions are satisfied) and the identities (27) and (28), we have

$$\begin{aligned} & \left| \frac{1}{12(\tau + 1)} \left[\omega_1 \mathcal{D}_{\tau+1}(\omega_1) + \frac{\omega_1 + \omega_2}{2} \mathcal{D}_{\tau+1}\left(\frac{\omega_1 + \omega_2}{2}\right) + \omega_2 \mathcal{D}_{\tau+1}(\omega_2) \right] - \frac{\mathcal{D}_\tau(\omega_2) - \mathcal{D}_\tau(\omega_1)}{\omega_2 - \omega_1} \right| \\ & \leq \frac{\omega_2 - \omega_1}{2} \left[(A_1(s) + A_3(s)) \left(\frac{\omega_1^2 \mathcal{D}_{\tau+2}(\omega_1)}{2(\tau + 1)(\tau + 2)} + \frac{\omega_1 \mathcal{D}_{\tau+1}(\omega_1)}{\tau + 1} \right) \right. \\ & \quad \left. + (A_2(s) + A_4(s)) \left(\frac{\omega_2^2 \mathcal{D}_{\tau+2}(\omega_2)}{2(\tau + 1)(\tau + 2)} + \frac{\omega_2 \mathcal{D}_{\tau+1}(\omega_2)}{\tau + 1} \right) \right], \end{aligned} \quad (29)$$

where $A_i(s)$, $i = 1, 2, 3, 4$ are as in Theorem 2.

Example 2. Let $0 < \omega_1 < \omega_2$ and $\tau > -1$. Then, by applying Corollary 4 (note that all assumptions are satisfied) and the identities (27) and (28), we have

$$\begin{aligned}
& \left| \frac{1}{12(\tau+1)} \left[\omega_1 \mathcal{D}_{\tau+1}(\omega_1) + \frac{\omega_1 + \omega_2}{2} \mathcal{D}_{\tau+1}\left(\frac{\omega_1 + \omega_2}{2}\right) + \omega_2 \mathcal{D}_{\tau+1}(\omega_2) \right] - \frac{\mathcal{D}_\tau(\omega_2) - \mathcal{D}_\tau(\omega_1)}{\omega_2 - \omega_1} \right| \\
& \leq \frac{\omega_2 - \omega_1}{2(s+1)^{(1/q)}} \left[\frac{2}{p+1} \left(\frac{2^{p+1} + 1}{6^{p+1}} \right) \right]^{(1/p)} \left[\left(\frac{\omega_1^2 \mathcal{D}_{\tau+2}(\omega_1)}{2(\tau+1)(\tau+2)} + \frac{\omega_1 \mathcal{D}_{\tau+1}(\omega_1)}{\tau+1} \right)^q \right. \\
& \quad \left. + \left(\frac{\omega_2^2 \mathcal{D}_{\tau+2}(\omega_2)}{2(\tau+1)(\tau+2)} + \frac{\omega_2 \mathcal{D}_{\tau+1}(\omega_2)}{\tau+1} \right)^q \right]^{(1/q)}.
\end{aligned} \tag{30}$$

Example 3. Let $0 < \omega_1 < \omega_2$ and $\tau > -1$. Then, by applying Corollary 6 (note that all assumptions are satisfied) and the identities (27) and (28), we have

$$\begin{aligned}
& \left| \frac{1}{12(\tau+1)} \left[\omega_1 \mathcal{D}_{\tau+1}(\omega_1) + \frac{\omega_1 + \omega_2}{2} \mathcal{D}_{\tau+1}\left(\frac{\omega_1 + \omega_2}{2}\right) + \omega_2 \mathcal{D}_{\tau+1}(\omega_2) \right] - \frac{\mathcal{D}_\tau(\omega_2) - \mathcal{D}_\tau(\omega_1)}{\omega_2 - \omega_1} \right| \\
& \leq \frac{\omega_2 - \omega_1}{2} \left(\frac{5}{36} \right)^{1-(1/q)} \left\{ \left[A_1(s) \left(\frac{\omega_1^2 \mathcal{D}_{\tau+2}(\omega_1)}{2(\tau+1)(\tau+2)} + \frac{\omega_1 \mathcal{D}_{\tau+1}(\omega_1)}{\tau+1} \right)^q \right. \right. \\
& \quad \left. \left. + A_2(s) \left(\frac{\omega_2^2 \mathcal{D}_{\tau+2}(\omega_2)}{2(\tau+1)(\tau+2)} + \frac{\omega_2 \mathcal{D}_{\tau+1}(\omega_2)}{\tau+1} \right)^q \right]^{(1/q)} + \left[A_3(s) \left(\frac{\omega_1^2 \mathcal{D}_{\tau+2}(\omega_1)}{2(\tau+1)(\tau+2)} + \frac{\omega_1 \mathcal{D}_{\tau+1}(\omega_1)}{\tau+1} \right)^q \right. \right. \\
& \quad \left. \left. + A_4(s) \left(\frac{\omega_2^2 \mathcal{D}_{\tau+2}(\omega_2)}{2(\tau+1)(\tau+2)} + \frac{\omega_2 \mathcal{D}_{\tau+1}(\omega_2)}{\tau+1} \right)^q \right]^{(1/q)} \right\},
\end{aligned} \tag{31}$$

where $A_i(s)$, $i = 1, 2, 3, 4$ are as in Theorem 2.

$$\mathcal{L}_r(\omega_1, \omega_2) = \left[\frac{\omega_2^{r+1} - \omega_1^{r+1}}{(r+1)(\omega_2 - \omega_1)} \right]^{(1/r)}, \quad r \in \mathbb{R} \setminus \{-1, 0\}. \tag{32}$$

3.2. Special Means. We consider the following two special means for different positive real numbers ω_1 and ω_2 :

(i) The arithmetic mean:

$$\mathcal{A}(\omega_1, \omega_2) = \frac{\omega_1 + \omega_2}{2}. \tag{32}$$

Proposition 1. Let $n \in \mathbb{N}$ and $0 < \omega_1 < \omega_2$. Then, we have for a fixed $s \in (0, 1]$

(ii) The generalized logarithmic mean:

$$\begin{aligned}
& \left| \sum_{k=0}^{n-1} \frac{1}{3n} \left[\mathcal{A} \left(\left(\frac{(n-k)\omega_1 + k\omega_2}{n} \right)^s, \left(\frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right)^s \right) \right. \right. \\
& \quad \left. \left. + \frac{2}{n^s} \mathcal{A}((2(n-k)-1)\omega_1, (2k+1)\omega_2) \right] - \mathcal{L}_s^s(\omega_1, \omega_2) \right| \leq \frac{2^{s-1}s}{n^{s+1}} (\omega_2 - \omega_1) \\
& \quad \times \sum_{k=0}^{n-1} \left[(A_1(s) + A_3(s)) \mathcal{A}^{s-1}((n-k)\omega_1, k\omega_2) + (A_2(s) + A_4(s)) \mathcal{A}^{s-1}((n-k-1)\omega_1, (k+1)\omega_2) \right],
\end{aligned} \tag{34}$$

where $A_i(s)$ for all $i = 1, 2, 3, 4$ are as defined in Theorem 2.

Proposition 2. Let $n \in \mathbb{N}$ and $0 < \omega_1 < \omega_2$. Then, we have for a fixed $s \in (0, 1]$

Proof. By applying Theorem 2 with $f(x) = x^s$, $x \in [\omega_1, \omega_2]$ and $s \in (0, 1]$, then we can obtain the desired result (34). $\square$

$$\begin{aligned}
& \left| \sum_{k=0}^{n-1} \frac{1}{3n} \left[\mathcal{A} \left(\left(\frac{(n-k)\omega_1 + k\omega_2}{n} \right)^s, \left(\frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right)^s \right) \right. \right. \\
& \quad \left. \left. + \frac{2}{n^s} \mathcal{A}((2(n-k)-1)\omega_1, (2k+1)\omega_2) \right] - \mathcal{L}_s^s(\omega_1, \omega_2) \right| \\
& \leq \frac{2^{s-1}s}{n^{s+1}} (\omega_2 - \omega_1) \left[\frac{2}{p+1} \left(\frac{2^{p+1}+1}{6^{p+1}} \right) \right]^{(1/p)} \left(\frac{1}{s+1} \right)^{(1/q)} \\
& \quad \times \sum_{k=0}^{n-1} \left[\mathcal{A}^{q(s-1)}((n-k)\omega_1, k\omega_2) + \mathcal{A}^{q(s-1)}((n-k-1)\omega_1, (k+1)\omega_2) \right]^{(1/q)},
\end{aligned} \tag{35}$$

where $(1/p) + (1/q) = 1$ with $q > 1$.

Proposition 3. Let $n \in \mathbb{N}$ and $0 < \omega_1 < \omega_2$. Then, we have for a fixed $s \in (0, 1]$

Proof. By applying Theorem 3 with $f(x) = x^s, x \in [\omega_1, \omega_2]$ and $s \in (0, 1]$, we can obtain the desired result (37) directly. $\square$

$$\begin{aligned}
& \left| \sum_{k=0}^{n-1} \frac{1}{3n} \left[\mathcal{A} \left(\left(\frac{(n-k)\omega_1 + k\omega_2}{n} \right)^s, \left(\frac{(n-k-1)\omega_1 + (k+1)\omega_2}{n} \right)^s \right) \right. \right. \\
& \quad \left. \left. + \frac{2}{n^s} \mathcal{A}((2(n-k)-1)\omega_1, (2k+1)\omega_2) \right] - \mathcal{L}_s^s(\omega_1, \omega_2) \right| \leq \frac{2^{s-1}s}{n^{s+1}} (\omega_2 - \omega_1) \left(\frac{5}{36} \right)^{1-(1/q)} \\
& \quad \times \sum_{k=0}^{n-1} \left\{ \left[A_1(s) \mathcal{A}^{q(s-1)}((n-k)\omega_1, k\omega_2) + A_2(s) \mathcal{A}^{q(s-1)}((n-k-1)\omega_1, (k+1)\omega_2) \right]^{(1/q)} \right. \\
& \quad \left. + \left[A_3(s) \mathcal{A}^{q(s-1)}((n-k)\omega_1, k\omega_2) + A_4(s) \mathcal{A}^{q(s-1)}((n-k-1)\omega_1, (k+1)\omega_2) \right]^{(1/q)} \right\},
\end{aligned} \tag{36}$$

where $A_i(s)$ for all $i = 1, 2, 3, 4$ are defined as in Theorem 2 and $q \geq 1$.

Proof. By applying Theorem 4 with $f(x) = x^s, x \in [\omega_1, \omega_2]$ and $s \in (0, 1]$, we can directly obtain the desired result (43). $\square$

3.3. Simpson Quadrature Formula. Let $\mathcal{P}$ be the partition of the points $\omega_1 = x_0 < x_1 < \dots < x_l = \omega_2$ of the interval $[\omega_1, \omega_2]$ and consider the quadrature formula

$$\int_{\omega_1}^{\omega_2} f(x) dx = \mathcal{S}_n^{(i)}(f, \mathcal{P}) + \mathcal{E}(f, \mathcal{P}), \tag{37}$$

where

$$\begin{aligned}
\mathcal{S}_n^{(i)}(f, \mathcal{P}) &:= \sum_{k=0}^{n-1} \frac{1}{6n} \left[f \left(\frac{(n-k)x_i + kx_{i+1}}{n} \right) \right. \\
& \quad \left. + 4f \left(\frac{(2(n-k)-1)x_i + (2k+1)x_{i+1}}{2n} \right) \right. \\
& \quad \left. + f \left(\frac{(n-k-1)x_i + (k+1)x_{i+1}}{n} \right) \right] (x_{i+1} - x_i)
\end{aligned} \tag{38}$$

is the Simpson version and $\mathcal{E}(f, \mathcal{P})$ denotes the associated approximation error. In this section, we are going to derive some new estimates for the Simpson quadrature formula.

Proposition 4. Let $0 < \omega_1 < \omega_2, s \in (0, 1]$ be fixed and $f: [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a differentiable function on (ω_1, ω_2) . If $f' \in \mathcal{L}^1[\omega_1, \omega_2]$ and $|f'|$ is an s -convex function on $[\omega_1, \omega_2]$, then we have

$$\begin{aligned}
|\mathcal{E}(f, \mathcal{P})| &\leq \sum_{i=0}^{l-1} \left(\frac{x_{i+1} - x_i}{n} \right)^2 \sum_{k=0}^{n-1} \left[(A_1(s) + A_3(s)) |f'(\mathbf{e}_{n,k}^{(i)})| \right. \\
& \quad \left. + (A_2(s) + A_4(s)) |f'(\mathbf{e}_{n,k+1}^{(i)})| \right],
\end{aligned} \tag{39}$$

where $A_i(s)$ for all $i = 1, 2, 3, 4$ are as defined in Theorem 2 and

$$\mathbf{e}_{n,j}^{(i)} := \frac{(n-j)x_i + jx_{i+1}}{n}, \quad \text{for } j = k, k+1. \tag{40}$$

Proof. By applying Theorem 2 on the subintervals $[x_i, x_{i+1}]$ ($i = 0, \dots, l-1$) of the partition $\mathcal{P}$, we can deduce

$$\begin{aligned}
& \left| \int_{x_i}^{x_{i+1}} f(x) dx - \sum_{k=0}^{n-1} \frac{1}{6n} \left[f\left(\frac{(n-k)x_i + kx_{i+1}}{n}\right) + 4f\left(\frac{(2(n-k)-1)x_i + (2k+1)x_{i+1}}{2n}\right) \right. \right. \\
& \quad \left. \left. + f\left(\frac{(n-k-1)x_i + (k+1)x_{i+1}}{n}\right) \right] (x_{i+1} - x_i) \right| \\
& \leq \left(\frac{x_{i+1} - x_i}{n} \right)^2 \sum_{k=0}^{n-1} \left[(A_1(s) + A_3(s)) |f'(\mathbf{e}_{n,k}^{(i)})| + (A_2(s) + A_4(s)) |f'(\mathbf{e}_{n,k+1}^{(i)})| \right].
\end{aligned} \tag{41}$$

Therefore, from (41), it follows that

$$\begin{aligned}
|\mathcal{E}(f, \mathcal{P})| &= \left| \int_{\omega_1}^{\omega_2} f(x) dx - \mathcal{S}_n^{(i)}(f, \mathcal{P}) \right| \leq \sum_{i=0}^{l-1} \left| \int_{x_i}^{x_{i+1}} f(x) dx - \sum_{k=0}^{n-1} \frac{1}{6n} \left[f\left(\frac{(n-k)x_i + kx_{i+1}}{n}\right) \right. \right. \\
& \quad \left. \left. + 4f\left(\frac{(2(n-k)-1)x_i + (2k+1)x_{i+1}}{2n}\right) + f\left(\frac{(n-k-1)x_i + (k+1)x_{i+1}}{n}\right) \right] (x_{i+1} - x_i) \right| \\
& \leq \sum_{i=0}^{l-1} \left(\frac{x_{i+1} - x_i}{n} \right)^2 \sum_{k=0}^{n-1} \left[(A_1(s) + A_3(s)) |f'(\mathbf{e}_{n,k}^{(i)})| + (A_2(s) + A_4(s)) |f'(\mathbf{e}_{n,k+1}^{(i)})| \right].
\end{aligned} \tag{42}$$

This gives our Proposition 4. $\square$

Proposition 5. Let $0 < \omega_1 < \omega_2$, $s \in (0, 1]$ be fixed and $f: [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a differentiable function on (ω_1, ω_2) . If

$f' \in \mathcal{L}^1[\omega_1, \omega_2]$ and $|f'|^q$ is an s -convex function on $[\omega_1, \omega_2]$, then we have for $q > 1$ and $(1/p) + (1/q) = 1$:

$$|\mathcal{E}(f, \mathcal{P})| \leq \left[\frac{2}{p+1} \left(\frac{2^{p+1} + 1}{6^{p+1}} \right) \right]^{(1/p)} \left(\frac{1}{s+1} \right)^{(1/q)} \sum_{i=0}^{l-1} \left(\frac{x_{i+1} - x_i}{n} \right)^2 \sum_{k=0}^{n-1} \left[|f'(\mathbf{e}_{n,k}^{(i)})|^q + |f'(\mathbf{e}_{n,k+1}^{(i)})|^q \right]^{(1/q)}, \tag{43}$$

where $\mathbf{e}_{n,k}^{(i)}$ and $\mathbf{e}_{n,k+1}^{(i)}$ are as defined in Proposition 4.

Proof. By the same method used for Proposition 4 and by using Theorem 3, one can prove this proposition directly. $\square$

Proposition 6. Let $0 < \omega_1 < \omega_2$, $s \in (0, 1]$ be fixed and $f: [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a differentiable function on (ω_1, ω_2) . If $f' \in \mathcal{L}^1[\omega_1, \omega_2]$ and $|f'|^q$ is an s -convex function on $[\omega_1, \omega_2]$, then we have for $q \geq 1$:

$$\begin{aligned}
|\mathcal{E}(f, \mathcal{P})| &\leq \left(\frac{5}{36} \right)^{1-(1/q)} \sum_{i=0}^{l-1} \left(\frac{x_{i+1} - x_i}{n} \right)^2 \sum_{k=0}^{n-1} \left\{ \left[A_1(s) |f'(\mathbf{e}_{n,k}^{(i)})|^q + A_2(s) |f'(\mathbf{e}_{n,k+1}^{(i)})|^q \right]^{(1/q)} \right. \\
& \quad \left. + \left[A_3(s) |f'(\mathbf{e}_{n,k}^{(i)})|^q + A_4(s) |f'(\mathbf{e}_{n,k+1}^{(i)})|^q \right]^{(1/q)} \right\},
\end{aligned} \tag{44}$$

where $\mathbf{e}_{n,k}^{(i)}$ and $\mathbf{e}_{n,k+1}^{(i)}$ are as defined in Proposition 4.

Proof. By the same method used for Proposition 4 and by using Theorem 4, one can prove this proposition directly. $\square$

4. Conclusion

We have obtained a new Simpson identity at first. By using this, we have established some interesting Simpson's type integral inequalities for s -convex functions. In order to illustrate the efficiency of our main results, some applications to the Bessel functions, special means, and error estimation

about Simpson quadrature formula are provided. To the best of our knowledge, these results are new in the literature. Since the class of convex (s -convex) functions have large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, quantum mechanics, special functions, mathematical inequalities, and related optimization theory and may stimulate further research in different areas of pure and applied sciences.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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