Research Article

Refinements of Some Integral Inequalities for (s, m) -Convex Functions

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In this paper, the refinements of integral inequalities for all those types of convex functions are given which can be obtained from (s, m) -convex functions. These inequalities not only provide refinements of bounds for unified integral operators but also for various associated fractional integral operators containing Mittag-Leffler function. At the same time, presented results give generalizations of many known fractional integral inequalities.

1. Introduction

The following fractional integral operator is the well-known Riemann–Liouville fractional integral operator.

Definition 1 (see [1]). Let $f \in L_1[a, b]$. Then, Riemann–Liouville fractional integrals of order μ where $\Re(\mu) > 0$ are defined as follows:

$$\begin{aligned} {}^u I_{a^+} f(x) &= \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t) dt, \quad x > a, \\ {}^u I_{b^-} f(x) &= \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t) dt, \quad x < b, \end{aligned} \quad (1)$$

where $\Gamma(\cdot)$ is the gamma function.

Next, generalizations of Riemann–Liouville fractional integral operators are given.

Definition 2 (see [2]). Let $f: [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on (a, b) , having a continuous derivative g' on (a, b) . The left-sided and the right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order μ where $\Re(\mu) > 0$ are defined by

$${}^u I_{g, a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (g(x) - g(t))^{\mu-1} g'(t) f(t) dt, \quad x > a, \quad (2)$$

$${}^u I_{g, b^-} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (g(t) - g(x))^{\mu-1} g'(t) f(t) dt, \quad x < b, \quad (3)$$

where $\Gamma(\cdot)$ is the gamma function.

A k -analogue of the above definition is given as follows.

Definition 3 (see [3]). Let $f: [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b) . The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order μ where $\Re(\mu), k > 0$ are defined by

$$\begin{aligned} {}^{\mu}I_{g^{-}b^{-}}^k f(x) &= \frac{1}{k\Gamma_k(\mu)} \int_a^x (g(x) - g(t))^{(\mu/k)-1} g'(t) f(t) dt, \quad x > a, \\ {}^{\mu}I_{g^{+}b^{+}}^k f(x) &= \frac{1}{k\Gamma_k(\mu)} \int_x^b (g(t) - g(x))^{(\mu/k)-1} g'(t) f(t) dt, \quad x < b, \end{aligned} \quad (4)$$

where $\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-(t^k/k)} dt$, $\Re(x) > 0$.

The following integral operator is given in [4].

Definition 4. Let $f, g: [a, b] \rightarrow \mathbb{R}$, $0 < a < b$, be the functions such that f be positive and $f \in L_1[a, b]$ and g be differentiable and strictly increasing. Also, let ϕ/x be an increasing function on $[a, \infty)$. Then, for $x \in [a, b]$, the left and right integral operators are defined by

$$\begin{aligned} (F_{a^{+}}^{\phi, g} f)(x) &= \int_a^x K_g(x, t; \phi) f(t) d(g(t)), \quad x > a, \\ (F_{b^{-}}^{\phi, g} f)(x) &= \int_x^b K_g(t, x; \phi) f(t) d(g(t)), \quad x < b, \end{aligned} \quad (5)$$

where $K_g(x, y; \phi) = ((\phi(g(x) - g(y)))/(g(x) - g(y)))$.

A fractional integral operator containing an extended generalized Mittag-Leffler function in its kernel is defined as follows.

Definition 5 (see [5]). Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, and $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a, b]$ and $x \in [a, b]$. Then, the generalized fractional integral operators $\epsilon_{\mu, \alpha, l, \omega, a^{+}}^{\gamma, \delta, k, c} f$ and $\epsilon_{\mu, \alpha, l, \omega, b^{-}}^{\gamma, \delta, k, c} f$ are defined by

$$(\epsilon_{\mu, \alpha, l, \omega, a^{+}}^{\gamma, \delta, k, c} f)(x; p) = \int_a^x (x - t)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x - t)^{\mu}; p) f(t) dt, \quad (6)$$

$$(\epsilon_{\mu, \alpha, l, \omega, b^{-}}^{\gamma, \delta, k, c} f)(x; p) = \int_x^b (t - x)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(t - x)^{\mu}; p) f(t) dt, \quad (7)$$

where

$$E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(t; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \quad (8)$$

is the extended generalized Mittag-Leffler function. For further study of the Mittag-Leffler function, see [6, 7]. $(c)_{nk}$ is the Pochhammer symbol defined by $(c)_{nk} = ((\Gamma(c + nk))/\Gamma(c))$, and β_p is the extended beta function given by

$$\beta_p(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-(p/(t(1-t)))} dt, \quad x, y, p \in \mathbb{R}_+. \quad (9)$$

The following identities for the constant function are obtained in [8] (see also [9]):

$$\begin{aligned} J_{\alpha, a^{+}}(x; p) &:= \left(\epsilon_{\mu, \alpha, l, \omega, a^{+}}^{\gamma, \delta, k, c} 1 \right)(x; p) = (x - a)^{\alpha} E_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega(x - a)^{\mu}; p), \\ J_{\beta, b^{-}}(x; p) &:= \left(\epsilon_{\mu, \beta, l, \omega, b^{-}}^{\gamma, \delta, k, c} 1 \right)(x; p) = (b - x)^{\beta} E_{\mu, \beta+1, l}^{\gamma, \delta, k, c}(\omega(b - x)^{\mu}; p). \end{aligned} \quad (10)$$

Recently, a unified integral operator is defined as follows.

Definition 6 (see [10]). Let $f, g: [a, b] \rightarrow \mathbb{R}$, $0 < a < b$, be the functions such that f be positive and $f \in L_1[a, b]$ and g be differentiable and strictly increasing. Also, let ϕ/x be an increasing function on $[a, \infty)$ and $\alpha, l, \gamma, c \in \mathbb{C}$, $p, \mu, \delta \geq 0$, and $0 < k \leq \delta + \mu$. Then, for $x \in [a, b]$, the left and the right integral operators are defined by

$$\left({}_g F_{\mu, \alpha, l, a^{+}}^{\phi, \gamma, \delta, k, c} f \right)(x, \omega; p) = \int_a^x K_x^{\gamma} \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}; g; \phi \right) f(y) d(g(y)), \quad (11)$$

$$\left({}_g F_{\mu, \alpha, l, b^{-}}^{\phi, \gamma, \delta, k, c} f \right)(x, \omega; p) = \int_x^b K_y^{\gamma} \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}; g; \phi \right) f(y) d(g(y)), \quad (12)$$

where the involved kernel is defined by

$$K_x^{\gamma} \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}; g; \phi \right) = \frac{\phi(g(x) - g(y))}{g(x) - g(y)} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(g(x) - g(y))^{\mu}; p). \quad (13)$$

The known fractional integrals studied in [2, 11–22] can be reproduced from the above definition, see [23], Remarks 6 and 7.

The aim of this study is to obtain the bounds of all known fractional integral operators defined in [2, 11–22] in a unified form for strongly (s, m) -convex functions. In the result, we get refinements of many known integral and fractional integral inequalities. Next, we recall definitions of convex, strongly convex, s -convex, m -convex, (s, m) -convex, and strongly (s, m) -convex functions.

Definition 7 (see [24]). A function $f: I \rightarrow \mathbb{R}$ is said to be a convex function if the inequality

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b), \quad (14)$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

The concept of a strongly convex function is defined as follows.

Definition 8 (see [25]). Let I be a nonempty convex subset of a normed space. A real-valued function f is said to be strongly convex with modulus $\lambda \geq 0$ on I if for each $a, b \in I$ and $t \in [0, 1]$, we have

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) - \lambda t(1-t)\|b - a\|^2. \quad (15)$$

A generalization of the convex function defined on the right half of the real line is called the s -convex function, and it is given as follows.

Definition 9 (see [26]). Let $s \in [0, 1]$. A function $f: [0, \infty) \rightarrow \mathbb{R}$ is said to be an s -convex function in the second sense if

$$f(tx + (1-t)b) \leq t^s f(a) + (1-t)^s f(b), \quad (16)$$

holds for all $a, b \in [0, \infty)$ and $t \in [0, 1]$.

The notion of the m -convex function and strongly m -convex function is defined as follows.

Definition 10 (see [27]). A function $f: [0, b] \rightarrow \mathbb{R}$ is said to be an m -convex function, where $m \in [0, 1]$ and $b > 0$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$, we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \quad (17)$$

Definition 11 (see [28]). A function $f: [0, +\infty) \rightarrow \mathbb{R}$ is said to be a strongly m -convex function with modulus λ if

$$f(tx + m(1-t)b) \leq tf(a) + m(1-t)f(b) - \lambda mt(1-t)(b-a)^2, \quad (18)$$

with $a, b \in [0, +\infty)$ and $m \in [0, 1]$.

A further generalized convexity is given as follows.

Definition 12 (see [29]). A function $f: [0, b] \rightarrow \mathbb{R}$ is said to be an (s, m) -convex function, where $(s, m) \in [0, 1]^2$ and $b > 0$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$, we have

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y). \quad (19)$$

The notion of the strongly (s, m) -convex function is defined as follows.

Definition 13 (see [30]). A function $f: [0, +\infty) \rightarrow \mathbb{R}$ is said to be a strongly (s, m) -convex function, with modulus $\lambda \geq 0$, for $(s, m) \in [0, 1]^2$, if

$$f(tx + m(1-t)b) \leq t^s f(a) + m(1-t)^s f(b) - \lambda t(1-t)|b-a|^2, \quad (20)$$

holds for all $a, b \in [0, +\infty)$ and $t \in [0, 1]$.

Using strongly (s, m) -convexity and utilizing fractional operators (6) and (7), some fractional integral inequalities are obtained as in [31]. The following result provides the bound of sum of left and right fractional integrals (6) and (7) for strongly (s, m) -convex functions at an arbitrary point.

Theorem 1 (see [31]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a real-valued function. If f is positive and strongly (s, m) -convex, then for $\alpha, \beta \geq 1$, the following fractional integral inequality holds:

$$\begin{aligned} & \left(\epsilon_{\mu, \alpha, l, \omega, a^+}^{\gamma, \delta, k, c} f \right)(x; p) + \left(\epsilon_{\mu, \beta, l, \omega, b^-}^{\gamma, \delta, k, c} f \right)(x; p) \leq \left(\frac{f(a) + mf(x/m)}{s+1} - \lambda \frac{(x-ma)^2}{6m^2} \right) \\ & (x-a) J_{\alpha-1, a^+}(x; p) + \left(\frac{f(b) + mf(x/m)}{s+1} - \lambda \frac{(mb-x)^2}{6m^2} \right) (b-x) \\ & J_{\beta-1, b^-}(x; p), \quad x \in [a, b]. \end{aligned} \quad (21)$$

The following Hadamard-type inequality holds for generalized fractional integral operators for strongly (s, m) -convex functions.

Theorem 2 (see [31]). Let $f: [a, b] \rightarrow \mathbb{R}$, $a > b$, be a real-valued function. If f is positive, strongly (s, m) -convex and $f((a+mb-x)/m) = f(x)$, then for $\alpha, \beta > 0$, the following fractional integral inequality holds:

$$\begin{aligned} & \frac{2^s}{1+m} \left(f\left(\frac{a+mb}{2}\right) (J_{\alpha+1, a^+}(b; p) + J_{\beta+1, b^-}(a; p)) + \frac{\lambda}{4m} ((b-a)^{\beta+2} J_{\beta+1, b^-}(a; p) \right. \\ & \left. - 2(1+m)(b-a)^{\beta+1} J_{\beta+2, b^-}(a; p) + 2(1+m)^2 J_{\beta+3, b^-}(a; p) + (b-a)^{\alpha+2} \right. \\ & \left. \times J_{\alpha+1, a^+}(b; p) - 2(1+m)(b-a)^{\alpha+1} J_{\alpha+2, a^+}(b; p) + 2(1+m)^2 J_{\alpha+3, a^+}(b; p) \right) \\ & \leq \left(\left(\epsilon_{\mu, \alpha+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right)(b; p) + \left(\epsilon_{\mu, \beta+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right)(a; p) \right) \\ & \leq [J_{\beta, b^-}(a; p) + J_{\alpha, a^+}(b; p)] (b-a) \left(\frac{f(b) + mf(a/m)}{s+1} - \lambda \frac{(mb-a)^2}{6m^2} \right). \end{aligned} \quad (22)$$

In the following, using the strongly (s, m) -convexity of $|f'|$, a modulus inequality is obtained.

Theorem 3 (see [31]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a real-valued function. If f is differentiable and $|f'|$ is strongly (s, m) -convex, then for $\alpha, \beta \geq 1$, the following fractional integral inequality holds:

$$\begin{aligned} & \left| \left(\epsilon_{\mu, \alpha+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x; p) + \left(\epsilon_{\mu, \beta+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x; p) - \left(J_{\alpha-1, a^+} (x; p) f(a) + J_{\beta-1, b^-} (x; p) f(b) \right) \right| \\ & \leq \left(\frac{|f'(a)| + m|f'(x/m)|}{s+1} - \lambda \frac{(x-ma)^2}{6m^2} \right) (x-a) J_{\alpha-1, a^+} (x; p) \\ & \quad + \left(\frac{|f'(b)| + m|f'(x/m)|}{s+1} - \lambda \frac{(mb-x)^2}{6m^2} \right) (b-x) J_{\beta-1, b^-} (x; p), \quad x \in [a, b]. \end{aligned} \quad (23)$$

In [32], we studied the properties of the kernel given in (13). Here, we are interested in the following property.

P: let g and ϕ/x be increasing functions. Then, for $x < t < y$, $x, y \in [a, b]$, the kernel $K_x^\gamma(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi)$ satisfies the following inequality:

$$K_t^x(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi) g'(t) \leq K_y^x(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi) g'(t). \quad (24)$$

This can be obtained from the following two straightforward inequalities:

$$\frac{\phi(g(t) - g(x))}{g(t) - g(x)} g'(t) \leq \frac{\phi(g(y) - g(x))}{g(y) - g(x)} g'(t),$$

$$E_{\mu, \alpha, l}^{\gamma, \delta, k, c} (\omega(g(t) - g(x))^\mu; p) \leq E_{\mu, \alpha, l}^{\gamma, \delta, k, c} (\omega(g(y) - g(x))^\mu; p). \quad (25)$$

The reverse of inequality (13) holds when g and ϕ/x are decreasing.

The upcoming section contains the results for unified integral operators dealing with the bounds of several

fractional integral operators in a compact form by utilizing strongly (s, m) -convex functions. A compact version of the Hadamard inequality is presented, and also a modulus inequality is given for the differentiable function f such that $|f'|$ is a strongly (s, m) -convex function. In the whole paper, we will use

$$I(a, b, g) = \frac{1}{b-a} \int_a^b g(t) dt. \quad (26)$$

2. Main Results

The following result provides the upper bound of unified integral operators.

Theorem 4. Let $f: [a, mb] \rightarrow \mathbb{R}$, $0 \leq a < mb$, be a positive integrable and strongly (s, m) -convex function, $m \neq 0$. Then, for unified integral operators (11) and (12), the following inequality holds:

$$\begin{aligned} & \left({}_g F_{\mu, \alpha, l, a^+}^{\phi, \gamma, \delta, k, c} f \right) (x, \omega; p) + \left({}_g F_{\mu, \beta, l, b^-}^{\phi, \gamma, \delta, k, c} f \right) (x, \omega; p) \leq K_x^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) \left(m f \left(\frac{x}{m} \right) g(x) - f(a) g(a) - \frac{\Gamma(s+1)}{(x-a)^s} \left(m f \left(\frac{x}{m} \right)^s I_{x^-} g(a) - f(a)^s I_{a^+} g(x) \right) \right. \\ & \quad + \frac{\lambda(x-ma)^2}{(x-a)} (2I(a, x, I_d g) - (a+x)I(a, x, g)) \Big) + K_b^x \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g; \phi \right) (f(b)g(b) \\ & \quad - m f \left(\frac{x}{m} \right) g(x) - \frac{\Gamma(s+1)}{(b-x)^s} (f(b)^s I_{b^-} g(x) - m f \left(\frac{x}{m} \right)^s I_{x^+} g(b)) \\ & \quad + \frac{\lambda(mb-x)^2}{(b-x)} (2I(x, b, I_d g) - (x+b)I(x, b, g)) \Big). \end{aligned} \quad (27)$$

Proof. By (P), the following inequalities hold:

$$K_x^t \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) g'(t) \leq K_x^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) g'(t), \quad a < t < x, \quad (28)$$

$$K_t^x \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) g'(t) \leq K_b^x \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g; \phi \right) g'(t), \quad x < t < b. \quad (29)$$

For a strongly (s, m) -convex function, the following inequalities hold for $a < t < x$ and $x < t < b$, respectively:

$$f(t) \leq \left(\frac{x-t}{x-a} \right)^s f(a) + m \left(\frac{t-a}{x-a} \right)^s f\left(\frac{x}{m}\right) - \frac{\lambda(x-t)(t-a)(x-ma)^2}{m^2(x-a)^2}, \quad (30)$$

$$f(t) \leq \left(\frac{t-x}{b-x} \right)^s f(b) + m \left(\frac{b-t}{b-x} \right)^s f\left(\frac{x}{m}\right) - \frac{\lambda(t-x)(b-t)(mb-x)^2}{m^2(b-x)^2}. \quad (31)$$

From (28) and (30), one can have

i.e.,

$$\begin{aligned} & \int_a^x K_x^t \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) f(t) d(g(t)) \leq f(a) K_x^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) \\ & \times \int_a^x \left(\frac{x-t}{x-a} \right)^s d(g(t)) + m f\left(\frac{x}{m}\right) K_x^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) \\ & \cdot \int_a^x \left(\frac{t-a}{x-a} \right)^s d(g(t)) \\ & - K_x^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) \frac{\lambda(x-ma)^2}{(x-a)^2} \int_a^x (x-t)(t-a) d(g(t)), \end{aligned} \quad (32)$$

$$\begin{aligned} & \left({}_g F_{\mu, \alpha, l, a^+}^{\phi, \gamma, \delta, k, c} f \right) (x, \omega; p) \leq K_x^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) \\ & \left(m f\left(\frac{x}{m}\right) g(x) - f(a) g(a) - \frac{\Gamma(s+1)}{(x-a)^s} \left(m f\left(\frac{x}{m}\right)^s I_{x^-} g(a) - f(a)^s I_{a^+} g(x) \right) \right. \\ & \left. + \frac{\lambda(x-ma)^2}{(x-a)} (2I(a, x, I_d g) - (a+x)I(a, x, g)) \right). \end{aligned} \quad (33)$$

On the other hand, from (29) and (31), one can have

$$\begin{aligned} & \int_x^b K_t^x \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g; \phi \right) f(t) d(g(t)) \leq f(b) K_b^x \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g; \phi \right) \\ & \times \int_x^b \left(\frac{t-x}{b-x} \right)^s d(g(t)) + m f\left(\frac{x}{m}\right) K_b^x \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g; \phi \right) \int_x^b \left(\frac{b-t}{b-x} \right)^s d(g(t)) \\ & - K_b^x \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g; \phi \right) \frac{\lambda(mb-x)^2}{m^2(b-x)^2} \int_x^b (t-x)(b-t) d(g(t)), \end{aligned} \quad (34)$$

i.e.,

$$\begin{aligned} \left({}_g F_{\mu, \beta, l, b^-}^{\phi, \gamma, \delta, k, c} f \right) (x, \omega; p) \leq & K_b^x \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g; \phi \right) \left(f(b)g(b) - mf\left(\frac{x}{m}\right)g(x) - \frac{\Gamma(s+1)}{(b-x)^s} \left(f(b)^s I_{b^-} g(x) - mf\left(\frac{x}{m}\right)^s I_{x^+} g(b) \right) \right. \\ & \left. + \frac{\lambda(mb-x)^2}{(b-x)} (2I(x, b, I_d g) - (x+b)I(x, b, g)) \right). \end{aligned} \quad (35)$$

By adding (33) and (35), (27) can be obtained. $\square$

Corollary 1. Setting $p = \omega = 0$ in (27), we can obtain the following inequality involving fractional integral operators defined in [4]:

$$\begin{aligned} \left(F_{\alpha, a^+}^{\phi} f \right) (x; p) + \left(F_{\beta, b^-}^{\phi} f \right) (x; p) \leq & K_g(a, x; \phi) \left(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a) - \frac{\Gamma(s+1)}{(x-a)^s} \left(mf\left(\frac{x}{m}\right)^s I_{x^-} g(a) - f(a)^s I_{a^+} g(x) \right) \right. \\ & \left. + \frac{\lambda(x-ma)^2}{(x-a)} (2I(a, x, I_d g) - (a+x)I(a, x, g)) \right) + K_g(x, b; \phi) (f(b)g(b) \\ & - mf\left(\frac{x}{m}\right)g(x) - \frac{\Gamma(s+1)}{(b-x)^s} (f(b)^s I_{b^-} g(x) - mf\left(\frac{x}{m}\right)^s I_{x^+} g(b)) \\ & \left. + \frac{\lambda(mb-x)^2}{(b-x)} (2I(x, b, I_d g) - (x+b)I(x, b, g)) \right). \end{aligned} \quad (36)$$

Remark 1

- (i) If we consider $\lambda = 0$ in (27), then Theorem 3.1 in [32] can be obtained, and for $\lambda > 0$, we get its refinement
- (ii) If we consider $\phi(t) = t^\alpha$ and $g(x) = x$ in (27), then Theorem 1 can be obtained
- (iii) If we consider $s = m = 1$ in the result of (ii), then Corollary 1 in [31] can be obtained
- (iv) If we consider $\alpha = \beta$ in the result of (ii), then Corollary 3 in [31] can be obtained
- (v) If we consider $f \in L_\infty[a, b]$ in the result of (ii), then Corollary 5 in [31] can be obtained
- (vi) If we consider $\alpha = \beta$ in the result of (v), then Corollary 7 in [31] can be obtained
- (vii) If we consider $s = 1$ in the result of (ii), then Corollary 5 in [31] can be obtained
- (viii) If we consider $(s, m) = (1, 1)$ in (27), then Theorem 2 in [33] is obtained
- (ix) If we consider $\alpha = \beta$, $\lambda = 0$, and $(s, m) = (1, 1)$ in (27), then Theorem 8 in [23] is obtained
- (x) If we consider $\lambda = 0$ and $p = \omega = 0$ in (27), then Theorem 1 in [34] is obtained
- (xi) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha)t^\alpha$, $p = \omega = 0$, and $(s, m) = (1, 1)$ in (27), then Theorem 1 in [35] is obtained

- (xii) If we consider $\alpha = \beta$ in the result of (xi), then Corollary 1 in [35] is obtained
- (xiii) If we consider $\lambda = 0$, $\phi(t) = t^\alpha$, $g(x) = x$, and $m = 1$ in (27), then Theorem 2.1 in [36] is obtained
- (xvi) If we consider $\alpha = \beta$ in the result of (xiii), then Corollary 2.1 in [36] is obtained
- (xv) If we consider $\lambda = 0$, $\phi(t) = ((\Gamma(\alpha) t^{(\alpha/k)})/(k\Gamma_k(\alpha)))$, $(s, m) = (1, 1)$, $g(x) = x$, and $p = \omega = 0$ in (27), then Theorem 1 in [37] can be obtained
- (xvi) If we consider $\alpha = \beta$ in the result of (xv), then Corollary 1 in [37] can be obtained
- (xvii) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha)t^\alpha$, $g(x) = x$, $p = \omega = 0$, and $(s, m) = (1, 1)$ in (27), then Theorem 1 in [38] is obtained
- (xviii) If we consider $\alpha = \beta$ in the result of (xvii), then Corollary 1 in [38] can be obtained
- (xviii) If we consider $\alpha = \beta = 1$ and $x = a$ or $x = b$ in the result of (xvii), then Corollary 2 in [38] can be obtained
- (xix) If we consider $\alpha = \beta = 1$ and $x = ((a+b)/2)$ in the result of (xvii), then Corollary 3 in [38] can be obtained

The following lemma is very helpful in the proof of the upcoming theorem, see [31].

Lemma 1. Let $f: [a, mb] \rightarrow \mathbb{R}$ be a strongly (s, m) -convex function, $0 \leq a < mb$. If f is $f((a + mb - x)/m) = f(x)$, $m \neq 0$, then the following inequality holds:

$$f\left(\frac{a + mb}{2}\right) \leq \frac{(1 + m)f(x)}{2^s} - \frac{\lambda}{4m}(a + mb - x - mx)^2. \quad (37)$$

In the literature, many mathematicians have established many types of Hadamard inequalities, and for their

generalizations, see [39–42]. This also motivates us to introduce the more generalized forms of Hadamard-type inequalities. So, by the help of the abovementioned lemma, the following result provides generalized Hadamard inequality for strongly (s, m) -convex functions.

Theorem 5. Under the assumptions of Theorem 4, in addition to $f(x) = f((a + mb - x)/m)$, the following inequality holds:

$$\begin{aligned} f\left(\frac{a + mb}{2}\right) \frac{2^s}{(1 + m)} & \left(\begin{aligned} & \left({}_g F_{\mu, \beta, l, a^+}^{\phi, \gamma, \delta, k, c} 1 \right) (b, \omega; p) + \frac{\lambda}{4m} \left({}_g F_{\mu, \beta, l, b^-}^{\phi, \gamma, \delta, k, c} (a + mb - x - mx)^2 \right) \\ & (a, \omega; p) + \left({}_g F_{\mu, \beta, l, b^-}^{\phi, \gamma, \delta, k, c} 1 \right) (a, \omega; p) + \frac{\lambda}{4m} \\ & \left({}_g F_{\mu, \beta, l, a^+}^{\phi, \gamma, \delta, k, c} (a + mb - x - mx)^2 \right) (b, \omega; p) \end{aligned} \right) \leq \left({}_g F_{\mu, \beta, l, a^+}^{\phi, \gamma, \delta, k, c} f \right) (b, \omega; p) + \left({}_g F_{\mu, \beta, l, b^-}^{\phi, \gamma, \delta, k, c} f \right) (a, \omega; p) \\ & \leq \left(K_b^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) + K_b^a \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g; \phi \right) \right) \\ & \quad \cdot \left(f(b)g(b) - mf\left(\frac{a}{m}\right)g(a) - \frac{\Gamma(s + 1)}{(b - a)^s} (f(b)^s I_{b^-} g(a) \right. \\ & \quad \left. - mf\left(\frac{a}{m}\right)^s I_{a^+} g(b) \right) + \frac{\lambda(mb - a)^2}{(b - a)} (2I(a, b, I_d g) \\ & \quad \left. - (a + b)I(a, b, g) \right). \end{aligned} \quad (38)$$

Proof. By (P), the following inequalities hold:

$$K_x^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) g'(x) \leq K_b^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) g'(x), \quad a < x < b, \quad (39)$$

$$K_b^x \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g; \phi \right) g'(x) \leq K_b^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) g'(x), \quad a < x < b. \quad (40)$$

A strongly (s, m) -convex function satisfying the following inequalities hold for $a < x < b$:

$$\begin{aligned} f(x) & \leq \left(\frac{x - a}{b - a} \right)^s f(b) + m \left(\frac{b - x}{b - a} \right)^s f\left(\frac{a}{m}\right) \\ & \quad - \frac{\lambda(b - x)(x - a)(b - ma)^2}{m^2(b - a)^2}. \end{aligned} \quad (41)$$

From (39) and (41), one can have

$$\begin{aligned} & \int_a^b K_x^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) f(x) d(g(x)) \\ & \leq mf\left(\frac{a}{m}\right) K_b^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) \int_a^b \left(\frac{b - x}{b - a} \right)^s d(g(x)) \\ & \quad + f(b) K_b^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) \int_a^b \left(\frac{x - a}{b - a} \right)^s d(g(x)) \\ & \quad - K_a^b \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) \\ & \quad \frac{\lambda(b - ma)^2}{m^2(b - a)^2} \int_a^b (x - a)(b - x) d(g(x)). \end{aligned} \quad (42)$$

Further, the aforementioned inequality takes the form which involves Riemann–Liouville fractional integrals in the right-hand side, and thus we have upper bound of the unified left-sided integral operator (2) as follows:

$$\begin{aligned} \left({}_g F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f\right)(a, \omega; p) &\leq K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi\right)\left(f(b)g(b) - mf\left(\frac{a}{m}\right)g(a) - \frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)^s I_{b^-}g(a) - mf\left(\frac{a}{m}\right)^s I_{a^+}g(b)\right)\right. \\ &\quad \left.+ \frac{\lambda(mb-a)^2}{(b-a)}(2I(a, b, I_d g) - (a+b)I(a, b, g))\right). \end{aligned} \quad (43)$$

On the other hand, from (39) and (41), the following inequality holds which involves Riemann–Liouville

fractional integrals on the right-hand side and gives the estimate of the integral operator (3):

$$\begin{aligned} \left({}_g F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f\right)(b, \omega; p) &\leq K_b^a\left(E_{\mu,\beta,l}^{\gamma,\delta,k,c}, g; \phi\right)\left(f(b)g(b) - mf\left(\frac{a}{m}\right)g(a) - \frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)^s I_{b^-}g(a) - mf\left(\frac{a}{m}\right)^s I_{a^+}g(b)\right)\right. \\ &\quad \left.+ \frac{\lambda(mb-a)^2}{(b-a)}(2I(a, b, I_d g) - (a+b)I(a, b, g))\right). \end{aligned} \quad (44)$$

By adding (43) and (44), the following inequality can be obtained:

$$\begin{aligned} \left({}_g F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f\right)(a, \omega; p) + \left({}_g F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f\right)(b, \omega; p) &\leq \left(K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi\right) + K_b^a\left(E_{\mu,\beta,l}^{\gamma,\delta,k,c}, g; \phi\right)\right)\left(f(b)g(b) - mf\left(\frac{a}{m}\right)g(a)\right. \\ &\quad \left.- \frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)^s I_{b^-}g(a) - mf\left(\frac{a}{m}\right)^s I_{a^+}g(b)\right) + \frac{\lambda(mb-a)^2}{(b-a)}(2I(a, b, I_d g) - (a+b)I(a, b, g))\right). \end{aligned} \quad (45)$$

Multiplying both sides of (37) by $K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)g'(x)$ and integrating over $[a, b]$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_a^b K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi\right) d(g(x)) \\ \leq \left(\frac{1}{2^s}\right)(1+m) \int_a^b K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi\right) f(x) d(g(x)) \\ - \frac{\lambda}{4m} \int_a^b K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi\right) (a+mb-x-mx)^2 d(g(x)). \end{aligned} \quad (46)$$

From Definition 6, the following inequality is obtained:

$$\begin{aligned} f\left(\frac{a+mb}{2}\right) \frac{2^s}{(1+m)} \left({}_g F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} 1\right)(a, \omega; p) \\ \leq \left({}_g F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f\right)(a, \omega; p) \\ - \frac{\lambda}{4m} \left({}_g F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} (a+mb-x-mx)^2\right)(a, \omega; p). \end{aligned} \quad (47)$$

Similarly, multiplying both sides of (37) by $K_b^x(E_{\mu,\beta,l}^{\gamma,\delta,k,c}, g; \phi)g'(x)$ and integrating over $[a, b]$, we have

$$\begin{aligned}
& f\left(\frac{a+mb}{2}\right) \frac{2^s}{(1+m)} \left({}_g F_{\mu,\beta,l,a^+}^{\phi,\gamma,\delta,k,c} 1\right)(b, \omega; p) \\
& \leq \left({}_g F_{\mu,\beta,l,a^+}^{\phi,\gamma,\delta,k,c} f\right)(b, \omega; p) \\
& - \frac{\lambda}{4m} \left({}_g F_{\mu,\beta,l,a^+}^{\phi,\gamma,\delta,k,c} (a+mb-x-mx)^2\right)(b, \omega; p).
\end{aligned} \tag{48}$$

By adding (47) and (48), the following inequality is obtained:

$$\begin{aligned}
& f\left(\frac{a+mb}{2}\right) \frac{2^s}{(1+m)} \left(\left({}_g F_{\mu,\beta,l,a^+}^{\phi,\gamma,\delta,k,c} 1\right)(b, \omega; p) + \frac{\lambda}{4m} \left({}_g F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} (a+mb-x-mx)^2\right)(a, \omega; p) + \left({}_g F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} 1\right)(a, \omega; p) \right. \\
& \left. + \frac{\lambda}{4m} \left({}_g F_{\mu,\beta,l,a^+}^{\phi,\gamma,\delta,k,c} (a+mb-x-mx)^2\right)(b, \omega; p) \right) \leq \left({}_g F_{\mu,\beta,l,a^+}^{\phi,\gamma,\delta,k,c} f\right)(b, \omega; p) + \left({}_g F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f\right)(a, \omega; p).
\end{aligned} \tag{49}$$

Using (45) and (49), inequality (38) can be obtained, which completes the proof. $\square$

Corollary 2. Setting $p = \omega = 0$ in (38), we can obtain the following inequality involving fractional integral operators defined in [4]:

$$\begin{aligned}
& f\left(\frac{a+mb}{2}\right) \frac{2^s}{(1+m)} \left(\begin{aligned} & \left(F_{\beta,a^+}^{\phi} 1\right)(b; p) + \frac{\lambda}{4m} \left(F_{\alpha,b^-}^{\phi} (a+mb-x-mx)^2\right) \\ & (a; p) + \left(F_{\alpha,b^-}^{\phi} 1\right)(a; p) + \frac{\lambda}{4m} \left(F_{\beta,a^+}^{\phi} (a+mb-x-mx)^2\right)(b; p) \end{aligned} \right) \\
& \leq \left(F_{\beta,a^+}^{\phi} f\right)(b; p) + \left(F_{\alpha,b^-}^{\phi} f\right)(a; p) \\
& \leq \left(K_g(a, b; \phi) + K_g(a, b; \phi)\right) \\
& \left(f(b)g(b) - mf\left(\frac{a}{m}\right)g(a) - \frac{\Gamma(s+1)}{(b-a)^s} \left(f(b)^s I_{b^-} g(a) - mf\left(\frac{a}{m}\right)^s I_{a^+} g(b) \right) \right. \\
& \left. + \frac{\lambda(mb-a)^2}{(b-a)} (2I(a, b, I_d g) - (a+b)I(a, b, g)) \right).
\end{aligned} \tag{50}$$

Remark 2

- (i) If we consider $\phi(t) = t^\alpha$ and $g(x) = x$ in (38), then Theorem 7 in [31] can be obtained
- (ii) If we consider $\lambda = 0$ in the result of (i), then Theorem 8 in [31] can be obtained
- (iii) If we consider $(s, m) = (1, 1)$ in (38), then Theorem 3 in [33] is obtained
- (iv) If we consider $\lambda = 0$ and $(s, m) = (1, 1)$ in (38), then Theorem 22 in [23] is obtained
- (v) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha)t^{\alpha+1}$, $p = \omega = 0$, and $(s, m) = (1, 1)$ in (38), then Theorem 3 in [35] is obtained
- (vi) If we consider $\alpha = \beta$ in the result of (v), then Corollary 3 in [35] is obtained

- (vii) If we consider $\lambda = 0$, $\phi(t) = t^{\alpha+1}$, $g(x) = x$, and $m = 1$ in (38), then Theorem 2.4 in [36] is obtained
- (viii) If we consider $\alpha = \beta$ in the result of (vii), then Corollary 2.6 in [36] is obtained
- (ix) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha)t^{(\alpha/k)+1}$, $(s, m) = (1, 1)$, $g(x) = x$, and $p = \omega = 0$ in (38), then Theorem 3 in [37] can be obtained
- (x) If we consider $\alpha = \beta$ in the result of (ix), then Corollary 6 in [37] can be obtained
- (xi) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha)t^{\alpha+1}$, $p = \omega = 0$, $(s, m) = 1$, and $g(x) = x$ in (38), then Theorem 3 in [38] can be obtained
- (xii) If we consider $\alpha = \beta$ in the result of (xi), then Corollary 6 in [38] can be obtained

Theorem 6. Let $f: [a, mb] \rightarrow \mathbb{R}$, $0 \leq a < mb$, be a differential function such that $|f'|$ is a strongly (s, m) -convex

function, $m \neq 0$. Then, for unified integral operators (11) and (12), the following inequality holds:

$$\begin{aligned} & \left| \left({}_g F_{\mu, \alpha, l, a^+}^{\phi, \gamma, \delta, k, c} f * g \right) (x, \omega; p) + \left({}_g F_{\mu, \beta, l, b^-}^{\phi, \gamma, \delta, k, c} f * g \right) (x, \omega; p) \right| \\ & \leq K_x^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) \left(\left(m \left| f' \left(\frac{x}{m} \right) \right| g(x) - |f'(a)| g(a) - \frac{\Gamma(s+1)}{(x-a)^s} \right) \right. \\ & \quad \left(m \left| f' \left(\frac{x}{m} \right) \right|^s I_{x^-} g(a) - |f'(a)|^s I_{a^+} g(x) \right) + \\ & \quad \left. \frac{\lambda(x-am)^2}{(x-a)} (2I(a, x, I_d g) - (a+x)I(a, x, g)) \right) \\ & \quad + K_b^x \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g; \phi \right) \left(\left(|f'(b)| g(b) - m \left| f' \left(\frac{x}{m} \right) \right| g(x) \right) - \frac{\Gamma(s+1)}{(b-x)^s} \right. \\ & \quad \left(|f'(b)|^s I_{b^-} g(x) - m \left| f' \left(\frac{x}{m} \right) \right|^s I_{x^+} g(b) \right) + \frac{\lambda(mb-x)^2}{(b-x)} \\ & \quad \left. (2I(x, b, I_d g) - (x+b)I(x, b, g)) \right) \end{aligned} \quad (51)$$

where

$$\begin{aligned} \left({}_g F_{\mu, \alpha, l, a^+}^{\phi, \gamma, \delta, k, c} f * g \right) (x, \omega; p) &:= \int_a^x K_x^t \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) f'(t) d(g(t)), \\ \left({}_g F_{\mu, \beta, l, b^-}^{\phi, \gamma, \delta, k, c} f * g \right) (x, \omega; p) &:= \int_x^b K_t^x \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) f'(t) d(g(t)). \end{aligned} \quad (52)$$

Proof. For a strongly (s, m) -convex function $|f'|$, the following inequalities hold for $a < t < x$ and $x < t < b$, respectively:

$$\begin{aligned} |f'(t)| &\leq \left(\frac{x-t}{x-a} \right)^s |f'(a)| + m \left(\frac{t-a}{x-a} \right)^s \left| f' \left(\frac{x}{m} \right) \right| \\ &\quad - \frac{\lambda(x-t)(t-a)(x-ma)^2}{m^2(x-a)^2}, \end{aligned} \quad (53)$$

$$\begin{aligned} |f'(t)| &\leq \left(\frac{t-x}{b-x} \right)^s |f'(b)| + m \left(\frac{b-t}{b-x} \right)^s \left| f' \left(\frac{x}{m} \right) \right| \\ &\quad - \frac{\lambda(t-x)(b-t)(mb-x)^2}{m^2(b-x)^2}. \end{aligned} \quad (54)$$

From (28) and (54), the following inequality is obtained:

$$\begin{aligned} & \left| \left({}_g F_{\mu, \alpha, l, a^+}^{\phi, \gamma, \delta, k, c} (f * g) \right) (x, \omega; p) \right| \leq K_x^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi \right) \\ & \left(\left(m \left| f' \left(\frac{x}{m} \right) \right| g(x) - |f'(a)| g(a) - \frac{\Gamma(s+1)}{(x-a)^s} \right) \right. \\ & \quad \left(m \left| f' \left(\frac{x}{m} \right) \right|^s I_{x^-} g(a) - |f'(a)|^s I_{a^+} g(x) \right) + \\ & \quad \left. \frac{\lambda(x-am)^2}{(x-a)} (2I(a, x, I_d g) - (a+x)I(a, x, g)) \right) \end{aligned} \quad (55)$$

Similarly, from (29) and (55), the following inequality is obtained:

$$\begin{aligned}
& \left| \left({}_g F_{\mu, \beta, l, b^-}^{\phi, \gamma, \delta, k, c} (f * g) \right) (x, \omega; p) \right| \leq K_x^a \left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g; \phi \right) \\
& \cdot \left(\left| f'(b) \right| g(b) - m \left| f' \left(\frac{x}{m} \right) \right| g(x) \right) - \frac{\Gamma(s+1)}{(b-x)^s} - \left(\left| f'(b) \right|^s I_{b^-} g(x) - m \left| f' \left(\frac{x}{m} \right) \right|^s I_{x^+} g(b) \right) \\
& + \frac{\lambda(mb-x)^2}{(b-x)} (2I(x, b, I_d g) - (x+b)I(x, b, g)).
\end{aligned} \tag{56}$$

By adding (56) and (57), inequality (52) can be achieved. $\square$

Corollary 3. Setting $p = \omega = 0$ in (52), we can obtain the following inequality involving fractional integral operators defined in [4]:

$$\begin{aligned}
& \left| \left(F_{\alpha, a^+}^{\phi} f * g \right) (x, p) + \left(F_{\beta, b^-}^{\phi} f * g \right) (x, p) \right| \leq K_g(a, x; \phi) \\
& \left(\begin{aligned} & \left(m \left| f' \left(\frac{x}{m} \right) \right| g(x) - \left| f'(a) \right| g(a) \right) - \frac{\Gamma(s+1)}{(x-a)^s} \\ & \left(m \left| f' \left(\frac{x}{m} \right) \right|^s I_{x^-} g(a) - \left| f'(a) \right|^s I_{a^+} g(x) \right) \\ & + \frac{\lambda(x-am)^2}{(x-a)} (2I(a, x, I_d g) - (a+x)I(a, x, g)) \end{aligned} \right) \\
& + K_b(x, b; \phi) \left(\begin{aligned} & \left(\left| f'(b) \right| g(b) - m \left| f' \left(\frac{x}{m} \right) \right| g(x) \right) - \frac{\Gamma(s+1)}{(b-x)^s} \\ & \left(\left| f'(b) \right|^s I_{b^-} g(x) - m \left| f' \left(\frac{x}{m} \right) \right|^s I_{x^+} g(b) \right) + \\ & \frac{\lambda(mb-x)^2}{(b-x)} (2I(x, b, I_d g) - (x+b)I(x, b, g)) \end{aligned} \right).
\end{aligned} \tag{57}$$

Remark 3

- (i) If we consider $\lambda = 0$ in (52), then Theorem 3.4 in [32] can be obtained
- (ii) If we consider $\phi(t) = t^\alpha$ and $g(x) = x$ in (52), then Theorem 6 in [31] can be obtained
- (iii) If we consider $s = m = 1$ in the result of (ii), then Corollary 13 in [31] can be obtained
- (iv) If we consider $\alpha = \beta$ in the result of (ii), then Corollary 11 in [31] can be obtained
- (v) If we consider $(s, m) = (1, 1)$ in (52), then Theorem 3 in [33] is obtained
- (vi) If we consider $\lambda = 0$ and $(s, m) = (1, 1)$ in (52), then Theorem 25 in [23] is obtained
- (vii) If we consider $\lambda = 0$ and $p = \omega = 0$ in (52), then Theorem 2 in [34] is obtained
- (viii) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha)t^{\alpha+1}$, $p = \omega = 0$, and $(s, m) = (1, 1)$ in (52), then Theorem 2 in [35] is obtained

- (ix) If we consider $\alpha = \beta$ in the result of (viii), then Corollary 2 in [35] is obtained
- (x) If we consider $\lambda = 0$, $\phi(t) = t^\alpha$, $g(x) = x$, and $m = 1$ in (52), then Theorem 2.3 in [36] is obtained
- (xi) If we consider $\alpha = \beta$ in the result of (x), then Corollary 2.5 in [36] is obtained
- (xii) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha)t^{\alpha/k+1}$, $(s, m) = (1, 1)$, $g(x) = x$, and $p = \omega = 0$ in (52), then Theorem 2 in [37] can be obtained
- (xiii) If we consider $\alpha = \beta$ in the result of (xii), then Corollary 4 in [37] can be obtained
- (xiv) If we consider $\alpha = \beta = k = 1$ and $x = ((a+b)/2)$ in the result of (xii), then Corollary 5 in [37] can be obtained
- (xv) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha)t^{\alpha+1}$, $g(x) = x$, $p = \omega = 0$, and $(s, m) = (1, 1)$ in (52), then Theorem 2 in [38] is obtained
- (xvi) If we consider $\alpha = \beta$ in the result of (xv), then Corollary 5 in [38] can be obtained

3. Concluding Remarks

In this paper, bounds of a unified integral operator for strongly (s, m) -convex functions are studied. The compact form of these bounds lead to further interesting consequences with respect to fractional integrals of various kinds for convex, (s, m) -convex, m -convex, s -convex, and convex functions. These findings are generalized in nature and give the refinements of many inequalities for unified and fractional integral operators via different types of convex functions.

Data Availability

No data are required for this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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