Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 8877129, 16 pages
https://doi.org/10.1155/2020/8877129

Research Article

Matrix Measure Approach for Stability and Synchronization of Complex-Valued Neural Networks with Deviating Argument

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Received 21 September 2020; Revised 17 October 2020; Accepted 27 October 2020; Published 16 November 2020

Academic Editor: Sanbo Ding

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This paper concentrates on global exponential stability and synchronization for complex-valued neural networks (CVNNs) with deviating argument by matrix measure approach. The Lyapunov function is no longer required, and some sufficient conditions are firstly obtained to ascertain the addressed system to be exponentially stable under different activation functions. Moreover, after designing a suitable controller, the synchronization of two complex-valued coupled neural networks is realized, and the derived condition is easy to be confirmed. Finally, some numerical examples are given to demonstrate the superiority and feasibility of the presented theoretical analysis and results.

1. Introduction

During the recent years, recurrent neural networks have always been an object with immense attention due to their specific self-learning ability, which makes neural networks have been extensively applied in combinatorial optimization, signal processing, parallel computing, and some other fields [1–8]. These widespread applications of neural networks are practically based on their affluent dynamic properties in theory, including stability, synchronization, dissipativity, and chaos. Therefore, some researchers have become more interested in the theoretical study about the dynamical behaviors of neural networks and achieved lots of excellent works [9–18].

In a large number of theoretical and applied results, complex signals frequently occur and complex-valued neural networks (CVNNs) are preferable. Actually, CVNNs could be regarded as an expansion about the real-valued neural networks (RVNNs) in a certain sense, in which the state variable, activation functions, and synaptic strength matrices are all complex-valued. Therefore, such networks have much more sophisticated features than the RVNNs in many aspects and accordingly make it capable to settle a lot of matters that cannot be settled by the real-valued ones [19]. As an example, when some real-world applications contain complex signals, complex-valued model can reduce and simplify the complexity of information processing because the phase and amplitude information of complex signal can be encoded simultaneously [20]. In consequence, it is very significant to research the dynamical behaviors of CVNNs, especially the stability and synchronization. In [21], master-slave CVNNs with time-varying delay have been researched, by the concepts of average impulsive gain and interval, and a suitable hybrid impulsive controller has been designed to get a condition for insuring the exponential synchronization of the coupled CVNNs. In [22], a class of memristor-based delayed CVNNs has been discussed, and some sufficient criteria have been obtained to achieve the exponential synchronization of such networks by the Lyapunov function method and the inequality technique. In [23], complex-valued BAM neural networks with time delays have been concerned, and based on the inequality techniques, upper right-hand Dini derivative concepts, and Halanay inequality, some novel conditions for the dissipativity are presented. In [24], the global asymptotic stability of fractional-order memristive delayed CVNNs has been investigated, and based on the Filippov framework and differential inclusion method, some sufficient conditions for the stability are proposed. Moreover, ultimate Mittag–Leffler synchronization between two fractional-order systems has been also

achieved by establishing fractional differential inequality. For more interesting works on the complex-valued neural networks, see [25–27] and the references therein.

Nonlinear system may be seriously affected by deviating arguments in the running process [28]. As a hybrid dynamical system, the nonlinear control system with deviating argument integrates the peculiarities of continuous systems and discrete systems. For many problems in ecology, demography, and economics, the present decisions or current behaviors are often influenced by the past and future information. By modeling the deviating arguments, the deviating function can be used to describe the past and future information. Akhmet et al. [29] propose the neural networks with deviating argument and establish stability conditions by the second Lyapunov method. Then, the neural network model with deviating argument has become a new and important research direction. In [30], a class of neural networks with impulsive and piecewise constant arguments has been studied, and stability criterion is derived for the existence and uniqueness of the periodic solution by using the Razumikhin-type technique and the Banach fixed-point theorem. In [31], the fractional-order neural networks with generalized piecewise constant arguments have been investigated, and the existence for the unique equilibrium point has been proved; besides that, some Mittag–Leffler stability conditions have also been presented. In [32], a kind of stochastic neural networks has been discussed, which can be affected by advanced and retarded argument. Several criteria have been provided to ascertain the uniqueness and existence of solution in the form of algebraic inequalities; in addition, the global mean square exponential stabilization for such system has also been implemented by the stability theory of stochastic differential equation.

Matrix measures can trade off the influences of positive and negative values of matrices, while noticing that many previous studies about stability and synchronization are expressed in norm or algebraic form, which make the range of nonnegative constants to be limited, and thus matrix measure approach is more effective when the synaptic strengths include both positive value and negative value. Compared with the Lyapunov function approach, the matrix measure method allows us to express the Lyapunov function directly by matrix measures of variables or error vector. It is well known that there are many papers for analyzing the dynamic characteristics of RVNNs by matrix measure approach [33–35]. In the recent few years, the matrix measure approach has been used in CVNNs [36–39]. In [36], the matrix measure approach is used to study the stability behaviors of CVNNs with time-varying delays, and several stability conditions are obtained based on the Halanay inequality. And in [38], the matrix measure method is also employed to investigate the synchronization problem of CVNNs with time-varying delays, by separating CVNNs to the real and imaginary parts, and some criteria are presented to ensure exponentially synchronization for the addressed CVNNs. However, as far as we know, the stability and synchronization of CVNNs with deviating argument have not been investigated by using the matrix measure approach.

Motivated by the above discussions, we try to use matrix measure approach to analyze global exponential stability and synchronization for CVNNs with deviating argument. The main results about this paper have been organized as follows: (1) Matrix measure approach is used firstly to analyze the dynamic behaviors of CVNNs with deviating argument, which is susceptive about the sign of the system matrix entries, hence more efficient and convenient to dispose the network. (2) By introducing the deviating argument, our model is more practical. Without structuring any Lyapunov auxiliary function, global exponential stability conditions for CVNNs with deviating argument under two activation functions are derived. (3) A proper complex-valued feedback controller is designed, and a sufficient condition is obtained to realize global exponential synchronization between drive and response CVNNs with deviating argument.

The remaining contents about this paper are constituted by the following four parts. In Section 2, model description is introduced, two assumptions on activation functions are given, and some lemmas and definitions are proposed. In Section 3, two stability conditions and a synchronous condition for CVNNs with deviating argument are made by matrix measure approach. Then, Section 4 gives numerical calculations to prove the effectiveness of the derived main results. At last, the conclusion of this paper is summarized in Section 5.

Notations: denote $\mathscr{R}^+$ and $\mathscr{N}$ as the sets of nonnegative real numbers and natural numbers, i.e., $\mathscr{R}^+ = [0, +\infty)$, $\mathscr{N} = \{0, 1, 2, \ldots\}$, respectively. $\mathscr{R}^n$, $\mathscr{R}^{m\times n}$, $\mathscr{C}^n$, and $\mathscr{C}^{m\times n}$ represent the sets of real vectors with n-dimensional, $m \times n$ real matrices, complex vectors with n-dimensional, and $m \times n$ complex matrices. Let i be the imaginary unit, i.e., $i = \sqrt{-1}$, and I is denoted as the identity matrix of corresponding dimension. Superscript "T" represents the transpose of a vector or matrix. Fix two sequences with real-valued $\{\delta_k\}$, $\{\zeta_k\}$, such that for all $k \in \mathscr{N}$, $\delta_k < \delta_{k+1}$, $\delta_k \le \zeta_k \le \delta_{k+1}$ with $\delta_k \longrightarrow +\infty$ as $k \longrightarrow +\infty$. M^R (M^I) represents the real (imaginary) part of matrix $M \in C^{m\times n}$.

2. Preliminaries

We are centered on the CVNNs with deviating argument described by the differential equations as follows:

$$\begin{cases} z'(t) = -\mathrm{A}z(t) + \mathrm{B}f(z(t)) + \mathrm{C}g(z(\beta(t))) + L, & t \ge t_0, \\ z(t_0) = z^0, \end{cases} \tag{1}$$

in which $z(t) = \{z_1(t), z_2(t), \ldots, z_n(t)\}^T \in \mathscr{C}^n$ means the state vector. $\beta(t) = \zeta_k$, for $k \in \mathscr{N}$, $t \in \mathscr{R}^+$, if $t \in [\delta_k, \delta_{k+1})$. $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\} \in \mathscr{R}^{n\times n}$ with $a_r > 0\,(r = 1, 2, \ldots, n)$ is the self-inhibition matrix. $B = (b_{rj})_{n\times n}$ and $C = (c_{rj})_{n\times n} \in \mathscr{C}^{n\times n}\,(r, j = 1, 2, \ldots, n)$ denote the synaptic strength matrices. And the complex-valued output functions are represented as $f(z(t)) = (f_1(z_1(t)), f_2(z_2(t)), \ldots, f_n(z_n(t)))^T$ and $g(z(\beta(t))) = \{g_1(z_1\ \{\beta(t)\}), tg_2n(z_2\ (\beta(t)))q, h\ldots, xg_n7(z_n(\beta(t)))\}^T$. $L = (L_1, L_2, \ldots,\ L_n)^T \in \mathscr{C}^n$ denotes the input vector. $z(t_0) = z^0 = (z_1^0, z_2^0, \ldots, z_n^0)^T$ is defined as the initial state.

Remark 1. For all $k \in \mathcal{N}$, $t \in [\delta_k, \delta_{k+1})$, function $\beta(t) = \zeta_k$. System (1) is said to be an advanced system if $t \in [\delta_k, \zeta_k)$. On the other hand, system (1) is said to be a retarded system if $t \in (\zeta_k, \delta_{k+1})$. Therefore, the deviating function $\beta(t)$ causes (1) to become a mixed system. For the derivation of the main results, some assumptions about the activation functions as follows are necessary.

Assumption 1. Denote $\alpha = \alpha_1 + i\alpha_2$ with $\alpha_1, \alpha_2 \in \mathcal{R}$. $f_r(\alpha)$ and $g_r(\alpha)$ satisfy $f_r(0) = g_r(0) = 0$. Moreover, they can be expressed as follows:

$$\begin{aligned} f_r(\alpha) &= f_r^R(\alpha_1) + \mathrm{i} f_r^I(\alpha_2), \\ g_r(\alpha) &= g_r^R(\alpha_1) + \mathrm{i} g_r^I(\alpha_2), \end{aligned} \tag{2}$$

where $r = 1, 2, \ldots, n$, and $f_r^R(\cdot)$, $f_r^I(\cdot)$, $g_r^R(\cdot)$, and $g_r^I(\cdot)$: $\mathcal{R} \longrightarrow \mathcal{R}$ satisfy the conditions as follows:

$$\begin{aligned} \left|f_r^R(\alpha_1) - f_r^R(\widehat{\alpha}_1)\right| &\le s_r|\alpha_1 - \widehat{\alpha}_1|, \\ \left|f_r^I(\alpha_2) - f_r^I(\widehat{\alpha}_2)\right| &\le m_r|\alpha_2 - \widehat{\alpha}_2|, \\ \left|g_r^R(\alpha_1) - g_r^R(\widehat{\alpha}_1)\right| &\le q_r|\alpha_1 - \widehat{\alpha}_1|, \\ \left|g_r^I(\alpha_2) - g_r^I(\widehat{\alpha}_2)\right| &\le h_r|\alpha_2 - \widehat{\alpha}_2|, \end{aligned} \tag{3}$$

in which $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ are arbitrary numbers in $\mathcal{R}$, and s_r, m_r, q_r, and h_r are known positive constants.

Furthermore, the nonlinear functions $f_r^R(\cdot)$, $f_r^I(\cdot)$, $g_r^R(\cdot)$, and $g_r^R(\cdot)$ are usually to be limited with greater condition.

Assumption 2. Let $\alpha = \alpha_1 + i\alpha_2$ with $\alpha_1, \alpha_2 \in \mathcal{R}$. $f_r(\alpha)$ and $g_r(\alpha)$ satisfy $f_r(0) = g_r(0) = 0$. Moreover, they can be expressed as follows:

$$\begin{aligned} f_r(\alpha) &= f_r^R(\alpha_1) + \mathrm{i} f_r^I(\alpha_2), \\ g_r(\alpha) &= g_r^R(\alpha_1) + \mathrm{i} g_r^I(\alpha_2), \end{aligned} \tag{4}$$

where $r = 1, 2, \ldots, n$, and $f_r^R(\cdot)$, $f_r^I(\cdot)$, $g_r^R(\cdot)$, and $g_r^I(\cdot)$ satisfy that $\forall \widehat{\alpha}_1, \widehat{\alpha}_2 \in \mathcal{R}$ with $\alpha_1 \ne \widehat{\alpha}_1, \alpha_2 \ne \widehat{\alpha}_2$:

$$\begin{aligned} 0 &\le \frac{f_r^R(\alpha_1) - f_r^R(\widehat{\alpha}_1)}{\alpha_1 - \widehat{\alpha}_1} \le s_r, \\ 0 &\le \frac{f_r^I(\alpha_2) - f_r^I(\widehat{\alpha}_2)}{\alpha_2 - \widehat{\alpha}_2} \le m_r, \\ 0 &\le \frac{g_r^R(\alpha_1) - g_r^R(\widehat{\alpha}_1)}{\alpha_1 - \widehat{\alpha}_1} \le q_r, \\ 0 &\le \frac{g_r^I(\alpha_2) - g_r^I(\widehat{\alpha}_2)}{\alpha_2 - \widehat{\alpha}_2} \le h_r, \end{aligned} \tag{5}$$

in which s_r, m_r, q_r, and h_r are known positive constants.

Remark 2. In many existing literature studies, the complex-valued activation function cannot be explicitly decomposed into real part and imaginary part. Under this condition, it is generally assumed that the activation function satisfies the Lipschitz condition on the complex domain. Then, the dynamic behaviors of CVNNs can be analyzed in the similar approach as RVNNs. However, if activation functions do not satisfy the Lipschitz condition, we have to separate the CVNNs into its real and imaginary parts. Moreover, activation functions are mostly restricted by certain conditions such as smooth, bounded, and nonconstant in RVNNs. But such restrictions for the activation functions of CVNNs cannot be allowed, the functions might become a scalar over the all complex field on the basis of Liouville's theorem. Therefore, choosing the right activation function for the CVNNs is one of the main challenges.

With the above assumptions, let $z(t) = \phi(t) + i\psi(t)$, we can rewrite system (1) as

$$\begin{cases} \phi'(t) = -A\phi(t) + B^R f^R(\phi(t)) - B^I f^I(\psi(t)) + C^R g^R(\phi(\beta(t))) - C^I g^I(\psi(\beta(t))) + L^R, \\ \psi'(t) = -A\psi(t) + B^R f^I(\psi(t)) + B^I f^R(\phi(t)) + C^R g^I(\psi(\beta(t))) + C^I g^R(\phi(\beta(t))) + L^I, \end{cases} \tag{6}$$

or in a more simplified form with the initial state

$$\begin{cases} \Delta'(t) = -\overline{A}\Delta(t) + B_1\overline{f}(\Delta(t)) + B_2\overline{\overline{f}}(\Delta(t)) + C_1\overline{g}(\Delta(\beta(t))) + C_2\overline{\overline{g}}(\Delta(\beta(t))) + \overline{L}, \quad t \ge t_0, \\ \Delta(t_0) = \Delta^0 = \begin{pmatrix} \phi^0 \\ \psi^0 \end{pmatrix}, \end{cases} \tag{7}$$

in which

$$\Delta(t)=\begin{pmatrix}\phi(t)\\ \psi(t)\end{pmatrix},$$
$$\overline{A}=\begin{pmatrix}A & \\ & A\end{pmatrix},$$
$$\overline{L}=\begin{pmatrix}L^R\\ L^I\end{pmatrix},$$
$$B_1=\begin{pmatrix}B^R & \\ & B^I\end{pmatrix},$$
$$B_2=\begin{pmatrix}-B^I & \\ & B^R\end{pmatrix},$$
$$C_1=\begin{pmatrix}C^R & \\ & C^I\end{pmatrix},\tag{8}$$
$$C_2=\begin{pmatrix}-C^I & \\ & C^R\end{pmatrix},$$
$$\overline{f}(\Delta(t))=\begin{pmatrix}f^R(\phi(t))\\ f^R(\phi(t))\end{pmatrix},$$
$$\overline{\overline{f}}(\Delta(t))=\begin{pmatrix}f^I(\psi(t))\\ f^I(\psi(t))\end{pmatrix},$$
$$\overline{g}(\Delta(\beta(t)))=\begin{pmatrix}g^R(\phi(\beta(t)))\\ g^R(\phi(\beta(t)))\end{pmatrix},$$
$$\overline{\overline{g}}(\Delta(\beta(t)))=\begin{pmatrix}g^I(\psi(\beta(t)))\\ g^I(\psi(\beta(t)))\end{pmatrix}.$$

For further discussion, several important definitions need to be presented and two useful lemmas are introduced as follows, which are employed for the later theoretical results.

Definition 1. (see [33]). Give a matrix $M=(m_{rj})_{n\times n}$ with $m_{\mathrm{rj}}\in\mathscr{R}$ and the following is the matrix measure of M:

$$\mu_p(M)=\lim_{\varepsilon\longrightarrow 0^+}\frac{\|\varepsilon M+I\|_p-1}{\varepsilon},\quad p=1,2,\infty.\tag{9}$$

The norms of matrix M are defined as follows:

$$\|M\|_1=\max_j\sum_{r=1}^{n}\left|m_{\mathrm{rj}}\right|,$$
$$\|M\|_2=\sqrt{\lambda_{\max}\left(M^TM\right)},\left|m_{\mathrm{rj}}\right|,\tag{10}$$
$$\|M\|_\infty=\max_r\sum_{j=1}^{n}$$

and then the matrix measure $\mu_p(M)$ can be given as follows:

$$\mu_1(M)=\max_j\left\{m_{\mathrm{jj}}+\sum_{r=1,r\neq j}^{n}\left|m_{\mathrm{rj}}\right|\right\},$$
$$\mu_2(M)=\frac{1}{2}\lambda_{\max}\left(M^T+M\right),\tag{11}$$
$$\mu_\infty(M)=\max_r\left\{m_{rr}+\sum_{j=1,j\neq r}^{n}\left|m_{\mathrm{rj}}\right|\right\}.$$

Definition 2. For any $t\geq t_0$, if there exist constants $\kappa>0$ and $\iota>0$ and the following inequality holds

$$\left\|\Delta(t)-\Delta^*\right\|_p\leq\kappa\left\|\Delta^0-\Delta^*\right\|_pe^{-\iota(t-t_0)},\tag{12}$$

then the equilibrium point $\Delta^*=((\phi^*)^T,(\psi^*)^T)^T$ of system (7) is called globally exponentially stable.

Lemma 1 (see [36]). *The following are the characteristics of matrix measure $\mu_p(M)$ mentioned in Definition 1 for all $M,N\in\mathscr{R}^{n\times n}$:*

(1) $-\|M\|_p\leq\mu_p(M)\leq\|M\|_p$
(2) $\mu_p(\delta M)=\delta\mu_p(M)$
(3) $\mu_p(M+N)\leq\mu_p(M)+\mu_p(N)$

Lemma 2 (see [36]). *The inequality $\mu_p(EG(\overline{\Delta}(t)))\leq\mu_p(E^*D)\,(p=1,\infty)$ holds, if*

$$0\leq\frac{g_r(\overline{\Delta}_r(t))}{\overline{\Delta}_r(t)}\leq d_r,\quad r=1,2,\ldots,n,\tag{13}$$

where $E=(e_{rj})_{n\times n}\in\mathscr{R}^{n\times n}$, $\overline{\Delta}=(\overline{\Delta}_1(t),\overline{\Delta}_2(t),\ldots,\overline{\Delta}_n(t))$, $D=diag\{d_1,d_2,\ldots,d_n\}$, $G(\overline{\Delta})=diag\{((g(\overline{\Delta}_1(t)))/(\overline{\Delta}_1(t))),((g(\overline{\Delta}_2(t)))/(\overline{\Delta}_2(t))),\ldots,((g_n(\overline{\Delta}_n(t)))/(\overline{\Delta}_n(t)))\}$, and $E^*=(e^*_{rj})_{n\times n}\in\mathscr{R}^{n\times n}$ with $e^*_{rj}=\max\{0,e_{rj}\}$ when $r=j$, otherwise, $e^*_{rj}=e_{rj}$.

3. Main Results

After the previous introduction and preparations, in this section, the dynamic performance of the CVNNs (1) with deviating argument is explored. Some conditions about the global exponential stability of our model are derived by the matrix measure approach and several lemmas. Moreover, by reasonably designing the feedback controller, the exponential synchronism of the master-slave CVNNs with deviating argument is achieved.

3.1. Global Exponential Stability. Suppose system (7) has an equilibrium point $\Delta^*=((\phi^*)^T,(\psi^*)^T)^T$; then

$$-\overline{A}\Delta^*+B_1\overline{f}(\Delta^*)+B_2\overline{\overline{f}}(\Delta^*)+C_1\overline{g}(\Delta^*)+C_2\overline{\overline{g}}(\Delta^*)+\overline{L}=0.\tag{14}$$

By setting

$$\overline{\Delta} = \Delta(t) - \Delta^* = \begin{pmatrix} \phi(t) - \phi^* \\ \psi(t) - \psi^* \end{pmatrix} = \begin{pmatrix} \overline{\phi}(t) \\ \overline{\psi}(t) \end{pmatrix}, \tag{15}$$

thus we can convert system (7) into the following differential equation:

$$\begin{cases} \overline{\Delta}'(t) = -\overline{A}\overline{\Delta}(t) + B_1\overline{F}(\overline{\Delta}(t)) + B_2\overline{\overline{F}}(\overline{\Delta}(t)) + C_1\overline{G}(\overline{\Delta}(\beta(t))) + C_2\overline{\overline{G}}(\overline{\Delta}(\beta(t))), \quad t \geq t_0, \\ \overline{\Delta}(t_0) = \overline{\Delta}^0 = \begin{pmatrix} \overline{\phi}^0 \\ \overline{\psi}^0 \end{pmatrix}, \end{cases} \tag{16}$$

where

$$\begin{aligned} \overline{F}(\overline{\Delta}(t)) &= \overline{f}(\overline{\Delta}(t) + \Delta^*) - \overline{f}(\Delta^*), \\ \overline{\overline{F}}(\overline{\Delta}(t)) &= \overline{\overline{f}}(\overline{\Delta}(t) + \Delta^*) - \overline{\overline{f}}(\Delta^*), \\ \overline{G}(\overline{\Delta}(\beta(t))) &= \overline{g}(\overline{\Delta}(\beta(t)) + \Delta^*) - \overline{g}(\Delta^*), \\ \overline{\overline{G}}(\overline{\Delta}(\beta(t))) &= \overline{\overline{g}}(\overline{\Delta}(\beta(t)) + \Delta^*) - \overline{\overline{g}}(\Delta^*). \end{aligned} \tag{17}$$

Lemma 3. *Assumed $\overline{\Delta}(t)$ be a solution of system (16), and there exists a scalar $\delta > 0$ such that $\delta_{k+1} - \delta_k \leq \delta$. If $\delta[\lambda_1(1 + \lambda_2\delta)e^{\lambda_1\delta} + \lambda_2] < 1$, then*

$$\|\overline{\Delta}(t)\|_p \leq \rho_1 \|\overline{\Delta}(t)\|_p, \tag{18}$$

for all $t \geq 0$, where $\lambda_1 = \|-\overline{A}\|_p + 2^{1/p}s\|B_1\|_p + 2^{1/p}m\|B_2\|_p$, $\lambda_2 = 2^{1/p}(q\|C_1\|_p + h\|C_2\|_p)$ with $s = \max_{1\leq r\leq n}\{s_r\}$, $m = \max_{1\leq r\leq n}\{m_r\}$, $q = \max_{1\leq r\leq n}\{q_r\}$, $h = \max_{1\leq r\leq n}\{h_r\}$, and $\rho_1 = \{1 - \delta[\lambda_1(1 + \lambda_2\delta)e^{\lambda_1\delta} + \lambda_2]\}^{-1}$.

Proof. For any $t \in [\delta_k, \delta_{k+1})$, $k \in \mathcal{N}$, it follows that

$$\begin{aligned} \|\overline{\Delta}(t)\|_p &= \left\| \overline{\Delta}(\zeta_k) + \int_{\zeta_k}^{t} \overline{\Delta}'(s)\mathrm{d}s \right\|_p \\ &\leq \|\overline{\Delta}(\zeta_k)\|_p + \int_{\zeta_k}^{t} \left\| -\overline{A}\overline{\Delta}(s) + B_1\overline{F}(\overline{\Delta}(s)) + B_2\overline{\overline{F}}(\Delta(s)) + C_1\overline{G}(\overline{\Delta}(\zeta_k)) + C_2\overline{\overline{G}}(\overline{\Delta}(\zeta_k)) \right\|_p \mathrm{d}s \\ &\leq \|\overline{\Delta}(\zeta_k)\|_p + \int_{\zeta_k}^{t} \Big[\|-\overline{A}\overline{\Delta}(s)\|_p + \|B_1\overline{F}(\overline{\Delta}(s))\|_p \\ &\quad + \|B_2\overline{\overline{F}}(\overline{\Delta}(s))\|_p + \|C_1\overline{G}(\overline{\Delta}(\zeta_k))\|_p \\ &\quad + \|C_2\overline{\overline{G}}(\overline{\Delta}(\zeta_k))\|_p \Big] \mathrm{d}s. \end{aligned} \tag{19}$$

Under Assumption 1, we have

$$\begin{aligned} \|\overline{F}(\overline{\Delta}(t))\|_p &= \left\| \begin{pmatrix} f^R(\overline{\phi}(t) + \phi^*) - f^R(\phi^*) \\ f^R(\overline{\phi}(t) + \phi^*) - f^R(\phi^*) \end{pmatrix} \right\|_p \\ &= \left[2\sum_{r=1}^{n} \left| f_r^R(\overline{\phi}_r(t) + \phi_r^*) - f_r^R(\phi_r^*) \right|^p \right]^{1/p} \\ &\leq \left[2\sum_{r=1}^{n} s_r^p |\overline{\phi}_r(t)|^p \right]^{1/p} \\ &\leq 2^{1/p} s \|\overline{\phi}_r(t)\|_p \leq 2^{1/p} s \|\overline{\Delta}(t)\|_p. \end{aligned} \tag{20}$$

The following inequalities can be obtained by using the same method as (20):

$$\|\overline{\overline{F}}(\overline{\Delta}(t))\|_p \leq 2^{1/p} m \|\overline{\Delta}(t)\|_p, \tag{21}$$

$$\|\overline{G}(\overline{\Delta}(\beta(t)))\|_p \leq 2^{1/p} q \|\overline{\Delta}(\beta(t))\|_p, \tag{22}$$

$$\|\overline{\overline{G}}(\overline{\Delta}(\beta(t)))\|_p \leq 2^{1/p} h \|\overline{\Delta}(\beta(t))\|_p. \tag{23}$$

By substituting inequalities (20)–(23) into (19), we have

$$\begin{aligned}\|\overline{\Delta}(t)\|_p &\le \|\overline{\Delta}(\zeta_k)\|_p + \int_{\zeta_k}^{t}\Big[\|-\overline{A\Delta}(s)\|_p + 2^{1/p}s\|B_1\overline{\Delta}(s)\|_p + 2^{1/p}m\|B_2\overline{\Delta}(s)\|_p + 2^{1/p}q\|C_1\overline{\Delta}(\zeta_k)\|_p \\ &\quad + 2^{1/p}h\|C_2\overline{\Delta}(\zeta_k)\|_p\Big]\mathrm{d}s \\ &\le \int_{\zeta_k}^{t}\Big(\|-\overline{A}\|_p + 2^{1/p}s\|B_1\|_p + 2^{1/p}m\|B_2\|_p\Big)\times\|\overline{\Delta}(s)\|_p\mathrm{d}s + \\ &\Big[1+\delta 2^{1/p}\Big(q\|C_1\|_p + h\|C_2\|_p\Big)\Big]\|\overline{\Delta}(\zeta_k)\|_p = (1+\lambda_2\delta)\|\overline{\Delta}(\zeta_k)\|_p + \int_{\zeta_k}^{t}\lambda_1\|\overline{\Delta}(s)\|_p\mathrm{d}s.\end{aligned} \tag{24}$$

By Gronwall's inequality, we have

$$\|\overline{\Delta}(t)\|_p \le (1+\lambda_2\delta)e^{\lambda_1\delta}\|\overline{\Delta}(\zeta_k)\|_p. \tag{25}$$

Furthermore, for $t \in [\delta_k, \delta_{k+1})$, it follows

$$\|\overline{\Delta}(\zeta_k)\|_p \le \|\overline{\Delta}(t)\|_p + \lambda_1\int_{\zeta_k}^{t}\|\overline{\Delta}(s)\|_p\mathrm{d}s + \lambda_2\int_{\zeta_k}^{t}\|\overline{\Delta}(\zeta_k)\|_p\mathrm{d}s. \tag{26}$$

Together with (25) and (26),

$$\|\overline{\Delta}(\zeta_k)\|_p \le \|\overline{\Delta}(t)\|_p + \delta\big[\lambda_1(1+\lambda_2\delta)e^{\lambda_1\delta} + \lambda_2\big]\|\overline{\Delta}(\zeta_k)\|_p, \tag{27}$$

then

$$\begin{aligned}\|\overline{\Delta}(\zeta_k)\|_p &\le \big\{1-\delta\big[\lambda_1(1+\lambda_2\delta)e^{\lambda_1\delta}+\lambda_2\big]\big\}^{-1}\|\overline{\Delta}(t)\|_p \\ &= \rho_1\|\overline{\Delta}(t)\|_p,\end{aligned} \tag{28}$$

for $t \in [\delta_k, \delta_{k+1})$. For all $t \in \mathscr{R}^+$, (18) holds. □

Theorem 1. *Let Assumption 1 hold, if the following condition satisfies*

$$\begin{aligned}&-\Big[\mu_p(-\overline{A}) + 2^{1/p}s\|B_1\|_p + 2^{1/p}m\|B_2\|_p \\ &+\rho_1\Big(2^{1/p}q\|C_1\|_p + 2^{1/p}h\|C_2\|_p\Big)\Big] > 0,\end{aligned} \tag{29}$$

where $p = 1, 2, \infty$, $s = \max_{1\le r\le n}\{s_r\}$, $m = \max_{1\le r\le n}\{m_r\}$, $q = \max_{1\le r\le n}\{q_r\}$, *and* $h = \max_{1\le r\le n}\{h_r\}$, *then the system (7) is globally exponentially stable.*

Proof. Differentiating $\|\overline{\Delta}\|_p$, we can derive that

$$\begin{aligned}&\lim_{\varepsilon\longrightarrow 0^+}\frac{\|\overline{\Delta}(t+\varepsilon)\|_p - \|\overline{\Delta}(t)\|_p}{\varepsilon} \\ &= \lim_{\varepsilon\longrightarrow 0^+}\frac{\|\varepsilon\overline{\Delta}'(t) + \overline{\Delta}(t) + o(\varepsilon)\|_p - \|\overline{\Delta}(t)\|_p}{\varepsilon},\end{aligned} \tag{30}$$

in which

$$\begin{aligned}&\|\varepsilon\overline{\Delta}'(t) + \overline{\Delta}(t) + o(\varepsilon)\|_p \\ &= \Big\|\varepsilon\Big[-\overline{A\Delta}(t) + B_1\overline{F}(\overline{\Delta}(t)) + B_2\overline{f}_2(\overline{\Delta}(t)) \\ &\quad + C_1\overline{G}(\overline{\Delta}(\beta(t))) + C_2\overline{\overline{G}}(\overline{\Delta}(\beta(t)))\Big] + \overline{\Delta}(t) + o(\varepsilon)_p\Big\| \\ &= \Big\|\varepsilon(-\overline{A})\overline{\Delta}(t) + \varepsilon B_1\overline{F}(\overline{\Delta}(t)) + \varepsilon B_2\overline{f}_2(\overline{\Delta}(t)) \\ &\quad + \varepsilon C_1\overline{G}(\overline{\Delta}(\beta(t))) + \varepsilon C_2\overline{\overline{G}}(\overline{\Delta}(\beta(t))) + \overline{\Delta}(t) + o(\varepsilon)_p\Big\| \\ &\le \|\varepsilon(-\overline{A}) + I\|_p\|\overline{\Delta}(t)\|_p + \varepsilon\|B_1\|_p\|\overline{F}(\overline{\Delta}(t))\|_p \\ &\quad + \varepsilon\|B_2\|_p\|\overline{f}_2(\overline{\Delta}(t))\|_p + \varepsilon\|C_1\|_p\|\overline{G}(\overline{\Delta}(\beta(t)))\|_p \\ &\quad + \varepsilon\|C_2\|_p\|\overline{\overline{G}}(\overline{\Delta}(\beta(t)))\|_p + \|o(\varepsilon)\|_p.\end{aligned} \tag{31}$$

Substitute inequalities (20)–(23) into (31), and it follows from (30) that

$$\begin{aligned}&\lim_{\varepsilon\longrightarrow 0^+}\frac{\|\overline{\Delta}(t+\varepsilon)\|_p - \|\overline{\Delta}(t)\|_p}{\varepsilon} \\ &\le \lim_{\varepsilon\longrightarrow 0^+}\frac{\|I+\varepsilon(-\overline{A})\|_p - 1}{\varepsilon}\|\overline{\Delta}(t)\|_p + 2^{1/p}s\|B_1\|_p\|\overline{\Delta}(t)\|_p \\ &\quad + 2^{1/p}m\|B_2\|_p\|\overline{\Delta}(t)\|_p + 2^{1/p}q\|C_1\|_p\|\overline{\Delta}(\beta(t))\|_p \\ &\quad + 2^{1/p}h\|C_2\|_p\|\overline{\Delta}(\beta(t))\|_p \\ &\le \Big[\mu_p(-\overline{A}) + 2^{1/p}s\|B_1\|_p + 2^{1/p}m\|B_2\|_p\Big]\|\overline{\Delta}(t)\|_p \\ &\quad + \Big(2^{1/p}q\|C_1\|_p + 2^{1/p}h\|C_2\|_p\Big)\|\overline{\Delta}(\beta(t))\|_p.\end{aligned} \tag{32}$$

By definition of upper right-hand Dini derivative and Lemma 3, one obtains

$$\begin{aligned}D^+\|\overline{\Delta}(t)\|_p &\le \Big[\mu_p(-\overline{A}) + 2^{1/p}s\|B_1\|_p + 2^{1/p}m\|B_2\|_p \\ &\quad + \rho_1\Big(2^{1/p}q\|C_1\|_p + 2^{1/p}h\|C_2\|_p\Big)\Big]\|\overline{\Delta}(t)\|_p \\ &= -\iota_1\|\overline{\Delta}(t)\|_p,\end{aligned} \tag{33}$$

where $\iota_1 = -[\mu_p(-\overline{A}) + 2^{1/p}s\|B_1\|_p + 2^{1/}\quad pm\|B_2\|_p + \rho_1(2^{1/p}q\|C_1\|_p + 2^{1/p}h\|C_2\|_p)]$, and then

$$\|\overline{\Delta}(t)\|_p \le \|\overline{\Delta}^0\|_p e^{-\iota_1(t-t_0)}, \quad t \ge t_0, \tag{34}$$

which can be claimed that system (6) achieves global exponential stability, that is, model (1) is also globally exponentially stable. □

Remark 3. In a finite-dimensional space, diverse norms can be considered equivalent and satisfy the relationship $\|\cdot\|_1 \geq \|\cdot\|_2 \geq \|\cdot\|_\infty$, whereas the corresponding matrix measure could not satisfy the same condition. Therefore, if we can find the matrix norm and measure that meet the inequality (29), then Theorems 1 holds.

Remark 4. As everyone knew that there are some differences between matrix norms and matrix measures. The matrix norm has to be nonnegative, but matrix measures can be assigned positive or negative values. Moreover, $\|M\|_p \equiv \|-M\|_p$, whereas matrix measures are generally sign-sensitive in that $\mu_p(M) \neq \mu_p(-M)$. If the connection weights include positive and negative values, the measure method will be more accurate and extensive than the norm form previously. Therefore, the results obtained by matrix measure in this paper are more excellent.

Theorem 2. *Under Assumption 2, the global exponential stability for system (7) can be guaranteed if the following condition satisfies*

$$\begin{aligned} &-\left[\mu_p(-\overline{A}) + \mu_p\left(\overline{B}^* \mathcal{S}\right) + (s+m)\left(\|B_1\|_p + \|B_2\|_p\right)\right. \\ &\left.+2^{1/p}\rho_1\left(q\|C_1\|_p + h\|C_2\|_p\right)\right] > 0, \end{aligned} \tag{35}$$

where $p = 1, \infty$, $\mathcal{S} = diag\{s_1, s_2, \ldots, s_n, m_1, m_2, \ldots, m_n\}$, $s = \max_{1\leq r\leq n}\{s_r\}$, $m = \max_{1\leq r\leq n}\{m_r\}$, $q = \max_{1\leq r\leq n}\{q_r\}$, $h = \max_{1\leq r\leq n}\{h_r\}$, $\overline{B} = (\overline{b}_{rj})_{2n\times 2n} = B_1 + B_2$, *and*

$$\overline{B}^* = \left(\overline{b}^*_{rj}\right)_{2n\times 2n} = \begin{cases} \max\{0, \overline{b}_{rj}\}, & r = j, \\ \overline{b}_{rj}, & r \neq j. \end{cases} \tag{36}$$

Proof. Define

$$\begin{aligned} \widetilde{G}(\overline{\Delta}(t)) = diag\Bigg\{ &\frac{f_1^R\left(\overline{\phi}_1(t) + \phi_1^*\right) - f_1^R(\phi_1^*)}{\overline{\phi}_1(t)}, \ldots, \\ &\frac{f_n^R\left(\overline{\phi}_n(t) + \phi_n^*\right) - f_n^R(\phi_n^*)}{\overline{\phi}_n(t)}, \\ &\frac{f_1^I\left(\overline{\psi}_1(t) + \psi_1^*\right) - f_1^I(\psi_1^*)}{\overline{\psi}_1(t)}, \ldots, \\ &\frac{f_n^I\left(\overline{\psi}_n(t) + \psi_n^*\right) - f_n^I(\psi_n^*)}{\overline{\psi}_n(t)}\Bigg\}. \end{aligned} \tag{37}$$

By rewriting $\overline{F}(\overline{\Delta}(t))$ by its definition, then

$$\begin{aligned} \overline{F}(\overline{\Delta}(t)) &= \begin{pmatrix} f^R\left(\overline{\phi}(t) + \phi^*\right) - f^R(\phi^*) \\ f^R\left(\overline{\phi}(t) + \phi^*\right) - f^R(\phi^*) \end{pmatrix} \\ &= K_1(t) - K_2(t) + K_3(t), \end{aligned} \tag{38}$$

where

$$\begin{aligned} K_1(t) &= \begin{pmatrix} f^R\left(\overline{\phi}(t) + \phi^*\right) - f^R(\phi^*) \\ f^I\left(\overline{\psi}(t) + \psi^*\right) - f^I(\psi^*) \end{pmatrix} = \widetilde{G}(\overline{\Delta}(t))\overline{\Delta}(t), \\ K_2(t) &= \begin{pmatrix} 0_{n\times 1} \\ f^I\left(\overline{\psi}(t) + \psi^*\right) - f^I(\psi^*) \end{pmatrix}, \\ K_3(t) &= \begin{pmatrix} f^R\left(\overline{\phi}(t) + \phi^*\right) - f^R(\phi^*) \\ 0_{n\times 1} \end{pmatrix}. \end{aligned} \tag{39}$$

From Assumption 2, the following inequalities can be hold

$$\begin{aligned} 0 &\leq \frac{f_r^R\left(\overline{\phi}_r(t) + \phi_r^*\right) - f_r^R(\phi_r^*)}{\overline{\phi}_r(t)} \leq s_r, \\ 0 &\leq \frac{f_r^I\left(\overline{\psi}_r(t) + \psi_r^*\right) - f_r^I(\psi_r^*)}{\overline{\psi}_r(t)} \leq m_r. \end{aligned} \tag{40}$$

Similarly, we can also have

$$\overline{\overline{F}}_2(\overline{\Delta}(t)) = \widetilde{G}(\overline{\Delta})(t)\overline{\Delta}(t) - K_4(t) + K_5(t), \tag{41}$$

where

$$\begin{aligned} K_4(t) &= \begin{pmatrix} f^R\left(\overline{\phi}(t) + \phi^*\right) - f^R(\phi^*) \\ 0_{n\times 1} \end{pmatrix}, \\ K_5(t) &= \begin{pmatrix} f^I\left(\overline{\psi}(t) + \psi^*\right) - f^I(\psi^*) \\ 0_{n\times 1} \end{pmatrix}. \end{aligned} \tag{42}$$

By substituting (38) and (41) into (31), and also considering (22) and (23), one has

$$\begin{aligned} &\left\|\overline{\Delta}(t) + \varepsilon\overline{\Delta}'(t) + o(\varepsilon)\right\|_p \\ &\quad= \Big\|\overline{\Delta}(t) + \varepsilon(-\overline{A})\overline{\Delta}(t) + \varepsilon B_1(\widetilde{G}(\overline{\Delta}(t))\overline{\Delta}(t)) \\ &\qquad -K_2(t) + K_3(t)) + \varepsilon B_2(\widetilde{G}(\overline{\Delta}(t))\overline{\Delta}(t)) \\ &\qquad -K_4(t) + K_5(t)) + \varepsilon C_1\overline{G}(\overline{\Delta}(\beta(t))) \\ &\qquad + \varepsilon C_2\overline{\overline{G}}(\overline{\Delta}(\beta(t))) + o(\varepsilon)_p\Big\| \\ &\quad\leq \left\|I + \varepsilon\left[-\overline{A} + (B_1 + B_2)\widetilde{G}(\overline{\Delta}(t))\right]\right\|_p \|\overline{\Delta}(t)\|_p \\ &\qquad + \varepsilon\|B_1\|_p\left(\|K_2(t)\|_p + \|K_3(t)\|_p\right) \\ &\qquad + \varepsilon\|B_2\|_p\left(\|K_4(t)\|_p + \|K_5(t)\|_p\right) \\ &\qquad + \varepsilon\|C_1\|_p\|\overline{G}(\overline{\Delta}(\beta(t)))\|_p \\ &\qquad + \varepsilon\|C_2\|_p\|\overline{\overline{G}}(\overline{\Delta}(\beta(t)))\|_p + \|o(\varepsilon)\|_p \\ &\quad\leq \left\|I + \varepsilon\left[-\overline{A} + (B_1 + B_2)\widetilde{G}(\overline{\Delta}(t))\right]\right\|_p \|\overline{\Delta}(t)\|_p \\ &\qquad + \varepsilon(s+m)\|B_1\|_p\|\overline{\Delta}(t)\|_p \\ &\qquad + \varepsilon(s+m)\|B_2\|_p\|\overline{\Delta}(t)\|_p \\ &\qquad + 2^{1/p}q\varepsilon\|C_1\|_p\|\overline{\Delta}(\beta(t))\|_p \\ &\qquad + 2^{1/p}h\varepsilon\|C_2\|_p\|\overline{\Delta}(\beta(t))\|_p \\ &\qquad + \|o(\varepsilon)\|_p. \end{aligned} \tag{43}$$

Consequently, from Lemma 3, we can get

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0^+}\frac{\|\overline{\Delta}(t+\varepsilon)\|_p-\|\overline{\Delta}(t)\|_p}{\varepsilon}\\
&\le \lim_{\varepsilon\longrightarrow 0^+}\frac{\left\|I+\varepsilon\left[-\overline{A}+(B_1+B_2)\widetilde{G}(\overline{\Delta}(t))\right]\right\|_p-1}{\varepsilon}\|\overline{\Delta}\|_p\\
&\quad+(s+m)\|B_1\|_p\|\overline{\Delta}(t)\|_p+(s+m)\|B_2\|_p\|\overline{\Delta}(t)\|_p\\
&\quad+2^{1/p}q\|C_1\|_p\|\overline{\Delta}(\beta(t))\|_p+2^{1/p}h\|C_2\|_p\|\overline{\Delta}(\beta(t))\|_p\\
&\le\Big[\mu_p\Big(-\overline{A}+(B_1+B_2)\widetilde{G}(\overline{\Delta}(t))\Big)\\
&\quad+(s+m)\Big(\|B_1\|_p+\|B_2\|_p\Big)\\
&\quad+2^{1/p}\rho_1\Big(q\|C_1\|_p+h\|C_2\|_p\Big)\Big]\|\overline{\Delta}(t)\|_p.
\end{aligned}\tag{44}$$

We have the following inequality by Lemmas 1 and 2:

$$\begin{aligned}
&\mu_p\Big(-\overline{A}+(B_1+B_2)\widetilde{G}(\overline{\Delta}(t))\Big)\\
&\quad\le\mu_p(-\overline{A})+\mu_p(B_1+B_2)\widetilde{G}(\overline{\Delta}(t))\\
&\quad\le\mu_p(-\overline{A})+\mu_p\Big(\overline{B}^*\mathcal{S}\Big).
\end{aligned}\tag{45}$$

Thus, one concludes that

$$\begin{aligned}
D^+\|\overline{\Delta}(t)\|_p&\le\Big[\mu_p(-\overline{A})+\mu_p\Big(\overline{B}^*\mathcal{S}\Big)\\
&\quad+(s+m)\Big(\|B_1\|_p+\|B_2\|_p\Big)\\
&\quad+2^{1/p}\rho_1\Big(q\|C_1\|_p+h\|C_2\|_p\Big)\Big]\|\overline{\Delta}(t)\|_p\\
&\le-\iota_2\|\overline{\Delta}(t)\|_p,
\end{aligned}\tag{46}$$

where $\iota_2=-[\mu_p(-\overline{A})+\mu_p(\overline{B}^*\mathcal{S})+(s+m)(\|B_1\|_p+\|B_2\|_p)+2^{1/p}\rho_1(q\|C_1\|_p+h\|C_2\|_p)]$, and then

$$\|\overline{\Delta}(t)\|_p\le\left\|\overline{\Delta}^0\right\|_p e^{-\iota_2(t-t_0)},\quad t\ge t_0.\tag{47}$$

The condition (47) guarantees that network (7) and network (1) are globally exponentially stable by Definition 2. □

3.2. Global Exponential Synchronization. Consider the corresponding response system of drive system (1) as following differential equations:

$$\begin{cases}\widetilde{z}'(t)=-A\widetilde{z}(t)+\mathrm{Bf}(\widetilde{z}(t))+\mathrm{Cg}(\widetilde{z}(\beta(t)))+L+U(t),\quad t\ge t_0\ge 0,\\ \widetilde{z}(t_0)=\widetilde{z}^0,\end{cases}\tag{48}$$

where $\widetilde{z}(t)=\widetilde{\phi}(t)+i\widetilde{\psi}(t)$ with $\widetilde{\phi}(t),\widetilde{\psi}(t)\in\mathscr{R}^n$, and $U(t)=(U_1(t),U_2(t),\ldots,U_n(t))^T$ denotes the controller.

Obviously, system (48) can also be divided into the following forms:

$$\begin{cases}\widetilde{\phi}'(t)=-A\widetilde{\phi}(t)+B^Rf^R(\widetilde{\phi}(t))-B^If^I(\widetilde{\psi}(t))+L^R+C^Rg^R(\widetilde{\phi}(\beta(t)))-C^Ig^I(\widetilde{\psi}(\beta(t)))+U^R(t),\\ \widetilde{\psi}'(t)=-A\widetilde{\psi}(t)+B^Rf^I(\widetilde{\psi}(t))+B^If^R(\widetilde{\phi}(t))+L^I+C^Rg^I(\widetilde{\psi}(\beta(t)))+C^Ig^R(\widetilde{\phi}(\beta(t)))+U^I(t).\end{cases}\tag{49}$$

Then subtract the master network (6) from the slave network (49), define synchronization errors $\widehat{\Delta}(t)=(\widetilde{\phi}(t)-\phi(t),\widetilde{\psi}(t)-\psi(t))^T=(\widehat{\phi}(t),\widehat{\psi}(t))^T$, and then the error system can be obtained as follows:

$$\begin{cases}\widehat{\phi}'(t)=-A\widehat{\phi}(t)+B^R\widetilde{f}(\widehat{\phi}(t))-B^I\widetilde{f}(\widehat{\psi}(t))+C^R\widetilde{g}(\widehat{\phi}(\beta(t)))-C^I\widetilde{g}(\widehat{\psi}(\beta(t)))+U^R(t),\\ \widehat{\psi}'(t)=-A\widehat{\psi}(t)+B^R\widetilde{f}(\widehat{\psi}(t))+B^I\widetilde{f}(\widehat{\phi}(t))+C^R\widetilde{g}(\widehat{\psi}(\beta(t)))+C^I\widetilde{g}(\widehat{\phi}(\beta(t)))+U^I(t),\end{cases}\tag{50}$$

in which

$$\begin{aligned}\widetilde{f}(\widehat{\phi}(t)) &= f^R(\widetilde{\phi}(t)) - f^R(\phi(t)),\\ \widetilde{f}(\widehat{\psi}(t)) &= f^I(\widetilde{\psi}(t)) - f^I(\psi(t)),\\ \widetilde{g}(\widehat{\phi}(\beta(t))) &= g^R(\widetilde{\phi}(\beta(t))) - g^R(\phi(\beta(t))),\\ \widetilde{g}(\widehat{\psi}(\beta(t))) &= g^I(\widetilde{\psi}(\beta(t))) - g^I(\psi(\beta(t))).\end{aligned} \tag{51}$$

The control input is designed as follows:

$$\begin{cases} U^R(t) = -S\widehat{\phi}(t) - Q\widehat{\psi}(t),\\ U^I(t) = -\mathbb{S}\widehat{\phi}(t) - \mathbb{Q}\widehat{\psi}(t),\end{cases} \tag{52}$$

where matrices S, $\mathbb{S}$, Q, and $\mathbb{Q}$ are positive definite and diagonal.

Definition 3. For all $t \geq t_0$, if there exist constants $\overline{\kappa} > 0$ and $\overline{\imath} > 0$ and the following inequality holds

$$\|\widehat{\Delta}(t)\|_p \leq \overline{\kappa}\left\|\widehat{\Delta}^0\right\|_p e^{-\overline{\imath}(t-t_0)}, \tag{53}$$

then the error system (50) achieves global exponential stability under controller (52), that is, the master systems (6) and the slave system (49) under the controller (52) are globally exponentially synchronizable.

Theorem 3. *Suppose there exists a scalar $\delta > 0$ such that $\delta_{k+1} - \delta_k \leq \delta$. If the following conditions hold*

(1) $\delta[\lambda_3(1+\lambda_4\delta)e^{\lambda_3\delta} + \lambda_4] < 1$

(2) $-[\mu_p(-\widetilde{A}) + \overline{sm}(\|B_3\|_p) + \rho_2\overline{qh}(\|C_3\|_p)] > 0$, *where* $p = 1, 2, \infty$, $\lambda_3 = \|-\widetilde{A}\|_p + \overline{sm}(\|B_3\|_p)$, $\lambda_4 = \overline{qh}(\|C_3\|_p)$, $\rho_2 = \left\{1 - \delta[\lambda_3(1+\lambda_4\delta)e^{\lambda_3\delta} + \lambda_4]\right\}^{-1}$, $\overline{sm} = \max_{1\leq r\leq n}\{s_r, m_r\}$, $\overline{qh} = \max_{1\leq r\leq n}\{q_r, h_r\}$,

$$\begin{aligned}\widetilde{A} &= \begin{pmatrix} A+S & Q\\ \mathbb{S} & A+\mathbb{Q}\end{pmatrix},\\ B_3 &= \begin{pmatrix} B^R & -B^I\\ B^I & B^R\end{pmatrix},\\ C_3 &= \begin{pmatrix} C^R & -C^I\\ C^I & C^R\end{pmatrix},\end{aligned} \tag{54}$$

then global exponential synchronization can be achieved for the master network (6) and slave network (49).

Proof. Assume $z(t)$ and $\widetilde{z}(t)$ are random solutions of network (1) and network (48), respectively. By differentiating $\|\widehat{\Delta}(t)\|_p$, we can easily get

$$\begin{aligned}&\lim_{\varepsilon\longrightarrow 0^+}\frac{\|\widehat{\Delta}(t+\varepsilon)\|_p - \|\widehat{\Delta}(t)\|_p}{\varepsilon}\\ &\quad= \lim_{\varepsilon\longrightarrow 0^+}\frac{\left\|\varepsilon\widehat{\Delta}'(t) + \widehat{\Delta}(t) + o(\varepsilon)\right\|_p - \|\widehat{\Delta}(t)\|_p}{\varepsilon},\end{aligned} \tag{55}$$

where

$$\begin{aligned}\left\|\varepsilon\hat{\Delta}'(t) + \hat{\Delta}(t) + o(\varepsilon)\right\|_p &= \left\|\varepsilon\begin{pmatrix}\hat{\phi}'(t)\\ \hat{\psi}'(t)\end{pmatrix} + \hat{\Delta}(t) + o(\varepsilon)\right\|_p\\ &\leq \left\|\varepsilon\begin{pmatrix}-A\hat{\phi}(t) + \widetilde{F}(\hat{\Delta}(t)) + \widetilde{G}(\hat{\Delta}(\beta(t))) + U^R(t)\\ -A\hat{\psi}(t) + \widetilde{\widetilde{F}}(\hat{\Delta}(t)) + \widetilde{\widetilde{G}}(\hat{\Delta}(\beta(t))) + U^I(t)\end{pmatrix}\right\|_p\\ &\quad + \|\hat{\Delta}(t) + o(\varepsilon)\|_p\\ &\leq \left\|\varepsilon\begin{pmatrix}-A\hat{\phi} + U^R(t)\\ -A\hat{\psi} + U^I(t)\end{pmatrix} + \hat{\Delta}(t)\right\|_p + \varepsilon\left\|\begin{pmatrix}\widetilde{F}(\hat{\Delta}(t))\\ \widetilde{\widetilde{F}}(\hat{\Delta}(t))\end{pmatrix}\right\|_p\\ &\quad + \varepsilon\left\|\begin{pmatrix}\widetilde{G}(\hat{\Delta}(\beta(t)))\\ \widetilde{\widetilde{G}}(\hat{\Delta}(\beta(t)))\end{pmatrix}\right\|_p + \|o(\varepsilon)\|_p,\end{aligned} \tag{56}$$

in which

$$\begin{aligned}
\widetilde{F}(\widehat{\Delta}(t)) &= B^R\widetilde{f}^R(\widehat{\phi}(t)) - B^I\widetilde{f}^I(\widehat{\psi}(t)),\\
\widetilde{\widetilde{F}}(\widehat{\Delta}(t)) &= B^R\widetilde{f}^I(\widehat{\psi}(t)) + B^I\widetilde{f}^R(\widehat{\phi}(t)),\\
\widetilde{G}(\widehat{\Delta}(\beta(t))) &= C^R\widetilde{g}^R(\widehat{\phi}(\beta(t))) - C^I\widetilde{g}^I(\widehat{\psi}(\beta(t))),\\
\widetilde{\widetilde{G}}(\widehat{\Delta}(\beta(t))) &= C^R\widetilde{g}^I(\widehat{\psi}(\beta(t))) + C^I\widetilde{g}^R(\widehat{\phi}(\beta(t))).
\end{aligned} \tag{57}$$

From Assumption 1, we can get

$$\begin{aligned}
\left\|\begin{pmatrix}\widetilde{F}(\widehat{\Delta}(t))\\ \widetilde{\widetilde{F}}(\widehat{\Delta}(t))\end{pmatrix}\right\|_p &= \left\|\begin{pmatrix}B^R & -B^I\\ B^I & B^R\end{pmatrix}\begin{pmatrix}\widetilde{f}^R(\widehat{\phi}(t))\\ \widetilde{f}^I(\widehat{\psi}(t))\end{pmatrix}\right\|_p\\
&\le \|B_3\|_p\left\|\begin{pmatrix}f^R(\widetilde{\phi}(t)) - f^R(\phi(t))\\ f^I(\widetilde{\psi}(t)) - f^I(\psi(t))\end{pmatrix}\right\|_p\\
&= \|B_3\|_p\left[\sum_{r=1}^{n}\left|f_r^R(\widetilde{\phi}_r(t)) - f_r^R(\phi_r(t))\right|^p\right.\\
&\quad\left.+\sum_{r=1}^{n}\left|f_r^I(\widetilde{\psi}_r(t)) - f_r^I(\psi_r(t))\right|^p\right]^{1/p}\\
&\le \|B_3\|_p\left[\sum_{r=1}^{n}s_r^p|\widehat{\phi}(t)|^p + \sum_{r=1}^{n}m_r^p|\widehat{\psi}(t)|^p\right]^{1/p}\\
&\le \overline{\mathrm{sm}}\|B_3\|_p\left[\sum_{r=1}^{n}|\widehat{\phi}(t)|^p + \sum_{r=1}^{n}|\widehat{\psi}(t)|^p\right]^{1/p}\\
&\le \overline{\mathrm{sm}}\|B_3\|_p\|\widehat{\Delta}(t)\|_p,
\end{aligned} \tag{58}$$

and the following inequality can be also obtained:

$$\left\|\begin{pmatrix}\widetilde{G}(\widehat{\Delta}(\beta(t)))\\ \widetilde{\widetilde{G}}(\widehat{\Delta}(\beta(t)))\end{pmatrix}\right\|_p \le \overline{qh}\|C_3\|_p\|\widehat{\Delta}(\beta(t))\|_p, \tag{59}$$

where $\overline{\mathrm{sm}} = \max_{1\le r\le n}\{s_r, m_r\}$, $\overline{\mathrm{qh}} = \max_{1\le r\le n}\{q_r, h_r\}$, and $B_3 = \begin{pmatrix}B^R & -B^I\\ B^I & B^R\end{pmatrix}$ and $C_3 = \begin{pmatrix}C^R & -C^I\\ C^I & C^R\end{pmatrix}$.

By substituting inequalities (56)–(59) into (55), we have

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0^+}\frac{\|\widehat{\Delta}(t+\varepsilon)\|_p - \|\widehat{\Delta}(t)\|_p}{\varepsilon}\\
&\le \lim_{\varepsilon\longrightarrow 0^+}\frac{\left\|\widehat{\Delta}(t) + \varepsilon\begin{pmatrix}-A\widehat{\phi} + U^R(t)\\ -A\widehat{\psi} + U^I(t)\end{pmatrix}\right\|_p - \|\widehat{\Delta}(t)\|_p}{\varepsilon}\\
&\quad + \left\|\begin{pmatrix}\widetilde{F}(\widehat{\Delta}(t))\\ \widetilde{\widetilde{F}}(\widehat{\Delta}(t))\end{pmatrix}\right\|_p + \left\|\begin{pmatrix}\widetilde{G}(\widehat{\Delta}(\beta(t)))\\ \widetilde{\widetilde{G}}(\widehat{\Delta}(\beta(t)))\end{pmatrix}\right\|_p\\
&\le \lim_{\varepsilon\longrightarrow 0^+}\frac{\left\|I + \varepsilon\begin{pmatrix}-A-S & -Q\\ -\mathbb{S} & -A-\mathbb{Q}\end{pmatrix}\right\|_p - 1}{\varepsilon}\|\widehat{\Delta}(t)\|_p\\
&\quad + \overline{\mathrm{sm}}\|B_3\|_p\|\widehat{\Delta}(t)\|_p + \overline{\mathrm{qh}}\|C_3\|_p\|\widehat{\Delta}(\beta(t))\|_p\\
&\le \mu_p(-\widetilde{A})\|\widehat{\Delta}(t)\|_p + \overline{\mathrm{sm}}B_3\|\widehat{\Delta}(t)\|_p\\
&\quad + \overline{\mathrm{qh}}C_3\|\widehat{\Delta}(\beta(t))\|_p,
\end{aligned} \tag{60}$$

in which $\widetilde{A} = \begin{pmatrix}-A-S & -Q\\ -\mathbb{S} & -A-\mathbb{Q}\end{pmatrix}$.

Fix $k\in N$, for any $t\in[\delta_k, \delta_{k+1})$, from inequalities (58) and (59), it follows that

$$\begin{aligned}
\|\widehat{\Delta}(t) * \|_p &= \left\|\widehat{\Delta}(\zeta_k) + \int_{\zeta_k}^{t}\widehat{\Delta}'(s)\mathrm{d}s\right\|_p\\
&\le \|\widehat{\Delta}(\zeta_k)\|_p + \int_{\zeta_k}^{t}\left\|\begin{pmatrix}-A-S & -Q\\ -\mathbb{S} & -A-\mathbb{Q}\end{pmatrix}\right\|_p\|\widehat{\Delta}(s)\|\mathrm{d}s\\
&\quad + \int_{\zeta_k}^{t}\overline{\mathrm{sm}}\|B_3\|_p\|\widehat{\Delta}(s)\|_p\mathrm{d}s\\
&\quad + \int_{\zeta_k}^{t}\overline{\mathrm{qh}}\|C_3\|_p\|\widehat{\Delta}(\zeta_k)\|_p\mathrm{d}s\\
&\le \|\widehat{\Delta}(\zeta_k)\|_p + \int_{\zeta_k}^{t}\left(\|-\widetilde{A}\|_p + \overline{\mathrm{sm}}\|B_3\|_p\right)\|\Delta(s)\|_p\mathrm{d}s\\
&\quad + \int_{\zeta_k}^{t}\overline{\mathrm{qh}}\|C_3\|_p\|\widehat{\Delta}(\zeta_k)\|_p\mathrm{d}s\\
&\le (1+\lambda_4\delta)\|\widehat{\Delta}(\zeta_k)\|_p + \int_{\zeta_k}^{t}\lambda_3\|\widehat{\Delta}(s)\|_p\mathrm{d}s,
\end{aligned} \tag{61}$$

in which $\lambda_3 = \overline{\mathrm{sm}}(\| - \widetilde{A}\|_p + \overline{\mathrm{sm}}\|B_3\|_p)$ and $\lambda_4 = \overline{\mathrm{qh}}\|C_3\|_p$.

Then the same as Lemma 3, finally we obtain

$$\begin{aligned}\|\widehat{\Delta}(\beta(t))\|_p &\le \left\{1 - \delta\left[\lambda_3\left(1 + \lambda_4\delta\right)e^{\lambda_3\delta} + \lambda_4\right]\right\}^{-1}\|\widehat{\Delta}(t)\|_p \\ &= \rho_2\|\widehat{\Delta}(t)\|_p,\end{aligned} \tag{62}$$

for $t \in [\delta_k, \delta_{k+1})$, and $\rho_2 = \{1 - \delta[\lambda_3(1 + \lambda_4\delta)e^{\lambda_3\delta} + \lambda_4]\}^{-1}$.

Further, from (60) and (62), we can get

$$\begin{aligned}D^+\|\widehat{\Delta}(t)\|_p &\le \left[\mu_p(-\widetilde{A}) + \overline{\mathrm{sm}}\left(\|B_3\|_p\right) + \rho_2\overline{\mathrm{qh}}\left(\|C_3\|_p\right)\right]\|\widehat{\Delta}(t)\|_p \\ &= -\bar{\imath}\|\widehat{\Delta}(t)\|_p,\end{aligned} \tag{63}$$

where $\bar{\imath} = -[\mu_p(-\widetilde{A}) + \overline{\mathrm{sm}}(\|B_3\|_p) + \rho_2\overline{\mathrm{qh}}(\|C_3\|_p)]$, then

$$\|\widehat{\Delta}(t)\|_p \le \left\|\widehat{\Delta}^0\right\|_p e^{-\bar{\imath}(t-t_0)}. \tag{64}$$

From the above discussions, the master network (6) and slave network (49) achieve global exponential synchronization by the feedback controller (52). □

Remark 5. Synchronization is different from the stabilization of single network, which mainly happens in multiple networks. Actually, some alterable coupling between networks must exist for synchronism achieved, and over the course of time, the errors between the coupled networks get more smaller until zero. We employ drive-response systems (1) and (48) to achieve exponential synchronization in the process of describing our model. Theoretically, the error system (50) can stabilize after a long enough time by several proper feedback control designs, and thus the coupled systems (1) and (48) can be said exponentially synchronized.

Remark 6. Based on the Lyapunov functions method, many papers on the dynamic characteristics of neural networks have been well done. In fact, choosing a desired Lyapunov function is considerably difficult, and the procedure of proofing the stability/synchronization for neural networks is very complicated and cumbersome. Matrix measure approach allows us to express the Lyapunov function directly in terms of matrix measures of variables or error vectors. Therefore, the criteria had been obtained more concisely.

Remark 7. It should be noted that CVNN (1) with deviating argument has more complex dynamic behaviours by the accession of deviating function $\beta(t)$. Compared with traditional neural networks, such system can not only rely on the present state, but also the previous and subsequent states, whereas many published works can only consider the present one. As a matter of fact, the deviating argument can be used to denote the past and future messages, that is to say, our theoretical analysis and results are more complex and comprehensive.

Remark 8. By decomposing the considered CVNNs into real and imaginary parts and then combining them in matrix form, we obtain an equivalent $2n$-dimensional RVNNs. Based on the definitions of matrix norm and measure, we use the matrix measure method to analyze the dynamic changes of such RVNNs, which increases the computation but greatly reduces the difficulty of analysis.

4. Illustrative Examples

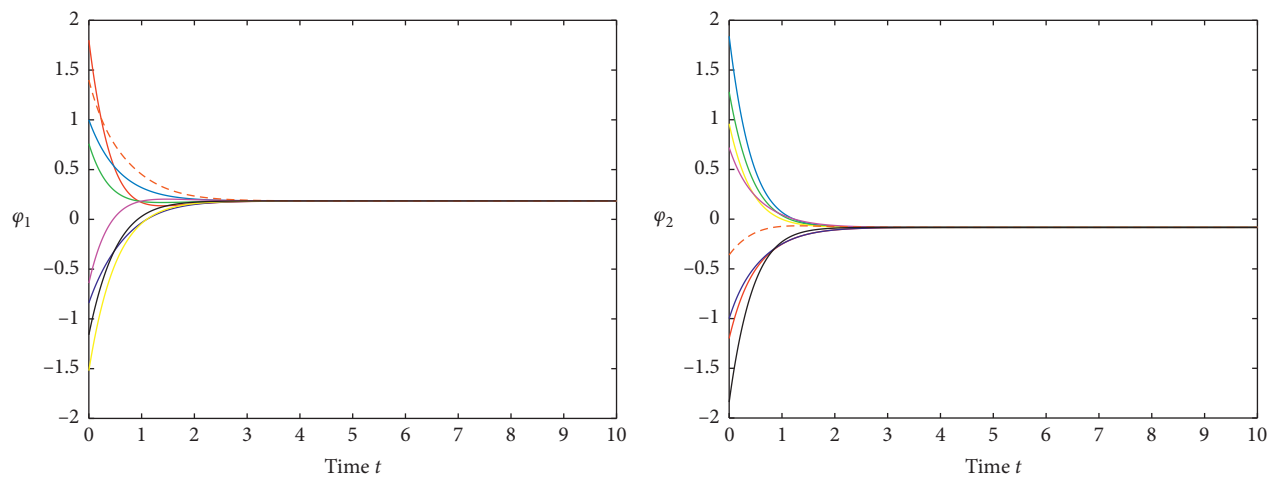
In order to verify the credibility of the derived main results, based on computer emulation, two examples are given in this section.

Example 1. We focus on a two-dimensional CVNN model described as follows:

$$\begin{aligned}z'(t) = &-\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}z_1(t)\\ z_2(t)\end{pmatrix} \\ &+\begin{pmatrix}-0.8+0.4i & 0.4+0.6i\\ 0.2+0.6i & -0.5-0.3i\end{pmatrix}\begin{pmatrix}f_1(z_1(t))\\ f_2(z_2(t))\end{pmatrix} \\ &+\begin{pmatrix}0.4-0.2i & 0.1+0.25i\\ -0.1-0.3i & 0.35+0.1i\end{pmatrix}\begin{pmatrix}g_1\left(\dfrac{z_1(\beta(t))}{2}\right)\\ g_2\left(\dfrac{z_2(\beta(t))}{2}\right)\end{pmatrix} \\ &+\begin{pmatrix}0.3-0.3i\\ -0.2-0.5i\end{pmatrix},\end{aligned} \tag{65}$$

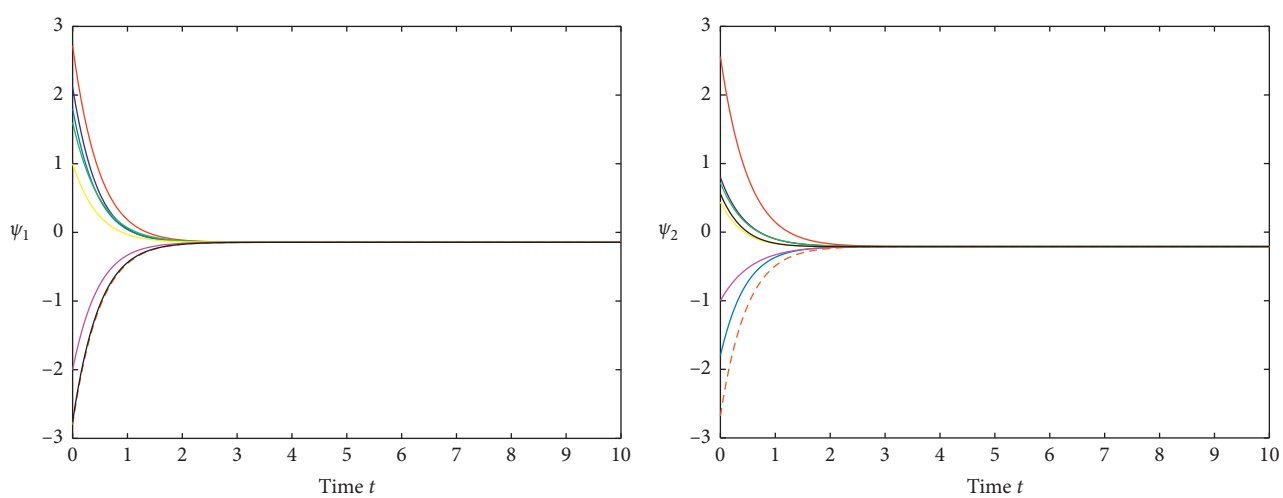
with two real-valued sequences $\{\delta_k\} = k/9$ and $\{\zeta_k\} = ((2k+1)/18)$, the deviating function $\beta(t) = \zeta_k$, if $t \in [\delta_k, \delta_{k+1})$, $t \in \mathscr{R}^+, k \in \mathscr{N}$, and for $z_r = \phi_r + i\psi_r$ with $\phi_r, \psi_r \in \mathscr{R}, r = 1, 2$, and the activation functions are as follows:

$$\begin{aligned}f_r^R(z_r) &= \frac{1}{10(r+2)}\sin(\phi_r), \\ f_r^I(z_r) &= -\frac{r}{10}\sin(\psi_r), \\ g_r^R(z_r) &= \frac{r}{8}\sin(\phi_r), \\ g_r^I(z_r) &= -\frac{r+1}{8}\sin(\psi_r).\end{aligned} \tag{66}$$



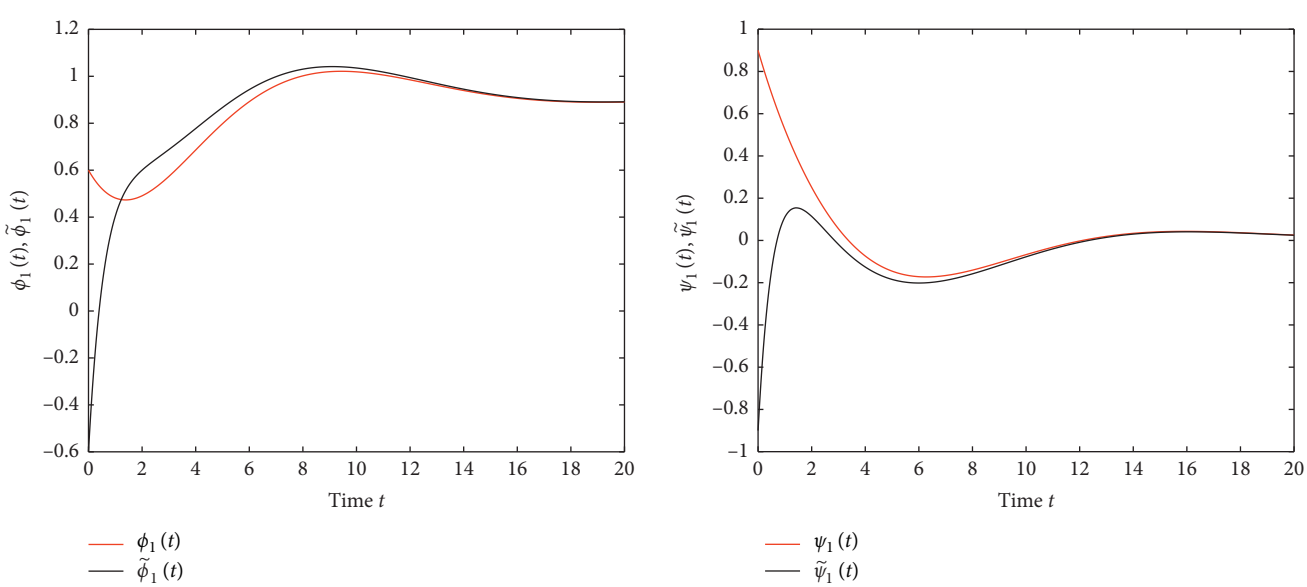
(a) (b)

Figure 1: Time response of the real parts $\phi(t)$ of the states $z(t)$ for (65).



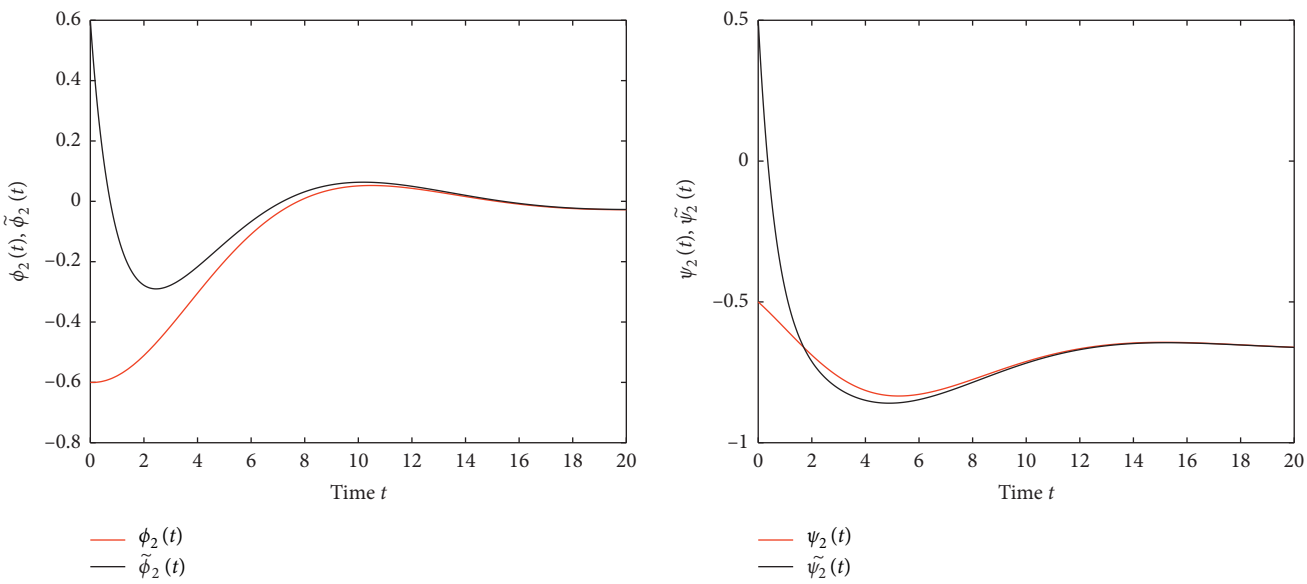
(a) (b)

Figure 2: Time response of the imaginary parts $\phi(t)$ of the states $z(t)$ for (65).



(a) (b)

Figure 3: Continued.



(c) (d)

Figure 3: Trajectories of the real parts and imaginary parts for master network (68) and slave network (69).

Then, we can easily get $\overline{A} = \mathrm{diag}\{2, 2, 2, 2\}$,

$$B_1 = \begin{pmatrix} -0.8 & 0.4 & 0 & 0 \\ 0.2 & -0.5 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.6 & -0.3 \end{pmatrix},$$
$$B_2 = \begin{pmatrix} -0.4 & -0.6 & 0 & 0 \\ -0.6 & 0.3 & 0 & 0 \\ 0 & 0 & -0.8 & 0.4 \\ 0 & 0 & 0.2 & -0.5 \end{pmatrix},$$
$$C_1 = \begin{pmatrix} 0.4 & 0.1 & 0 & 0 \\ -0.1 & 0.35 & 0 & 0 \\ 0 & 0 & -0.2 & 0.25 \\ 0 & 0 & -0.3 & 0.1 \end{pmatrix}, \quad (67)$$
$$C_2 = \begin{pmatrix} 0.2 & -0.25 & 0 & 0 \\ 0.3 & -0.1 & 0 & 0 \\ 0 & 0 & 0.4 & 0.1 \\ 0 & 0 & -0.1 & 0.35 \end{pmatrix}.$$

It is obvious that Assumption 1 holds with $s = (1/30), m = (1/5), q = (1/4)$, and $h = (3/8)$. Here we take $p = 1$, then $\|\overline{A}\|_1 = \|-\overline{A}\|_1 = 2$, $\mu_1(-\overline{A}) = -2$, $\|B_1\|_1 = \|B_2\|_1 = 1$, $\delta = (1/9)$, and $\|C_1\|_1 = \|C_2\|_1 = 0.5$. Then, $\lambda_1 = 2.47$, $\lambda_2 = 0.625$, and $\rho_1 = 1.85$.

Hence, we have $-[\mu_1(-\overline{A}) + 2s\|B_1\|_1 + 2m\|B_2\|_1 + \rho_1(2q\|C_1\|_1 + 2h\|C_2\|_1)] = 0.37 > 0$. According to Theorem 1, system (65) is globally exponentially stable.

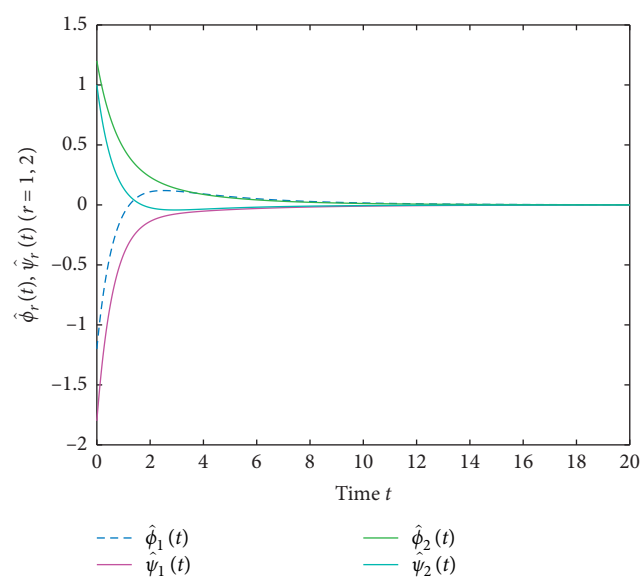


Figure 4: Trajectories of synchronization errors $\hat{z}_r(t) = \tilde{z}_r(t) - z_r(t)\,(r = 1, 2)$.

The evolutionary behaviors of the solutions for network (65) under the following eight initial states are described in Figures 1 and 2. Case 1: $z_1(t) = 1.8 + 2.72i$, $z_2(t) = -1.2 + 2.56i$; Case 2: $z_1(t) = -0.84 + 2.12i, z_2(t) = -1 + 0.8i$; Case 3: $z_1(t) = 1 + 1.8i, z_2(t) = 1.84 - 1.8i$; Case 4: $z_1(t) = 0.76 + 1.6i, z_2(t) = 1.28 + 0.72i$; Case 5: $z_1(t) = -1.52 + i, z_2(t) = 0.96 + 0.44i$; Case 6: $z_1(t) = -0.64 - 2i, z_2(t) = 0.72 - i$; Case 7: $z_1(t) = 1.4 - 2.8i$, $z_2(t) = -0.36 - 2.68i$; and Case 8: $z_1(t) = -1.16 - 2.76i$, $z_2(t) = -1.84 + 0.56i$.

Example 2. We focus on the following two-dimensional complex-valued master and slave networks with deviating argument.

$$\begin{aligned} z'(t) = &-\begin{pmatrix} 0.6 & 0 \\ 0 & 0.6 \end{pmatrix}\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \\ &+\begin{pmatrix} -0.1+0.7i & 0.2+0.45i \\ 0.15+0.55i & -0.25 \end{pmatrix}\begin{pmatrix} f_1(z_1(t)) \\ f_2(z_2(t)) \end{pmatrix} \\ &+\begin{pmatrix} -0.4+0.1i & 0.2+0.3i \\ 0.1+0.3i & -0.25-0.15i \end{pmatrix}\begin{pmatrix} g_1\left(\dfrac{z_1(\beta(t))}{2}\right) \\ g_2\left(\dfrac{z_2(\beta(t))}{2}\right) \end{pmatrix} \\ &+\begin{pmatrix} 0.2-0.5i \\ -0.1-0.4i \end{pmatrix}, \end{aligned} \tag{68}$$

$$\begin{aligned} \widetilde{z}'(t) = &-\begin{pmatrix} 0.6 & 0 \\ 0 & 0.6 \end{pmatrix}\begin{pmatrix} \widetilde{z}_1(t) \\ \widetilde{z}_2(t) \end{pmatrix} \\ &+\begin{pmatrix} -0.1+0.7i & 0.2+0.45i \\ 0.15+0.55i & -0.25 \end{pmatrix}\begin{pmatrix} f_1(\widetilde{z}_1(t)) \\ f_2(\widetilde{z}_2(t)) \end{pmatrix} \\ &+\begin{pmatrix} -0.4+0.1i & 0.2+0.3i \\ 0.1+0.3i & -0.25-0.15i \end{pmatrix}\begin{pmatrix} g_1\left(\dfrac{\widetilde{z}_1(\beta(t))}{2}\right) \\ g_2\left(\dfrac{\widetilde{z}_2(\beta(t))}{2}\right) \end{pmatrix} \\ &+\begin{pmatrix} 0.2-0.5i \\ -0.1-0.4i \end{pmatrix} + U(t), \end{aligned} \tag{69}$$

with two real-valued sequences $\{\delta_k\} = (k/10)$ and $\{\zeta_k\} = ((2k+1)/20)$, the deviating function $\beta(t) = \zeta_k$, if $t \in [\delta_k, \delta_{k+1})$, $t \in \mathscr{R}^+$, $k \in \mathscr{N}$, and control input is taken as $U(t) = -S\widehat{\phi}(t) - Q\widehat{\psi}(t) + i(-\mathbb{S}\widehat{\phi}(t) - \mathbb{Q}\widehat{\psi}(t))$, with $S = Q = \mathbb{S} = \mathbb{Q} = \mathrm{diag}\{1.2, 1.2\}$.

For $z_r = \phi_r + i\psi_r$ with $\phi_r, \psi_r \in \mathscr{R}$, the activation functions are as follows:

$$\begin{aligned} f_r^R(z_r) &= g_r^R(z_r) = \frac{r}{3}\tanh(\phi_r), \\ f_r^I(z_r) &= g_r^I(z_r) = \frac{r}{3}\tanh(\psi_r), \quad r = 1, 2. \end{aligned} \tag{70}$$

It is clear that Assumption 1 holds with $\overline{\mathrm{sm}} = \overline{\mathrm{qh}} = s = m = q = h = (2/3)$.

Here we take $p = 1$, and it is obvious that $\|\widetilde{A}\|_1 = \|-\widetilde{A}\|_1 = 3$, $\|B_3\|_1 = 1.5$, $\|C_3\|_1 = 0.9$, $\mu_1(-\overline{A}) = -3$, and $\delta = (1/10)$. Then, $\lambda_3 = 4$, $\lambda_4 = 0.6$, and $\rho_2 = 3.29$.

Hence, we have $-[\mu_1(-\widetilde{A}) + \overline{\mathrm{sm}}(\|B_3\|_1) + \rho_2\overline{\mathrm{qh}}(\|C_3\|_1)] = 0.026 > 0$. On the basis of Theorem 3, the exponential synchronism of master network (68) and slave network (69) can be derived. The trajectories of the solutions $\phi_r(t), \psi_r(t), \widetilde{\phi}_r(t), \widetilde{\psi}_r(t)\,(r = 1, 2)$ are demonstrated in Figures 3 and 4 depicts synchronization errors between $\widetilde{z}_r(t)$ and $z_r(t)\,(r = 1, 2)$.

5. Conclusion

The investigation of the dynamic behavior of CVNNs has attracted widespread considerations. However, based on the matrix measure approach, the dynamic analysis about CVNNs with deviating argument has scarcely been researched. We explore the stability/synchronization problem for CVNNs with deviating argument via matrix measure method. As a hybrid dynamical system, the CVNNs with deviating argument integrate the peculiarity of successive and discrete networks, and it has the prospective to represent the property of its inward principle. In addition, most of the research studies about the dynamic performance of neural networks are achieved by Lyapunov functions method. Under the characteristics of the matrix measure method, we do not need to structure any auxiliary functions, and under two nonlinear activation functions, some criteria for ascertaining the stability of CVNNs with deviating argument are obtained. Moreover, we achieved the global exponential synchronism between the master network and the slave network. Furthermore, we are considering extending this method to fractional-order networks with deviating argument to solve more practical problems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The work of the first author for master student Wenbo Zhou was done under the direction of Assistant Professor Jin-E Zhang. This work was supported by the Natural Science Foundation of China under Grants 61976084 and 61773152.

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