A New Algorithm for Solving Large-Scale Generalized Eigenvalue Problem Based on Projection Methods

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1. Introduction

Many problems arising in different fields of science and engineering can be reduced to the generalized eigenvalue problem [1–3]:

\[ Ax = \lambda Bx, \]  

(1)

where \( A, B \) are \( n \times n \) real or complex, large, sparse, and only a few of the eigenvalues are desired. Also, when is \( B = I \) (identity matrix), we have a standard eigenvalue problem. Computing eigenpairs \((\lambda, x)\) of the generalized and standard eigenvalue problems is one of the important problems in many scientific applications [4–7]. There are several methods for solving such eigenvalue problems [8]. Among these methods, the iterative methods are used to generate a subspace that contains the desired eigenvectors. Techniques based on the Krylov subspaces are powerful tools for building desired subspaces for large-scale eigenvalue problems [9–11].

Expressed methods in this article find all of the zeros that lie in a circle using numerical integration. In this paper, we briefly describe moment-based method in Section 2, Rayleigh–Ritz with contour integral method in Section 3, block version of the Sakurai–Sugiura method in Section 4, and block version of the SS method with Rayleigh–Ritz procedure in Section 5 for solving generalized eigenvalue problem (1). In Section 6, we provide four numerical tests for comparing four methods, and in Section 6, we apply the BSSRR method with selected matrices from different application areas, and finally, we draw some conclusions in Section 7.

2. Moment-Based Method (SS Method)

For solving (1), we consider computing entire poles of a rational function:

\[ f(z) = u^H (zB - A)^{-1} v, \quad v, u \in \mathbb{C}^n / \{0\}. \]  

(2)

Those are eigenvalues \( \lambda \) of equation (1) and lie in a circle using numerical integration. Let \( \Gamma \) be positively oriented closed Jordan curve [12] in the complex plane and \( \lambda_1, \ldots, \lambda_n \) be distinct eigenvalues that lie in the \( \Gamma \). Let

\[ \mu_k = \frac{1}{2\pi i} \int_{\Gamma} (z - \gamma)^{k-1} f(z)dz, \quad k = 0, 1, \ldots, \]  

(3)

where \( \gamma \) is located inside \( \Gamma \) and the \( m \times m \) Hankel matrices \( H_m, H_m^\infty \) be \( H_m = [\mu_{i+j-1}]_{i,j=1}^m \) and \( H_m^\infty = [\mu_{i+j-1}]_{i,j=1}^m \). Also, let

\[ H_m, H_m^\infty \]
\[ s_k = \frac{1}{2\pi i} \int_{\Gamma} (z - \gamma)^k (zB - A)^{-1} dz, \quad k = 0, 1, \ldots \]  

Then, we have the following theorem.

**Theorem 1.** If \( \nu_j = 0 \) for \( 1 \leq j \leq m \), then the eigenvalues of the pencil \( H_m^\lambda - \lambda H_m \) are given by \( \lambda_1 - \gamma, \ldots, \lambda_m - \gamma \).

**Proof.** In [13], by approximating the integral of equation (3) via the N-point trapezoidal rule, we obtain

\[ \mu_k = \tilde{\mu}_k = \frac{1}{N} \sum_{j=0}^{N-1} (\omega_j - \gamma)^{k+1} f(\omega_j), \quad k = 0, 1, \ldots \]  

Let \( \tilde{\xi}_1, \ldots, \tilde{\xi}_m \) be the eigenvalues of pencil \( H_m^\lambda - \lambda H_m \). We regard \( \tilde{\lambda}_j = \gamma + \tilde{\xi}_j, 1 \leq j \leq m \) as the approximations for \( \lambda_1, \ldots, \lambda_m \), and

\[ (\omega_j B - A)\tilde{y}_j = \nu_j, \quad j = 0, 1, \ldots, N - 1, \]  

\[ \tilde{s}_k = \frac{1}{N} \sum_{j=0}^{N-1} (\omega_j - \gamma)^{k+1} \tilde{y}_j, \quad k = 0, 1, \ldots \]  

Also, let \( \tilde{V}_m \) be the Vandermonde matrix for \( \tilde{\xi}_1, \ldots, \tilde{\xi}_m \). Then, the approximations for the eigenvectors are obtained by

\[ [\tilde{q}_1, \ldots, \tilde{q}_m] = [\tilde{s}_0, \ldots, \tilde{s}_{m-1}]^T \tilde{V}_m. \]  

**3. Rayleigh–Ritz with Contour Integral Method (CIRR Method)**

We consider (1), let \( A, B \in \mathbb{R}^{n \times n} \) be symmetric and let \( B \) be positive definite and \( (\lambda_j, x_j), 1 \leq j \leq n \) be eigenpairs of the matrix pencil \((A, B)\). We apply a Rayleigh–Ritz procedure with an orthonormal basis \( Q \in \mathbb{R}^{n \times n} \). The projected matrices are given by \( A = QT^T AQ \) and \( B = QT^T BQ \). \( Q \in \mathbb{R}^{n \times n} \) is used to generate a sequence of subspace containing approximations to the desired eigenvector. The Ritz values of the projected pencil \((\tilde{A}, t\tilde{B})\) are taken as approximate eigenvalues for original pencil \((A, B)\) with corresponding Ritz vectors. In this method, by applying the Rayleigh–Ritz procedure moments are not explicitly used [17]. The algorithm is as follows.

Rayleigh–Ritz procedure

1. Construct an orthonormal basis \( Q \)
2. Form \( \tilde{A} = QT^T AQ \) and \( \tilde{B} = QT^T BQ \)
3. Compute the eigenpairs \((\theta_j, \omega_j), 1 \leq j \leq m\) of \((\tilde{A}, \tilde{B})\)
4. Set \( p_j = Qw_j, j = 1, \ldots, M \)

**Theorem 2.** Let \( s_k \) be defined by (4). Suppose that \( \nu \) is expanded by the eigenvectors \( \{x_1, \ldots, x_n\} \) as

\[ \nu = \sum_{j=1}^{n} \alpha_j x_j. \]  

Then,

\[ s_k = \sum_{j=1}^{m} \alpha_j (\lambda_j - \gamma)^k x_j, \quad j = 0, 1, \ldots, m - 1. \]  

**Proof.** It follows from (4) and (8) that

\[ s_k = \frac{1}{2\pi i} \int_{\Gamma} \sum_{j=1}^{n} \alpha_j (z - \gamma)^k (zB - A)^{-1} Bx_j dz. \]  

Since \((\lambda_j, x_j)\) is an eigenpair of the matrix pencil \((A, B)\), we have \((zB - A)x_j = (z - \lambda_j)Bx_j \) and thus \((zB - A)^{-1} Bx_j = (z - \lambda_j)^{-1} x_j \), and thus

\[ s_k = \frac{1}{2\pi i} \int_{\Gamma} \sum_{j=1}^{n} \alpha_j (z - \gamma)^k (zB - A)^{-1} Bx_j dz, \quad k = 0, 1, \ldots, m - 1. \]  

By the residue theorem, we obtain the result.
We define the \( m \times m \) Vandermonde matrix with \( \lambda_1 - \gamma, \ldots, \lambda_m - \gamma \) by

\[ V = \begin{bmatrix} 1 & (\lambda_1 - \gamma) & \cdots & (\lambda_1 - \gamma)^{m-1} \\ 1 & (\lambda_2 - \gamma) & \cdots & (\lambda_2 - \gamma)^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (\lambda_m - \gamma) & \cdots & (\lambda_m - \gamma)^{m-1} \end{bmatrix}. \]  

From the equation (9), we have

\[ S = XDV, \]  

where \( S = [s_0, \ldots, s_{m-1}] \), \( X = [x_1, \ldots, x_m] \), and \( D = \text{diag}(\alpha_1, \ldots, \alpha_m) \).

**Theorem 3.** If \( \lambda_1, \ldots, \lambda_m \) are distinct and \( \alpha_j \neq 0 \) for \( 1 \leq j \leq m \), then

\[ \text{span}\{q_1, \ldots, q_m\} = \text{span}\{x_1, \ldots, x_m\}. \]  

**Proof.** Since \( \lambda_1, \ldots, \lambda_m \) are mutually distinct and \( \alpha_j \neq 0 \) for \( 1 \leq j \leq m \). \( V \) and \( D \) are nonsingular. Therefore, it follows from (13) that

\[ \text{span}\{s_0, \ldots, s_{m-1}\} = \text{span}\{x_1, \ldots, x_m\}. \]  

Since the vectors \( \{q_1, \ldots, q_m\} \) are orthonormal basis of \( \text{span}\{s_0, \ldots, s_{m-1}\} \), equation (14) holds.

For nonzero vector \( \nu \in \mathbb{R}^n \), we define the moments:

\[ \mu_k = \frac{1}{2\pi i} \int_{\Gamma} (z - \gamma)^k (Bv)^T (zB - A)^{-1} Bvdz, \quad k = 0, 1, \ldots \]  

where \( \gamma \) is located inside \( \Gamma \). Also, we obtain the following approximations via the N-point trapezoidal rule:
\[ \mu_k = \tilde{\mu}_k = \frac{1}{N} \sum_{j=0}^{N-1} (\omega_j - \gamma)^{k+1} (Bv) \left( \frac{(\omega_jB - A)}{k+1} Bv \right), \quad k = 0, 1, \ldots, \]

\[ \tilde{s}_k = \frac{1}{N} \sum_{j=0}^{N-1} (\omega_j - \gamma)^{k+1} (\omega_jB - A)^{-1} Bv, \quad k = 0, 1, \ldots. \]

**4. Block Sakurai–Sugiura Method (BSS Method)**

In this method, for solving (1), we reformulate the SS method in the context of the resolvent theory. This method has the potential to resolve degenerated eigenvalues.

**Theorem 4.** Let \( zB - A \) be a regular pencil of order \( N \). Then, there exist nonsingular matrices \( \tilde{P}, \tilde{Q} \in \mathbb{C}^{N \times N} \) such that

\[ \tilde{P}(zB - A)\tilde{Q} = \begin{bmatrix} zI_k - I_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & zI_k - I_d \end{bmatrix}, \]

where \( I_1, I_d \in \mathbb{C}^{k \times k} \) are Jordan blocks, \( N_i \) is nilpotent, and \( I_k \) denotes the identity matrix of order \( k \).

**Proof.** In [12].

Here, because \( \tilde{P}, \tilde{Q} \) are the regular matrices, we can define \( P = \tilde{P} \) and \( Q = \tilde{Q}^{-1} \). According to (18), we will partition row vectors in \( \tilde{P} \) and \( \tilde{Q} \) into \( P, Q \in \mathbb{C}^{N \times N} \), and column vectors in \( P \) and \( Q \) into \( P_i, Q_i \in \mathbb{C}^{N \times k} \), respectively, for \( i = 1, \ldots, r \).

**Theorem 5.** Resolvent of the regular pencil \((zB - A)^{-1}\) is decomposed into

\[ (zB - A)^{-1} = \sum_{i=1}^{d} Q_i (zI_k - I_i)^{-1} \tilde{P}_i + \sum_{i=d+1}^{r} Q_i (zN_i - I_k)^{-1} \tilde{P}_i, \]

where \( \alpha_i \) is an eigenvalue of Jordan block \( I_i \).

**Proof.** Let \( W = \tilde{P}(zB - A)\tilde{Q} \). According to Theorem 4, we have

\[ (zB - A)^{-1} = QW^{-1} \tilde{P} = \sum_{i=1}^{d} Q_i (zI_k - I_i)^{-1} \tilde{P}_i + \sum_{i=d+1}^{r} Q_i (zN_i - I_k)^{-1} \tilde{P}_i. \]

Using the resolvent of the Jordan block,

\[ R(z, I_i) = (zI_k - I_i)^{-1} = \sum_{m=0}^{k-1} \left( \frac{z}{\alpha_i} \right)^m I_i, \]

and \( zN_i - I_k \), we get the result.

**Theorem 6.** The localized moment matrix is written as

\[ M_n = \sum_{i, \alpha_i \in G} Q_i \tilde{Q}_i. \]

**Proof.** In [18].

**Definition 1.** Let \( C \) and \( D \) be arbitrary \( N \times m \) matrices, where \( N > m \geq k_f \). A size-reduced moment matrix is defined as

\[ M_n = C^H M_n D \in \mathbb{C}^{m \times m}. \]

**Theorem 7.** If ranks of both \( C^H Q_i \) and \( \tilde{Q}_i D \) are \( k_f \), nonsingular part of a matrix pencil \( zM_0 - M_i \) is equivalent to \( zI_{k_f} - J_f \).

**Proof.** In [18].

**Theorem 8.** The right eigenvectors of the original matrix pencil \( zB - A \) are given by \( Q_f = M_0 D Q_f \), and its adjoint is given by \( \tilde{Q}_f = \tilde{P}_f C^H M_0 \).

**Proof.** In [18].

**Theorem 9.** If elements of \( \tilde{Q}_f^v \) and \( v^H Q_f \) are nonzero, and there is no degeneracy in \( J_f \), then nonsingular part of a matrix pencil \( zH_m - H_m^x \) is equivalent to \( zI_{k_x} - J_f \).

**Proof.** By choosing row vectors of \( C^H \) and vectors of \( D \) to be

\[ \left( C^H \right)_{i, s} = v^H J_f^{-1} Q_f^v, \]

and \( D_{i, s} = Q_f H_m^{-1} \tilde{Q}_f^v \) for \( i = 1, 2, \ldots, m \), respectively, we have \( H = M_0 \) and \( H_m = M_1 \). As for the rank of \( \tilde{Q}_f D \), we consider that column vectors of \( \tilde{Q}_f D \) from the Krylov series of \( J_f \) starting from \( \tilde{Q}_f^v \). Because \( I_f \) has not degenerated, and elements of \( \tilde{Q}_f^v \) are nonzero, these column vectors are linearly independent, and thus the rank of \( \tilde{Q}_f D \) is \( f k_f \).

**5. Block Version of the SS Method with the Rayleigh–Ritz Procedure (BSSRR Method)**

We suggest a new algorithm for computing all poles of analytic function (2) with the use of the algorithm in [19]. As the eigenpairs \( (\lambda_i, x_i) \) of equation (1) can be obtained from \( H^{(1)} \) \( u_0 = \theta_i H^{(2)} u_0 \), where \( H^{(1)} \) and \( H^{(2)} \) are small \( M \times M \) Hankel matrices. Let \( V \in \mathbb{C}^{n \times M} \), a random matrix, and \( S = [S_0, S_1, \ldots, S_{M-1}] \in \mathbb{C}^{n \times LM} \), where
Theorem 10. Let \( \delta_m \) be the subspace of the block version of the SS method with the Rayleigh–Ritz procedure defined by (26). Then, we have
\[
\delta_m = \kappa^*_m(S_T, P_T V).
\]

Proof. From the definition of \( S_k \) (25) and Lemma 1, we have \( S_k = S_k^T P_T V \). Therefore,

\[
S_k = [s_k^{(1)}, \ldots, s_k^{(L)}] = \frac{1}{2\pi i} \int_\Gamma (z - \gamma)^i (zB - A)^{-1} Bdz, \quad k = 0, 1, \ldots, M - 1.
\]

(25)

Then, the block version of the SS method with Rayleigh–Ritz procedure [20] constructs the LM-dimensional subspace:
\[
\delta_M = \text{span}\{ S \} = \text{span}\{ s_0^{(1)}, s_0^{(L)}, s_1^{(1)}, s_1^{(L)}, \ldots, s_{M-1}^{(1)}, s_{M-1}^{(L)} \}.
\]

(26)

For the Rayleigh–Ritz procedure, the subspace \( \delta_M \) contains all eigenvectors of (1) \( \delta_M = \text{span}\{ x_1, x_2, \ldots, x_m \} \) for \( m \leq LM \). With using N-point trapezoidal rule for equation (25), we have
\[
S_k \approx \tilde{S}_k = \sum_{j=1}^N \omega_j (z_j - \gamma)(z_jB - A)^{-1} BV.
\]

(27)

where \( z_j \) is the quadrature point and \( \omega_j \) is the corresponding weight [21].

Based on Theorem 4, we analyse the relationship between the contour integral spectral projection and the Krylov subspace.

\[
\sum_{i: \lambda \in \Omega} Q_i(J_i - \gamma I)^\ell \tilde{Q}_i = \left( \sum_{i: \lambda \in \Omega} Q_i(J_i - \gamma I) \tilde{Q}_i \right)^\ell = \left( \sum_{i: \lambda \in \Omega} Q_i \tilde{Q}_i \right)^\ell = S_k^\ell P_T, \quad \ell = 0, 1, \ldots
\]

(31)

\[
\delta_M = \text{span}\{ [S_0, S_1, \ldots, S_k] \}
\]

\[
= \text{span}\{ [P_T V, S_T P_T V, \ldots, S_{M-1} P_T V] \} = \kappa^*_M(S_T, P_T V).
\]

(35)

Therefore, Theorem 10 is proved. \( \square \)

5.1. An Arnoldi-Based Interpretation of the Contour Integral Spectral Projection. Since the matrices \( P, Q \) are nonsingular, we define \( P = P^{-1}, Q = Q^{-1} \). According to the Jordan block structure of (18), we partition row vectors in \( P, Q \) into \( P_i, Q_i \in \mathbb{C}^{n \times n} \) and column vectors in \( P, Q \) into \( P_i, Q_i \in \mathbb{C}^{n \times n} \) respectively, for \( i = 1, 2, \ldots, r \). Then, we can derive the following lemma and theorem.

Lemma 1. Let \( \psi_k(z) \) be a \( k \)-degree polynomial. Then, we have
\[
\frac{1}{2\pi i} \int_\Gamma (z - \gamma)(zB - A)^{-1} Bdz = \psi_k(S_T) P_T,
\]

(28)

where
\[
S_T = \sum_{i: \lambda_i \in \Omega} Q_i(J_i - \gamma I)\tilde{Q}_i, \quad P_T = \sum_{i: \lambda_i \in \Omega} Q_i\tilde{Q}_i.
\]

(29)

Proof. From Theorem 6 and the binomial theorem
\[
(z - \gamma)^\ell = \sum_{\ell = 0}^{\ell} \binom{\ell}{j} z^{\ell-j} \gamma^j, \quad \ell = 0, 1, \ldots
\]

(30)

Here, since \( \tilde{Q}_i \tilde{Q}_j = I \) and \( \tilde{Q}_i Q_j = 0 \) for \( i \neq j \),

\[
\delta_M = \text{span}\{ [S_0, S_1, \ldots, S_k] \}
\]

\[
= \text{span}\{ [P_T V, S_T P_T V, \ldots, S_{M-1} P_T V] \} = \kappa^*_M(S_T, P_T V).
\]

(35)

Therefore, Theorem 10 is proved. \( \square \)

Remark 1. Theorem 10 shows that the block version of the SS method with the Rayleigh–Ritz procedure can be regarded as the Rayleigh–Ritz procedure based on the block Krylov subspace \( \kappa^*_M(S_T, P_T V) \). Here, we note that, in the block version of the SS method with the Rayleigh–Ritz procedure, the basis vectors of \( \kappa^*_M(S_T, P_T V) \) are explicitly computed by (25) and the QR decomposition of \( S \) (Algorithm 1).

And \( (\lambda_i, x_i) = (\theta_i, W u_i) \) for \( i = 1, 2, \ldots, LM \).

6. Numerical Experiments

In this section, we have provided five numerical examples. In Examples 1–4, we have discussed stability Algorithm 2 (SS method), Algorithm 3 (CIRR method), Algorithm 4 (BSS method), and Algorithm 1 (BSSRR method), and in Example 5, we have applied the BSS method with selected matrices in the fields of engineering sciences. Also, we have used
Input: $L, M, N \in \mathbb{N}, V \in \mathbb{C}^{n \times l}, y, (z_j, \omega_j)$ for $j = 1, 2, \ldots, N$
Output: eigenpairs $(\lambda_i, x_i)$ for $i = 1, 2, \ldots, LM$
(1) Solve $(z, B - A)Y_j = BV$ for $j = 1, 2, \ldots, N$.[14–16]
(2) Compute $X_k = \sum_{j=1}^{N} \omega_j (z_j - \gamma)^j Y_j$ for $k = 0, 1, \ldots, M - 1$
(3) Compute QR decomposition of $S = [S_0, S_1, \ldots, S_{M-1}]; S = WR$
(4) Compute eigenpairs $(\theta_j, u_j)$ of the matrix pencil $(W^H A W, W^H B W)$

Algorithm 1: BSSRR method.

Input: $u, v \in \mathbb{C}^{n \times l}, n, m, \gamma, \rho$
Output: $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m, \tilde{q}_1, \ldots, \tilde{q}_m$
(1) Set $\omega_j \leftarrow y + \rho (2\pi j / N)$, $j = 0, 1, \ldots, N - 1$
(2) Solve $(\omega, B - A)Y_j = V$ for $y_j$, $j = 0, 1, \ldots, N - 1$, [14–16]
(3) Set $\mu_j = -\omega_j^2 / (2\rho)$
(4) Compute $\mu_k$, $k = 0, \ldots, 2m - 1$ by (5)
(5) Compute the eigenvalues $\xi_1, \ldots, \xi_m$ of the pencil $\tilde{H}_m^\omega - \lambda \tilde{H}_m$
(6) Compute $\tilde{q}_1, \ldots, \tilde{q}_m$ by (7)
(7) Set $\tilde{\lambda}_j \leftarrow y + \xi_j$, $j = 1, \ldots, m$

Algorithm 2: SS method.

Input: $v \in \mathbb{R}^{n \times l}, N, M, \gamma, \rho$
Output: $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m, \tilde{\xi}_1, \ldots, \tilde{\xi}_m$
(1) Set $\omega_j \leftarrow \gamma + \rho (2\pi j / N)$, $j = 0, 1, \ldots, N - 1$
(2) Solve $(\omega, B - A)Y_j = V$ for $y_j$, $j = 0, 1, \ldots, N - 1$, [14–16]
(3) Compute $\tilde{S}_k$, $k = 0, 1, \ldots, M - 1$ by (17)
(4) Construct an orthonormal basis $Q$ from $[\tilde{S}_0, \ldots, \tilde{S}_{M-1}]
(5) Form \tilde{A} = Q^T A Q$ and $\tilde{B} = Q^T B Q$
(6) Compute the eigenpairs $(\theta_j, \omega_j)$ ($1 \leq j \leq M$) of $(\tilde{A}, t \tilde{B})$
(7) Set $p_j = Q \omega_j$, $j = 1, \ldots, M$
(8) Select the approximate eigenpairs $(\tilde{\lambda}_1, \tilde{\xi}_1), \ldots, (\tilde{\lambda}_m, \tilde{\xi}_m)$ from $(\theta_j, p_j)$ ($1 \leq j \leq M$

Algorithm 3: CIRR method.

Input: $V \in \mathbb{C}^{n \times l}, [z_j, \omega_j]$ for $j = 1, 2, \ldots, M$
Output: $\alpha_k$ for $k = 1, 2, \ldots, K$
(1) Solve $(z, B - A) \tilde{V}_j = BV$ [14–16] and calculate $V_j = V^H \tilde{V}_j \in \mathbb{C}^{n \times l}$
(2) Compute $\gamma_j = \sum_{j=1}^{M} \omega_j z_j^* V_j$
(3) Construct Hankel matrices $H_m$ and $H_m^\omega \in \mathbb{C}^{m \times m \times m \times m}$
(4) Perform singular value decomposition, $H_m = W u U$
(5) Construct $H = s^{-1/2} W^H H_m^\omega U^H s^{-1/2} \in \mathbb{C}^{k \times k}$
(6) Compute eigenvalues of $H$ to have $\alpha_k$
(7) Compute $\tilde{S}_m = \sum_{j=1}^{M} \omega_j z_j^* \tilde{V}_j$
(8) Compute $(Q_1, \ldots, Q_k) = (\tilde{S}_0, \ldots, \tilde{S}_{M-1}) U^H s^{-1/2} (q_1, \ldots, q_k)$

Algorithm 4: BSS method.

$A x_j + \lambda_j B x_j / \Delta x_j + \Delta x_j (\lambda_j |Bx_j|$) to compute relative residual for all of the methods. In computational results tables, the number of eigenpairs has been named NE.

Example 1. A real symmetric matrix $A \in \mathbb{R}^{n \times n}$ was prepared, which has five primary eigenvalues $-12.03, -12.02, -12.01, -12.00, -11.99$. In the range of $[-12.5, -11.5]$, other eigenvalues were taken randomly in the range of $[-40, 40]$, and a random unitary matrix was prepared to construct $A$. An identity matrix was used for $B$. After applying Algorithms 1–4, we obtained numerical results that have been shown in Table 1.
Table 1: The minimum relative residual of Algorithms 1–4 in Example 1.

<table>
<thead>
<tr>
<th>N.A, N.B</th>
<th>m = 16, N = 32</th>
<th>M = 18, N = 32</th>
<th>m = 16</th>
<th>M = 32, m = 4, l = 4</th>
<th>Threshold = 1.1 × 10^{-18}</th>
<th>M = 4, L = 4, N = 32</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS</td>
<td>NE</td>
<td>CIRR</td>
<td>NE</td>
<td>BSS</td>
<td>NE</td>
<td>BSSRR</td>
</tr>
<tr>
<td>n = 200</td>
<td>0.82114</td>
<td>16</td>
<td>8.857E−008</td>
<td>16</td>
<td>1.990E−008</td>
<td>16</td>
</tr>
<tr>
<td>n = 400</td>
<td>0.7919</td>
<td>16</td>
<td>3.2359E−06</td>
<td>16</td>
<td>5.1353E−09</td>
<td>16</td>
</tr>
<tr>
<td>n = 600</td>
<td>0.8393</td>
<td>16</td>
<td>3.2571E−05</td>
<td>16</td>
<td>1.7353E−07</td>
<td>16</td>
</tr>
<tr>
<td>n = 800</td>
<td>0.8665</td>
<td>16</td>
<td>8.8640E−05</td>
<td>16</td>
<td>2.6448E−05</td>
<td>16</td>
</tr>
<tr>
<td>n = 1000</td>
<td>0.8508</td>
<td>16</td>
<td>7.7542E−05</td>
<td>16</td>
<td>2.7825E−05</td>
<td>16</td>
</tr>
<tr>
<td>n = 1500</td>
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<td>16</td>
<td>5.6523E−05</td>
<td>16</td>
<td>8.7373E−05</td>
<td>16</td>
</tr>
</tbody>
</table>

Table 2: The minimum relative residual of Algorithms 1–4 in Example 2.

<table>
<thead>
<tr>
<th>N.A, N.B</th>
<th>m = 16, N = 32</th>
<th>M = 18, N = 32</th>
<th>m = 16</th>
<th>M = 32, m = 4, l = 4</th>
<th>Threshold = 1.1 × 10^{-18}</th>
<th>M = 4, L = 4, N = 32</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS</td>
<td>NE</td>
<td>CIRR</td>
<td>NE</td>
<td>BSS</td>
<td>NE</td>
<td>BSSRR</td>
</tr>
<tr>
<td>n = 200</td>
<td>0.9307</td>
<td>16</td>
<td>—</td>
<td>9.919E−05</td>
<td>16</td>
<td>6.1079E−16</td>
</tr>
<tr>
<td>n = 400</td>
<td>0.89555</td>
<td>16</td>
<td>—</td>
<td>1.3066E−04</td>
<td>16</td>
<td>8.3165E−16</td>
</tr>
<tr>
<td>n = 600</td>
<td>0.8826</td>
<td>16</td>
<td>—</td>
<td>6.6731E−04</td>
<td>16</td>
<td>9.0355E−16</td>
</tr>
<tr>
<td>n = 800</td>
<td>0.8802</td>
<td>16</td>
<td>—</td>
<td>1.600E−004</td>
<td>16</td>
<td>8.6891E−16</td>
</tr>
<tr>
<td>n = 1000</td>
<td>0.8749</td>
<td>16</td>
<td>—</td>
<td>3.2000E−004</td>
<td>16</td>
<td>1.2295E−15</td>
</tr>
<tr>
<td>n = 1500</td>
<td>0.8432</td>
<td>16</td>
<td>—</td>
<td>1.6400E−004</td>
<td>16</td>
<td>1.9151E−15</td>
</tr>
</tbody>
</table>

Figure 1: Results of comparing the relative residual of Algorithms 1–4 in Example 2 with n = 1000.

Table 3: The minimum relative residual of Algorithm 1–4 and in Example 3.

<table>
<thead>
<tr>
<th>N.A, N.B</th>
<th>m = 16, N = 64</th>
<th>M = 18, N = 64</th>
<th>m = 16</th>
<th>M = 64, m = 4, l = 4</th>
<th>Threshold = 1.1 × 10^{-18}</th>
<th>M = 4, L = 4, N = 64</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS</td>
<td>NE</td>
<td>CIRR</td>
<td>NE</td>
<td>BSS</td>
<td>NE</td>
<td>BSSRR</td>
</tr>
<tr>
<td>n = 200</td>
<td>0.6600</td>
<td>16</td>
<td>5.7595E−13</td>
<td>16</td>
<td>3.4279E−11</td>
<td>8</td>
</tr>
<tr>
<td>n = 400</td>
<td>0.6028</td>
<td>16</td>
<td>7.2639E−11</td>
<td>16</td>
<td>6.1907E−11</td>
<td>10</td>
</tr>
<tr>
<td>n = 600</td>
<td>0.6509</td>
<td>16</td>
<td>3.4114E−11</td>
<td>16</td>
<td>3.0774E−09</td>
<td>11</td>
</tr>
<tr>
<td>n = 800</td>
<td>0.5966</td>
<td>16</td>
<td>6.2826E−11</td>
<td>16</td>
<td>3.0774E−08</td>
<td>12</td>
</tr>
<tr>
<td>n = 1000</td>
<td>0.6065</td>
<td>16</td>
<td>6.8571E−10</td>
<td>16</td>
<td>9.7887E−08</td>
<td>12</td>
</tr>
<tr>
<td>n = 1500</td>
<td>0.6737</td>
<td>16</td>
<td>3.6078E−12</td>
<td>16</td>
<td>7.9782E−08</td>
<td>11</td>
</tr>
</tbody>
</table>
Figure 2: Results of comparing the relative residual of Algorithms 1–4 in Example 3 with $n = 1000$.

Table 4: The minimum relative residual of Algorithms 1–4 in Example 4.

<table>
<thead>
<tr>
<th>$\gamma = 0, \rho = 0.15$</th>
<th>$M = 18$, $N = 32$, $m = 16$</th>
<th>$M = 32$, $m = 4$, $I = 4$, Threshold = $1.1 \times 10^{-14}$</th>
<th>$M = 4$, $L = 4$, $N = 32$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N.A, N.B</td>
<td>SS</td>
<td>NE</td>
<td>CIRR</td>
</tr>
<tr>
<td>$n = 200$</td>
<td>0.7389</td>
<td>16</td>
<td>0.0035</td>
</tr>
<tr>
<td>$n = 400$</td>
<td>0.7350</td>
<td>16</td>
<td>0.0057</td>
</tr>
<tr>
<td>$n = 600$</td>
<td>0.7370</td>
<td>16</td>
<td>0.0049</td>
</tr>
<tr>
<td>$n = 800$</td>
<td>0.7389</td>
<td>16</td>
<td>0.0068</td>
</tr>
<tr>
<td>$n = 1000$</td>
<td>0.7240</td>
<td>16</td>
<td>0.0069</td>
</tr>
<tr>
<td>$n = 1500$</td>
<td>0.7393</td>
<td>16</td>
<td>0.0065</td>
</tr>
</tbody>
</table>

Figure 3: Results of comparing the relative residual of Algorithms 1–4 in Example 4 with $n = 1000$. 
Example 2. We let that $A$, $B$ were complex, random matrices, and $B$ was positive definite. After applying Algorithms 1–4, we obtained numerical results that have been shown in Table 2. Also, the relative residual for described methods has been drawn in Figure 1 for $n = 1000$.

Example 3. In this example, $A$, $B$ were taken sparse, symmetric, and random, and $A$ was positive definite. After applying Algorithms 1–4, we obtained numerical results that have been shown in Table 3. Also, the relative residual for described methods has been drawn in Figure 2 for $n = 1000$.

Example 4. We consider matrices:

$$A = I_n,$$

$$B = \begin{bmatrix}
-7 & -3 & -1 & -1 & -1 \\
-3 & -8 & -3 & -1 & -1 \\
-1 & -3 & -8 & -3 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & -3 & -8 & -3 \\
-1 & -1 & -1 & -1 & -7 \\
\end{bmatrix},$$

(36)

After applying Algorithms 1–4, we obtained numerical results that have been shown in Table 4. Also, the relative residual for described methods has been drawn in Figure 3 for $n = 1000$.

Table 5: Application areas, names, and properties 17 matrices selected.

<table>
<thead>
<tr>
<th>Application</th>
<th>Name</th>
<th>N</th>
<th>NZ</th>
<th>CON</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized eigenvalue problem</td>
<td>lund_a</td>
<td>147</td>
<td>2449</td>
<td>5.4430E + 006</td>
</tr>
<tr>
<td>Generalized eigenvalue problem</td>
<td>lund_b</td>
<td>147</td>
<td>2449</td>
<td>6.0317E + 004</td>
</tr>
<tr>
<td>Chemical kinetics</td>
<td>fs_183_1</td>
<td>183</td>
<td>998</td>
<td>1.5122E + 013</td>
</tr>
<tr>
<td>Square dielectric wave guide</td>
<td>dw256B</td>
<td>512</td>
<td>2500</td>
<td>4.047</td>
</tr>
<tr>
<td>Chemical kinetics</td>
<td>fs_760_1</td>
<td>760</td>
<td>5739</td>
<td>1.1234E + 016</td>
</tr>
<tr>
<td>Large helicopter model</td>
<td>rotor2</td>
<td>791</td>
<td>10685</td>
<td>1.2651E + 013</td>
</tr>
<tr>
<td>Dynamic analysis in structural engineering</td>
<td>bcsstk19</td>
<td>817</td>
<td>6853</td>
<td>2.8130E + 011</td>
</tr>
<tr>
<td>Unsymmetric basis from LP problem</td>
<td>bcsstk12</td>
<td>1473</td>
<td>34241</td>
<td>5.2502E + 008</td>
</tr>
<tr>
<td>Power systems simulation</td>
<td>qh882</td>
<td>822</td>
<td>4841</td>
<td>8.6511E + 006</td>
</tr>
<tr>
<td>Oil reservoir simulation</td>
<td>Sherman1</td>
<td>1000</td>
<td>3750</td>
<td>2.2575E + 004</td>
</tr>
<tr>
<td>Aeroelasticity</td>
<td>tols1090</td>
<td>1090</td>
<td>3546</td>
<td>2.1164E + 006</td>
</tr>
<tr>
<td>Alfvén spectra in magneto hydrodynamic</td>
<td>mdh1280</td>
<td>1280</td>
<td>47906</td>
<td>9.9827E + 004</td>
</tr>
<tr>
<td>Nuclear reactor models</td>
<td>nnc261</td>
<td>1374</td>
<td>8588</td>
<td>4.1082E + 015</td>
</tr>
<tr>
<td>Dynamic analysis in structural engineering</td>
<td>bcsstk13</td>
<td>2003</td>
<td>83883</td>
<td>4.5698E + 010</td>
</tr>
<tr>
<td>Dynamic analysis in structural engineering</td>
<td>bcsstk12</td>
<td>147</td>
<td>34241</td>
<td>5.2502E + 008</td>
</tr>
<tr>
<td>Dynamic analysis in structural engineering</td>
<td>bcsstk13</td>
<td>2003</td>
<td>83883</td>
<td>4.5698E + 010</td>
</tr>
</tbody>
</table>

Table 6: Calculation relative residual 2-norm of the BSSRR method.

<table>
<thead>
<tr>
<th>Name matrix</th>
<th>NE</th>
<th>Minimum residual</th>
<th>Maximum residual</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>lund_a</td>
<td>16</td>
<td>1.2115E − 015</td>
<td>3.2651E − 015</td>
<td>0.4351</td>
</tr>
<tr>
<td>lund_b</td>
<td>16</td>
<td>7.7260E − 015</td>
<td>4.9013E − 015</td>
<td>0.4034</td>
</tr>
<tr>
<td>fs_183_1</td>
<td>16</td>
<td>9.4890E − 016</td>
<td>7.6258E − 010</td>
<td>0.6897</td>
</tr>
<tr>
<td>dw256B</td>
<td>16</td>
<td>3.0480E − 015</td>
<td>1.13825E − 013</td>
<td>37.9230</td>
</tr>
<tr>
<td>fs_760_1</td>
<td>16</td>
<td>1.9950E − 015</td>
<td>1.1748E − 014</td>
<td>114.1690</td>
</tr>
<tr>
<td>rotor2</td>
<td>16</td>
<td>2.9661E − 015</td>
<td>7.6861E − 011</td>
<td>73.6634</td>
</tr>
<tr>
<td>bcsstk19</td>
<td>16</td>
<td>1.7704E − 015</td>
<td>4.3297E − 015</td>
<td>80.9064</td>
</tr>
<tr>
<td>bp_1600</td>
<td>16</td>
<td>4.1415E − 015</td>
<td>8.8643E − 015</td>
<td>118.4747</td>
</tr>
<tr>
<td>qh882</td>
<td>16</td>
<td>8.3663E − 011</td>
<td>8.1529E − 010</td>
<td>91.1155</td>
</tr>
<tr>
<td>Sherman1</td>
<td>16</td>
<td>2.0279E − 015</td>
<td>6.7531E − 015</td>
<td>145.8976</td>
</tr>
<tr>
<td>tols1090</td>
<td>16</td>
<td>6.3521E − 15</td>
<td>1.0472E − 12</td>
<td>174.7583</td>
</tr>
<tr>
<td>mdh1280</td>
<td>16</td>
<td>9.1528E − 015</td>
<td>1.1804E − 012</td>
<td>362.0046</td>
</tr>
<tr>
<td>nnc261</td>
<td>16</td>
<td>2.4955E − 015</td>
<td>1.387E − 014</td>
<td>362.3879</td>
</tr>
<tr>
<td>bcsstk12</td>
<td>16</td>
<td>1.7482E − 015</td>
<td>4.9099E − 015</td>
<td>546.7578</td>
</tr>
<tr>
<td>utm1700</td>
<td>16</td>
<td>5.0652E − 015</td>
<td>2.6436E − 014</td>
<td>960.8218</td>
</tr>
<tr>
<td>Plat1919</td>
<td>16</td>
<td>3.1237E − 015</td>
<td>1.2013E − 013</td>
<td>1191.317</td>
</tr>
<tr>
<td>bcsstk13</td>
<td>16</td>
<td>2.8611E − 015</td>
<td>5.3720E − 014</td>
<td>1363.4479</td>
</tr>
</tbody>
</table>
Some general conclusions are given as follows: the numerical results presented in the previous section. Several specific conclusions were drawn in connection with symptoms parameters. 7. General Conclusions and Plans for Future Work

Several specific conclusions were drawn in connection with the numerical results presented in the previous section. Some general conclusions are given as follows:

1. All numerical experiments indicate that CIRR, BSS, and BSSRR methods have higher stability than respect to the SS method
2. BSSRR method has less relative residual respect to SS, CIRR, and BSS methods
3. If $\gamma$ is used for calculation of relative residual in the BSSRR method, then we have higher accuracy and less consuming time

Example 5. In this example, we selected seventeen matrices from the UF sparse matrix collection. Two major requirements were used in the selection procedure: matrices with different parameters and matrices arising in different application areas were chosen. We consider the following symptoms parameters.

The order $N$, the number of nonzero elements $NZ$, and the condition number $CON$. The application areas of the selected matrices are listed in Table 5. Matrices from the different areas were selected and thus obtained results by running the matrices will be typical in several scientific fields. We applied the BSSRR method for the calculation of relative residual generalized eigenvalue problem (1) when $A$ is one of the selected matrices in Table 5 and $B$ is identity matrix. As the dimension $A$, $B$ is equal. We computed relative residual respect to $\gamma$. In Table 6 and computed respect to other norms in Table 7, too, the number of eigenpairs for each matrix was sixteen in Table 7.

### Table 7: Calculation minimum relative residual of the BSSRR method with other norms.

<table>
<thead>
<tr>
<th>Name matrix</th>
<th>$|\cdot|_1$</th>
<th>Time</th>
<th>$|\cdot|_\infty$</th>
<th>Time</th>
<th>$|\cdot|_{\infty}$</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>lund_a</td>
<td>1.6280E-15</td>
<td>0.4641</td>
<td>1.464E-015</td>
<td>0.4751</td>
<td>1.0893E-015</td>
<td>0.4429</td>
</tr>
<tr>
<td>lund_b</td>
<td>1.2158E-015</td>
<td>0.4329</td>
<td>1.1523E-015</td>
<td>0.4289</td>
<td>9.5626E-016</td>
<td>0.4587</td>
</tr>
<tr>
<td>fs_183_1</td>
<td>1.3389E-015</td>
<td>0.9160</td>
<td>1.9050E-015</td>
<td>0.9926</td>
<td>5.7331E-015</td>
<td>0.7920</td>
</tr>
<tr>
<td>fs_760_1</td>
<td>1.4451E-014</td>
<td>115.7364</td>
<td>5.8661E-015</td>
<td>116.8876</td>
<td>9.4203E-015</td>
<td>116.0084</td>
</tr>
<tr>
<td>rotor2</td>
<td>8.5274E-015</td>
<td>74.3794</td>
<td>3.6814E-015</td>
<td>79.6705</td>
<td>1.5234E-015</td>
<td>75.7969</td>
</tr>
<tr>
<td>bcsstk19</td>
<td>4.6544E-015</td>
<td>83.6374</td>
<td>1.1299E-015</td>
<td>84.1706</td>
<td>1.6893E-015</td>
<td>83.6942</td>
</tr>
<tr>
<td>bcsstk17</td>
<td>2.7482E-010</td>
<td>98.5387</td>
<td>1.8575E-010</td>
<td>93.9466</td>
<td>1.0929E-010</td>
<td>94.3459</td>
</tr>
<tr>
<td>tols1090</td>
<td>6.4706E-014</td>
<td>203.0137</td>
<td>7.8958E-14</td>
<td>186.5922</td>
<td>8.0820E-14</td>
<td>175.2136</td>
</tr>
<tr>
<td>mdh1280</td>
<td>1.1013E-014</td>
<td>357.0195</td>
<td>3.8742E-015</td>
<td>363.0094</td>
<td>6.2742E-015</td>
<td>357.0245</td>
</tr>
<tr>
<td>nmc261</td>
<td>2.8400E-015</td>
<td>391.5121</td>
<td>2.9017E-015</td>
<td>369.4831</td>
<td>3.2134E-015</td>
<td>363.7672</td>
</tr>
<tr>
<td>bcsstk12</td>
<td>1.0533E-014</td>
<td>615.8051</td>
<td>8.1737E-015</td>
<td>624.6021</td>
<td>1.8975E-015</td>
<td>582.1993</td>
</tr>
<tr>
<td>utm1700</td>
<td>1.1256E-014</td>
<td>985.1695</td>
<td>6.1700E-015</td>
<td>1031.4640</td>
<td>5.5940E-015</td>
<td>968.1327</td>
</tr>
</tbody>
</table>

Designing quadrature points with higher performance and a more precise error analysis of the BSSRR method is a part of our future work.

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### Acknowledgments

The support of Eng. Akbar Shahidzadeh Arabani is gratefully acknowledged.

### References


