Research Article

A New Algorithm for Solving Large-Scale Generalized Eigenvalue Problem Based on Projection Methods

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Received 17 September 2020; Revised 10 November 2020; Accepted 29 November 2020; Published 14 December 2020

Academic Editor: Li-Tao Zhang

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In this paper, we consider four methods for determining certain eigenvalues and corresponding eigenvectors of large-scale generalized eigenvalue problems which are located in a certain region. In these methods, a small pencil that contains only the desired eigenvalue is derived using moments that have obtained via numerical integration. Our purpose is to improve the numerical stability of the moment-based method and compare its stability with three other methods. Numerical examples show that the block version of the moment-based (SS) method with the Rayleigh–Ritz procedure has higher numerical stability than respect to other methods.

1. Introduction

Many problems arising in different fields of science and engineering can be reduced to the generalized eigenvalue problem [1–3]:

$$Ax = \lambda Bx, \quad (1)$$

where A, B are $n \times n$ real or complex, large, sparse, and only a few of the eigenvalues are desired. Also, when is $B = I$ (identity matrix), we have a standard eigenvalue problem. Computing eigenpairs (λ, x) of the generalized and standard eigenvalue problems is one of the important problems in many scientific applications [4–7]. There are several methods for solving such eigenvalue problems [8]. Among these methods, the iterative methods are used to generate a subspace that contains the desired eigenvectors. Techniques based on the Krylov subspaces are powerful tools for building desired subspaces for large-scale eigenvalue problems [9–11]. Expressed methods in this article find all of the zeros that lie in a circle using numerical integration. In this paper, we briefly describe moment-based method in Section 2, Rayleigh–Ritz with contour integral method in Section 3, block version of the Sakurai–Sugiura method in Section 4, and block version of the SS method with

Rayleigh–Ritz procedure in Section 5 for solving generalized eigenvalue problem (1). In Section 6, we provide four numerical tests for comparing four methods, and in Section 6, we apply the BSSRR method with selected matrices from different application areas, and finally, we draw some conclusions in Section 7.

2. Moment-Based Method (SS Method)

For solving (1), we consider computing entire poles of a rational function:

$$f(z) = u^H (zB - A)^{-1} v, \quad v, u, v \in \mathbb{C}^n \setminus \{0\}. \quad (2)$$

Those are eigenvalues λ of equation (1) and lie in a circle using numerical integration. Let Γ be positively oriented closed Jordan curve [12] in the complex plane and $\lambda_1, \dots, \lambda_n$ be distinct eigenvalues that lie in the Γ . Let

$$\mu_k = \frac{1}{2\pi i} \int_{\Gamma} (z - \gamma)^k f(z) dz, \quad k = 0, 1, \dots, \quad (3)$$

where γ is located inside Γ and the $m \times m$ Hankel matrices $H_m, H_m^<$ be $H_m = [\mu_{i+j-2}]_{i,j=1}^m$ and $H_m^< = [\mu_{i+j-1}]_{i,j=1}^m$. Also, let

$$s_k = \frac{1}{2\pi i} \int_{\Gamma}^{-} (z - \gamma)^k (zB - A)^{-1} v dz, \quad k = 0, 1, \dots \quad (4)$$

Then, we have the following theorem.

Theorem 1. If $v_j = 0$ for $1 \leq j \leq m$, then the eigenvalues of the pencil $H_m^< - \lambda H_m$ are given by $\lambda_1 - \gamma, \dots, \lambda_m - \gamma$.

Proof. In [13], by approximating the integral of equation (3) via the N-point trapezoidal rule, we obtain

$$\mu_k \approx \hat{\mu}_k = \frac{1}{N} \sum_{j=0}^{N-1} (\omega_j - \gamma)^{k+1} f(\omega_j), \quad k = 0, 1, \dots \quad (5)$$

Let $\xi_1, \dots, \xi_m$ be the eigenvalues of pencil $H_m^< - \lambda H_m$. We regard $\hat{\lambda}_j = \gamma + \xi_j$, $1 \leq j \leq m$ as the approximations for $\lambda_1, \dots, \lambda_m$ and

$$(\omega_j B - A)y_j = v, \quad j = 0, 1, \dots, N-1, \quad (6)$$

$$\hat{s}_k = \frac{1}{N} \sum_{j=0}^{N-1} (\omega_j - \gamma)^{k+1} y_j, \quad k = 0, 1, \dots$$

Also, let $\hat{V}_m$ be the Vandermonde matrix for $\xi_1, \dots, \xi_m$. Then, the approximations for the eigenvectors are obtained by

$$[\hat{q}_1, \dots, \hat{q}_m] = [\hat{s}_0, \dots, \hat{s}_{m-1}] \hat{V}_m^{-T}. \quad (7)$$

3. Rayleigh–Ritz with Contour Integral Method (CIRR Method)

We consider (1), let $A, B \in \mathbb{R}^{n \times n}$ be symmetric and let B be positive definite and (λ_j, x_j) , $1 \leq j \leq n$ be eigenpairs of the matrix pencil (A, B) . We apply a Rayleigh–Ritz procedure with an orthonormal basis $Q \in \mathbb{R}^{n \times m}$. The projected matrices are given by $\tilde{A} = Q^T A Q$ and $\tilde{B} = Q^T B Q$. $Q \in \mathbb{R}^{n \times m}$ is used to generate a sequence of subspace containing approximations to the desired eigenvector. The Ritz values of the projected pencil $(\tilde{A}, t\tilde{B})$ are taken as approximate eigenvalues for original pencil (A, B) with corresponding Ritz vectors. In this method, by applying the Rayleigh–Ritz procedure moments are not explicitly used [17]. The algorithm is as follows.

Rayleigh–Ritz procedure

- (1) Construct an orthonormal basis Q
- (2) Form $\tilde{A} = Q^T A Q$ and $\tilde{B} = Q^T B Q$
- (3) Compute the eigenpairs (θ_j, w_j) ($1 \leq j \leq m$) of $(\tilde{A}, t\tilde{B})$
- (4) Set $p_j = Q w_j$, $j = 1, \dots, M$

Theorem 2. Let s_k be defined by (4). Suppose that v is expanded by the eigenvectors $\{x_1, \dots, x_n\}$ as

$$v = \sum_{j=1}^n \alpha_j x_j. \quad (8)$$

Then,

$$s_k = \sum_{j=1}^m \alpha_j (\lambda_j - \gamma)^k x_j, \quad j = 0, 1, \dots, m-1. \quad (9)$$

Proof. It follows from (4) and (8) that

$$s_k = \frac{1}{2\pi i} \int_{\Gamma}^{-} \sum_{j=1}^n \alpha_j (z - \gamma)^k (zB - A)^{-1} B x_j dz. \quad (10)$$

Since (λ_j, x_j) is an eigenpair of the matrix pencil (A, B) , we have $(zB - A)x_j = (z - \lambda_j)Bx_j$ and thus $(zB - A)^{-1}Bx_j = (z - \lambda_j)^{-1}x_j$, and thus

$$s_k = \sum_{j=1}^n \frac{1}{2\pi i} \int_{\Gamma}^{-} \frac{\alpha_j (z - \gamma)^k}{z - \lambda_j} x_j dz, \quad k = 0, 1, \dots, m-1. \quad (11)$$

By the residue theorem, we obtain the result.

We define the $m \times m$ Vandermonde matrix with $\lambda_1 - \gamma, \dots, \lambda_m - \gamma$ by

$$V = \begin{bmatrix} 1 & (\lambda_1 - \gamma) & \dots & (\lambda_1 - \gamma)^{m-1} \\ 1 & (\lambda_2 - \gamma) & \dots & (\lambda_2 - \gamma)^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (\lambda_m - \gamma) & \dots & (\lambda_m - \gamma)^{m-1} \end{bmatrix}. \quad (12)$$

From the equation (9), we have

$$S = XDV, \quad (13)$$

where $S = [s_0, \dots, s_{m-1}]$, $X = [x_1, \dots, x_m]$, and $D = \text{diag}(\alpha_1, \dots, \alpha_m)$.

Theorem 3. If $\lambda_1, \dots, \lambda_m$ are distinct and $\alpha_j \neq 0$ for $1 \leq j \leq m$, then

$$\text{span}\{q_1, \dots, q_m\} = \text{span}\{x_1, \dots, x_m\}. \quad (14)$$

Proof. Since $\lambda_1, \dots, \lambda_m$ are mutually distinct and $\alpha_j \neq 0$ for $1 \leq j \leq m$. V and D are nonsingular. Therefore, it follows from (13) that

$$\text{span}\{s_0, \dots, s_{m-1}\} = \text{span}\{x_1, \dots, x_m\}. \quad (15)$$

Since the vectors $\{q_1, \dots, q_m\}$ are orthonormal basis of $\text{span}\{s_0, \dots, s_{m-1}\}$, equation (14) holds. $\square$

For nonzero vector $v \in \mathbb{R}^n$, we define the moments:

$$\mu_k = \frac{1}{2\pi i} \int_{\Gamma}^{-} (z - \gamma)^k (Bv)^T (zB - A)^{-1} Bv dz, \quad k = 0, 1, \dots, \quad (16)$$

where γ is located inside Γ . Also, we obtain the following approximations via the N-point trapezoidal rule:

$$\begin{aligned}\mu_k &\simeq \hat{\mu}_k = \frac{1}{N} \sum_{j=0}^{N-1} (\omega_j - \gamma)^{k+1} (Bv)^T (\omega_j B - A)^{-1} Bv, \quad k = 0, 1, \dots, \\ \hat{s}_k &= \frac{1}{N} \sum_{j=0}^{N-1} (\omega_j - \gamma)^{k+1} (\omega_j B - A)^{-1} Bv, \quad k = 0, 1, \dots.\end{aligned}\quad (17)$$

4. Block Sakurai–Sugiura Method (BSS Method)

In this method, for solving (1), we reformulate the SS method in the context of the resolvent theory. This method has the potential to resolve degenerated eigenvalues.

Theorem 4. Let $zB - A$ be a regular pencil of order N . Then, there exist nonsingular matrices $\tilde{P}, Q \in \mathbb{C}^{N \times N}$ such that

$$\tilde{P}(zB - A)Q = \begin{bmatrix} zI_{k_1} - J_1 & - & - & - & - & - \\ - & \ddots & - & - & - & - \\ - & - & zI_{k_d} - J_d & - & - & - \\ - & - & - & zN_{d+1} - I_{k_{d+1}} & - & - \\ - & - & - & - & \ddots & - \\ - & - & - & - & - & zN_r - I_{k_r} \end{bmatrix}, \quad (18)$$

where $J_i, N_i \in \mathbb{C}^{k_i \times k_i}$ are Jordan blocks, N_i is nilpotent, and I_k denotes the identity matrix of order k .

Proof. In [12].

Here, because $\tilde{P}, Q$ are the regular matrices, we can define $P = \tilde{P}^{-1}$ and $\tilde{Q} = Q^{-1}$. According to the (18), we will partition row vectors in $\tilde{P}$ and $\tilde{Q}$ into $\tilde{P}_i, \tilde{Q}_i \in \mathbb{C}^{k_i \times N}$, and column vectors in P and Q into $P_i, Q_i \in \mathbb{C}^{N \times k_i}$, respectively, for $i = 1, \dots, r$.

Theorem 5. Resolvent of the regular pencil $(zB - A)^{-1}$ is decomposed into

$$(zB - A)^{-1} = \sum_{i=1}^d Q_i \left\{ \sum_{m=0}^{k_i-1} \frac{(J_i - \alpha_i I_{k_i})^m}{(z - \alpha_i)^{m+1}} \right\} \tilde{P}_i - \sum_{i=d+1}^r Q_i \left\{ \sum_{m=0}^{k_i-1} z^m N_i^m \right\} \tilde{P}_i, \quad (19)$$

where α_i is an eigenvalue of Jordan block J_i .

Proof. Let $W = \tilde{P}(zB - A)Q$. According to Theorem 4, we have

$$\begin{aligned}(zB - A)^{-1} &= QW^{-1}\tilde{P} = \sum_{i=1}^d Q_i (zI_{k_i} - J_i)^{-1} \tilde{P}_i \\ &\quad + \sum_{i=d+1}^r Q_i (zN_i - I_{k_i})^{-1} \tilde{P}_i.\end{aligned}\quad (20)$$

Using the resolvent of the Jordan block,

$$R(z, J_i) = (zI_{k_i} - J_i)^{-1} = \sum_{m=0}^{k_i-1} \frac{(J_i - \alpha_i I_{k_i})^m}{(z - \alpha_i)^{m+1}} \quad (21)$$

And $(zN_i - I_{k_i})^{-1} = -z^{-1}R(z^{-1}, N_i)$, we get the result.

Theorem 6. The localized moment matrix is written as

$$M_n = \sum_{i: \alpha_i \in G} Q_i J_i^n \tilde{Q}_i. \quad (22)$$

Proof. In [18].

Definition 1. Let C and D be arbitrary $N \times m$ matrices, where $N > m \geq k_r$. A size-reduced moment matrix is defined as

$$M_n = C^H M_n D \in \mathbb{C}^{m \times m}. \quad (23)$$

Theorem 7. If ranks of both $C^H Q_\Gamma$ and $\tilde{Q}_\Gamma D$ are k_r , nonsingular part of a matrix pencil $zM_0 - M_1$ is equivalent to $zI_{k_r} - J_\Gamma$.

Proof. In [18]. □

Theorem 8. The right eigenvectors of the original matrix pencil $zB - A$ are given by $Q_\Gamma = M_0 D Q_\Gamma$, and its adjoint is given by $\tilde{Q}_\Gamma = \tilde{P}_\Gamma C^H M_0$.

Proof. In [18]. □

Theorem 9. If all elements of $\tilde{Q}_\Gamma v$ and $v^H Q_\Gamma$ are nonzero, and there is no degeneracy in J_Γ , then nonsingular part of a matrix pencil $zH_m - H_m^<$ is equivalent to $zI_{k_r} - J_\Gamma$.

Proof. By choosing row vectors of C^H and vectors of D to be

$$(C^H)_{i,*} = v^H Q_\Gamma J_\Gamma^{i-1} \tilde{Q}_\Gamma. \quad (24)$$

And $D_{i,*} = Q_\Gamma J_\Gamma^{i-1} \tilde{Q}_\Gamma v$ for $i = 1, 2, \dots, m$, respectively, we have $H = M_0$ and $H_m^< = M_1$. As for the rank of $\tilde{Q}_\Gamma D$, we consider that column vectors of $\tilde{Q}_\Gamma D$ from the Krylov series of J_Γ starting from $\tilde{Q}_\Gamma v$. Because J_Γ has not degenerated, and elements of $\tilde{Q}_\Gamma v$ are nonzero, these column vectors are linearly independent, and thus the rank of $\tilde{Q}_\Gamma D$ is $f k_r$.

5. Block Version of the SS Method with the Rayleigh–Ritz Procedure (BSSRR Method)

We suggest a new algorithm for computing all poles of analytic function (2) with the use of the algorithm in [19]. As the eigenpairs (λ_i, x_i) of equation (1) can be obtained from $H_M^{(1)} u_i = \theta_i H_M^{(2)} u_i$, where $H_M^{(1)}$ and $H_M^{(2)}$ are small $M \times M$ Hankel matrices. Let $V \in \mathbb{C}^{N \times L} \setminus \{0\}$, a random matrix, and $S = [S_0, S_1, \dots, S_{M-1}] \in \mathbb{C}^{N \times LM}$, where

$$S_k = [s_k^{(1)}, \dots, s_k^{(L)}]$$

$$= \frac{1}{2\pi i} \oint_{\Gamma} (z - \gamma)^k (zB - A)^{-1} B V dz, \quad k = 0, 1, \dots, M-1. \quad (25)$$

Then, the block version of the SS method with Rayleigh–Ritz procedure [20] constructs the LM-dimensional subspace:

$$\delta_M = \text{span}\{S\} = \text{span}\{s_0^{(1)}, \dots, s_0^{(L)}, s_1^{(1)}, \dots, s_1^{(L)}, \dots, s_{M-1}^{(1)}, \dots, s_{M-1}^{(L)}\}. \quad (26)$$

For the Rayleigh–Ritz procedure, the subspace δ_M contains all eigenvectors of (1) $\delta_M = \text{span}\{x_1, x_2, \dots, x_m\}$ for $m \leq LM$. With using N-point trapezoidal rule for equation (25), we have

$$S_k \approx \tilde{S}_k = \sum_{j=1}^N \omega_j (z_j - \gamma)^k (z_j B - A)^{-1} B V. \quad (27)$$

where z_j is the quadrature point and ω_j is the corresponding weight [21].

Based on Theorem 4, we analyse the relationship between the contour integral spectral projection and the Krylov subspace.

5.1. An Arnoldi-Based Interpretation of the Contour Integral Spectral Projection. Since the matrices $\tilde{P}, Q$ are nonsingular, we define $P = \tilde{P}^{-1}, \tilde{Q} = Q^{-1}$. According to the Jordan block structure of (18), we partition row vectors in $\tilde{P}, \tilde{Q}$ into $\tilde{P}_i, \tilde{Q}_i \in \mathbb{C}^{n_i \times n}$ and column vectors in P, Q into $P_i, Q_i \in \mathbb{C}^{n \times n_i}$, respectively, for $i = 1, 2, \dots, r$. Then, we can derive the following lemma and theorem.

Lemma 1. Let $\psi_k(z)$ be a k -degree polynomial. Then, we have

$$\frac{1}{2\pi i} \oint_{\Gamma} \psi_k(z - \gamma) (zB - A)^{-1} B dz = \psi_k(S_{\Gamma}) P_{\Gamma}, \quad (28)$$

where

$$S_{\Gamma} = \sum_{i: \lambda_i \in \Omega} Q_i (J_i - \gamma I) \tilde{Q}_i, P_{\Gamma} = \sum_{i: \lambda_i \in \Omega} Q_i \tilde{Q}_i. \quad (29)$$

Proof. From Theorem 6 and the binomial theorem

$$(z - \gamma)^{\ell} = \sum_{j=0}^{\ell} \binom{\ell}{j} z^{\ell-j} \gamma^j, \text{ we have the following relation:}$$

$$\frac{1}{2\pi i} \oint_{\Gamma} (z - \gamma)^{\ell} (zB - A)^{-1} B dz = \sum_{i: \lambda_i \in \Omega} Q_i (J_i - \gamma I)^{\ell} \tilde{Q}_i, \quad \ell = 0, 1, \dots \quad (30)$$

Here, since $\tilde{Q}_i Q_i = I$ and $\tilde{Q}_i Q_j = 0$ for $i \neq j$,

$$\sum_{i: \lambda_i \in \Omega} Q_i (J_i - \gamma I)^{\ell} \tilde{Q}_i = \left(\sum_{i: \lambda_i \in \Omega} Q_i (J_i - \gamma I) \tilde{Q}_i \right)^{\ell} \left(\sum_{i: \lambda_i \in \Omega} Q_i \tilde{Q}_i \right) = S_{\Gamma}^{\ell} P_{\Gamma}, \quad \ell = 0, 1, \dots \quad (31)$$

Therefore, Lemma 1 is proved. $\square$

Definition 2. Let $A \in \mathbb{C}^{n \times n}$, $V = [v_1, v_2, \dots, v_L] \in \mathbb{C}^{n \times L}$, and let $\kappa_m(A, v_1)$ the Krylov subspace defined by

$$\kappa_m(A, v_1) = \text{span}\{v_1, Av_1, \dots, A^{m-1}v_1\}. \quad (32)$$

Then, the subspace $\kappa_m^*(A, V)$ is defined by the sum of the Krylov subspaces, i.e.,

$$\begin{aligned} \kappa_m^*(A, V) &= \kappa_m(A, v_1) + \kappa_m(A, v_2) + \dots + \kappa_m(A, v_L) \\ &= \text{span}\{[V, AV, \dots, A^{m-1}V]\}. \end{aligned} \quad (33)$$

Theorem 10. Let δ_m be the subspace of the block version of the SS method with the Rayleigh–Ritz procedure defined by (26). Then, we have

$$\delta_m = \kappa_m^*(S_{\Gamma}, P_{\Gamma}V). \quad (34)$$

Proof. From the definition of S_k (25) and Lemma 1, we have $S_k = S_{\Gamma}^k P_{\Gamma}V$. Therefore,

$$\begin{aligned} \delta_M &= \text{span}\{[S_0, S_1, \dots, S_k]\} \\ &= \text{span}\{[P_{\Gamma}V, S_{\Gamma}P_{\Gamma}V, \dots, S_{\Gamma}^{M-1}P_{\Gamma}V]\} = \kappa_M^*(S_{\Gamma}, P_{\Gamma}V). \end{aligned} \quad (35)$$

Therefore, Theorem 10 is proved. $\square$

Remark 1. Theorem 10 shows that the block version of the SS method with the Rayleigh–Ritz procedure can be regarded as the Rayleigh–Ritz procedure based on the block Krylov subspace $\kappa_M^*(S_{\Gamma}, P_{\Gamma}V)$. Here, we note that, in the block version of the SS method with the Rayleigh–Ritz procedure, the basis vectors of $\kappa_M^*(S_{\Gamma}, P_{\Gamma}V)$ are explicitly computed by (25) and the QR decomposition of S (Algorithm 1).

And $(\lambda_i, x_i) = (\theta_i, Wu_i)$ for $i = 1, 2, \dots, LM$.

6. Numerical Experiments

In this section, we have provided five numerical examples. In Examples 1–4, we have discussed stability Algorithm 2 (SS method), Algorithm 3 (CIRR method), Algorithm 4 (BSS method), and Algorithm 1 (BSSRR method), and in Example 5, we have applied the BSSRR method with selected matrices in the fields of engineering sciences. Also, we have used

Input: $L, M, N \in \mathbb{N}, V \in \mathbb{C}^{n \times L}, \gamma, (z_j, \omega_j)$ for $j = 1, 2, \dots, N$
 Output: eigenpairs (λ_i, x_i) for $i = 1, 2, \dots, LM$
 (1) Solve $(z_j B - A)Y_j = BV$ for $j = 1, 2, \dots, N$, [14–16]
 (2) Compute $S_k = \sum_{j=1}^N \omega_j (z_j - \gamma)^k Y_j$ for $k = 0, 1, \dots, M-1$
 (3) Compute QR decomposition of $S = [S_0, S_1, \dots, S_{M-1}]$: $S = WR$
 (4) Compute eigenpairs (θ_i, u_i) of the matrix pencil $(W^H A W, W^H B W)$

ALGORITHM 1: BSSRR method.

Input: $u, v \in \mathbb{C}^n, N, m, \gamma, \rho$
 Output: $\hat{\lambda}_1, \dots, \hat{\lambda}_m, \hat{q}_1, \dots, \hat{q}_m$
 (1) Set $\omega_j \leftarrow \gamma + \rho(2\pi i j/N)$, $j = 0, 1, \dots, N-1$
 (2) Solve $(\omega_j B - A)y_j = v$, $j = 0, 1, \dots, N-1$, [14–16]
 (3) Set $f_j \leftarrow u^H y_j$, $j = 0, 1, \dots, N-1$
 (4) Compute $\hat{\mu}_k$, $k = 0, \dots, 2m-1$ by (5)
 (5) Compute the eigenvalues $\xi_1, \dots, \xi_m$ of the pencil $\hat{H}_m^< - \lambda \hat{H}_m$
 (6) Compute $\hat{q}_1, \dots, \hat{q}_m$ by (7)
 (7) Set $\hat{\lambda}_j \leftarrow \gamma + \xi_j$, $j = 1, \dots, m$

ALGORITHM 2: SS method.

Input: $v \in \mathbb{R}^n, N, M, \gamma, \rho$
 Output: $\hat{\lambda}_1, \dots, \hat{\lambda}_m, \hat{x}_1, \dots, \hat{x}_m$
 (1) Set $\omega_j = \gamma + \rho \exp(2\pi i(j + 1/2)/N)$, $j = 0, 1, \dots, N-1$
 (2) Solve $(\omega_j B - A)y_j = Bv$ for y_j , $j = 0, 1, \dots, N-1$, [14–16]
 (3) Compute $\hat{s}_k$, $k = 0, 1, \dots, M-1$ by (17)
 (4) Construct an orthonormal basis Q from $\{\hat{s}_0, \dots, \hat{s}_{M-1}\}$
 (5) Form $\tilde{A} = Q^T A Q$ and $\tilde{B} = Q^T B Q$
 (6) Compute the eigenpairs (θ_j, ω_j) ($1 \leq j \leq M$) of $(\tilde{A}, t\tilde{B})$
 (7) Set $p_j = Q\omega_j$, $j = 1, \dots, M$
 (8) Select the approximate eigenpairs $(\hat{\lambda}_1, \hat{x}_1), \dots, (\hat{\lambda}_m, \hat{x}_m)$ from (θ_j, p_j) ($1 \leq j \leq M$)

ALGORITHM 3: CIRR method.

Input: $V \in \mathbb{C}^{N \times L}, \{z_j, \omega_j\}$ for $j = 1, 2, \dots, M$
 Output: α_k for $k = 1, 2, \dots, K$
 (1) Solve $(z_j B - A)\tilde{V}_j = BV$ [14–16] and calculate $V_j = V^H \tilde{V}_j \in \mathbb{C}^{L \times L}$
 (2) Compute $\bar{\mu}_n = \sum_{j=1}^M \omega_j z_j^n V_j$
 (3) Construct Hankel matrices H_m and $H_m^< \in \mathbb{C}^{ml \times ml}$
 (4) Perform singular value decomposition, $H_m = WsU$
 (5) Construct $H = s^{-1/2} W^H H_m^< U^H s^{-1/2} \in \mathbb{C}^{K \times K}$
 (6) Compute eigenvalues of H to have α_k
 (7) Compute $\bar{S}_n = \sum_{j=1}^M \omega_j z_j^n \tilde{V}_j$
 (8) Compute $(Q_1, \dots, Q_K) = (\bar{S}_0, \dots, \bar{S}_{m-1}) U^H s^{-1/2} (q_1, \dots, q_K)$

ALGORITHM 4: BSS method.

$Ax_i + \lambda_i Bx_{i2} / (Ax_{i2} + |\lambda_i| Bx_{i2})$ to compute relative residual for all of the methods. In computational results tables, the number of eigenpairs has been named NE.

Example 1. A real symmetric matrix $A \in \mathbb{R}^{n \times n}$ was prepared, which has five primary eigenvalues $-12.03, -12.02, -12.01,$

$-12.00, -11.99$. In the range of $[-12.5, -11.5]$, other eigenvalues were taken randomly in the range of $[-40, 40]$, and a random unitary matrix was prepared to construct A . An identity matrix was used for B . After applying Algorithms 1–4, we obtained numerical results that have been shown in Table 1.

TABLE 1: The minimum relative residual of Algorithms 1–4 in Example 1.

$\gamma = -0.2 + 0.6i, \rho = 0.5$								
N.A	$m = 16, N = 32$		$M = 18, N = 32 \quad m = 16$		$M = 32, m = 4, l = 4$ Threshold = 1.1×10^{-18}		$M = 4, L = 4, N = 32$	
	SS	NE	CIRR	NE	BSS	NE	BSSRR	NE
$n = 200$	0.82114	16	$8.8577E-008$	16	$1.9902E-008$	16	$7.9051E-016$	16
$n = 400$	0.7919	16	$3.2359E-06$	16	$5.1353E-09$	16	$9.8608E-16$	16
$n = 600$	0.8393	16	$3.2571E-05$	16	$1.1753E-07$	16	$1.3489E-15$	16
$n = 800$	0.8665	16	$8.8640E-05$	16	$2.6448E-05$	16	$1.6268E-15$	16
$n = 1000$	0.8508	16	$7.7542E-05$	16	$2.7825E-05$	16	$2.2567E-15$	16
$n = 1500$	0.8671	16	$5.6523E-05$	16	$8.7373E-005$	16	$1.9151E-15$	16

TABLE 2: The minimum relative residual of Algorithms 1–4 in Example 2.

$\gamma = -0.1 + 0.5i, \rho = 0.4$								
N.A, N.B	$m = 16, N = 32$		$M = 18, N = 32 \quad m = 16$		$M = 32, m = 4, l = 4$ Threshold = 1.1×10^{-14}		$M = 4, L = 4, N = 32$	
	SS	NE	CIRR	NE	BSS	NE	BSSRR	NE
$n = 200$	0.9307	16	—	—	$9.919E-05$	16	$6.1079E-16$	16
$n = 400$	0.89555	16	—	—	$1.3066E-04$	16	$8.3165E-16$	16
$n = 600$	0.8926	16	—	—	$6.6731E-004$	16	$9.0355E-16$	16
$n = 800$	0.8802	16	—	—	$1.600E-004$	16	$8.6891E-16$	16
$n = 1000$	0.8749	16	—	—	$3.2000E-004$	16	$1.2295E-15$	16
$n = 1500$	0.8432	16	—	—	$1.6400E-004$	16	$1.9151E-15$	16

FIGURE 1: Results of comparing the relative residual of Algorithms 1–4 in Example 2 with $n = 1000$.

TABLE 3: The minimum relative residual of Algorithm 1–4 and in Example 3.

$\gamma = 0, \rho = 0.14$								
N.A, N.B	$m = 16, N = 64$		$M = 18, N = 64 \quad m = 16$		$M = 64, m = 4, l = 4$ Threshold = 1.1×10^{-18}		$M = 4, L = 4, N = 64$	
	SS	NE	CIRR	NE	BSS	NE	BSSRR	NE
$n = 200$	0.6600	16	$5.7595E-13$	16	$3.4279E-11$	8	$6.3585E-15$	16
$n = 400$	0.6028	16	$7.2639E-11$	16	$6.1907E-11$	10	$3.1440E-15$	16
$n = 600$	0.6509	16	$3.4114E-11$	16	$3.0774E-09$	11	$3.3606E-15$	16
$n = 800$	0.5966	16	$6.2826E-11$	16	$3.0774E-08$	12	$8.2243E-15$	16
$n = 1000$	0.6065	16	$6.8571E-10$	16	$9.7887E-08$	12	$3.3916E-15$	16
$n = 1500$	0.6737	16	$3.6078E-12$	16	$7.9782E-08$	11	$2.3400E-15$	16

FIGURE 2: Results of comparing the relative residual of Algorithms 1–4 in Example 3 with $n = 1000$.

TABLE 4: The minimum relative residual of Algorithms 1–4 in Example 4.

$\gamma = 0, \rho = 0.15$								
N.A, N.B	$m = 16, N = 32$		$M = 18, N = 32, m = 16$		$M = 32, m = 4, l = 4$ Threshold = 1.1×10^{-14}		$M = 4, L = 4, N = 32$	
	SS	NE	CIRR	NE	BSS	NE	BSSRR	NE
$n = 200$	0.7389	16	0.0035	16	0.0315	16	$2.4592E - 015$	16
$n = 400$	0.7350	16	0.0057	16	0.0337	16	$4.2091E - 15$	16
$n = 600$	0.7370	16	0.0049	16	0.0349	16	$5.2251E - 15$	16
$n = 800$	0.7389	16	0.0068	16	0.0323	16	$6.0494E - 15$	16
$n = 1000$	0.7240	16	0.0069	16	0.0329	16	$6.4196E - 15$	16
$n = 1500$	0.7393	16	0.0065	16	0.0384	16	$7.9371E - 15$	16

FIGURE 3: Results of comparing the relative residual of Algorithms 1–4 in Example 4 with $n = 1000$.

TABLE 5: Application areas, names, and properties 17 matrices selected.

Application	Name	N	NZ	CON
Generalized eigenvalue problem	lund_a	147	2449	5.4430E+006
Generalized eigenvalue problem	lund_b	147	2449	6.0317E+004
Chemical kinetics	fs_183_1	183	998	1.5122E+013
Square dielectric wave guide	dw256B	512	2500	4.5047
Chemical kinetics	fs_760_1	760	5739	1.1234E+016
Large helicopter model	rotor2	791	10685	1.2651E+013
Dynamic analysis in structural engineering	bcsstk19	817	6853	2.8130E+011
Unsymmetric basis from LP problem	bp_1600	822	4841	8.6511E+006
Power systems simulation	qh882	882	47906	9.9827E+024
Oil reservoir simulation	Sherman1	1000	3750	2.2575E+004
Aeroelasticity	tols1090	1090	3546	2.1164E+006
Alfven spectra in magneto hydrodynamic	mdh1280	1280	47906	9.9827E+024
Nuclear reactor models	nnc261	1374	8588	4.1082E+015
Dynamic analysis in structural engineering	bcsstk12	1473	34241	5.2502E+008
Plasma physics	utm1700	1700	21500	1.6380E+007
Splatzman symmetric finite difference three ocean model	Plat1919	1919	32399	5.1323E+018
Dynamic analysis in structural engineering	bcsstk13	2003	83883	4.5698E+010

TABLE 6: Calculation relative residual 2-norm of the BSSRR method.

$\gamma = 0, \rho = 0.12, M = 4, L = 4, N = 32$				
BSSRR				
Name matrix	NE	Minimum residual	Maximum residual	Time
lund_a	16	1.2115E-015	3.2651E-015	0.4351
lund_b	16	7.7260E-015	4.9013E-015	0.4034
fs_183_1	16	9.4890E-016	7.6258E-010	0.6897
dw256B	16	3.0480E-015	1.13825E-013	37.9230
fs_760_1	16	1.9950E-015	1.1748E-014	114.1690
rotor2	16	2.9661E-015	7.6861E-011	73.6634
bcsstk19	16	1.7704E-015	4.3297E-015	80.9064
bp_1600	16	4.1415E-015	8.8643E-015	118.4747
qh882	16	8.3663E-011	8.1529E-010	91.1155
Sherman1	16	2.0279E-015	6.7531E-015	145.8976
tols1090	16	6.3521E-15	1.0472E-12	174.7583
mdh1280	16	9.1528E-015	1.1804E-012	362.0046
nnc261	16	2.4955E-015	1.387E-014	362.3879
bcsstk12	16	1.7482E-015	4.9099E-015	546.7578
utm1700	16	5.0652E-015	2.6436E-014	960.8218
Plat1919	16	3.1237E-015	1.2013E-013	1191.3179
bcsstk13	16	2.8611E-015	5.3720E-014	1363.4479

Example 2. We let that A, B were complex, random matrices, and B was positive definite. After applying Algorithms 1–4, we obtained numerical results that have been shown in Table 2. Also, the relative residual for described methods has been drawn in Figure 1 for $n = 1000$.

Example 3. In this example, A, B were taken sparse, symmetric, and random, and A was positive definite. After applying Algorithms 1–4, we obtained numerical results that have been shown in Table 3. Also, the relative residual for described methods has been drawn in Figure 2 for $n = 1000$.

Example 4. We consider matrices:

$$A = I_n,$$

$$B = \begin{bmatrix} -7 & -3 & -1 & -1 & -1 & -1 \\ -3 & -8 & -3 & -1 & -1 & -1 \\ -1 & -3 & -8 & -3 & -1 & -1 \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -3 & -8 & -3 & -1 \\ -1 & -1 & -1 & -3 & -8 & -3 \\ -1 & -1 & -1 & -1 & -3 & -7 \end{bmatrix}, \quad (36)$$

After applying Algorithms 1–4, we obtained numerical results that have been shown in Table 4. Also, the relative residual for described methods has been drawn in Figure 3 for $n = 1000$.

TABLE 7: Calculation minimum relative residual of the BSSRR method with other norms.

$\gamma = 0, \rho = 0.12, M = 4, L = 4, N = 32$						
Name matrix	BSSRR					
	$\ -\ _1$	Time	$\ -\ _{\infty}$	Time	$\ -\ _{fro}$	Time
lund_a	1.6280E-15	0.4641	1.464E-015	0.4751	1.0893E-015	0.4429
lund_b	1.2158E-015	0.4329	1.1523E-015	0.4289	9.5626E-016	0.4587
fs_183_1	1.3389E-015	0.9160	1.9050E-015	0.9926	5.7331E-015	0.7920
dw256B	1.0643E-014	38.3247	4.8503E-015	38.5182	4.8285E-015	39.0655
fs_760_1	1.4451E-014	115.7634	5.8661E-015	116.8876	9.4203E-015	116.0084
rotor2	8.5274E-015	74.3794	3.6814E-015	79.6705	1.5234E-015	75.7969
bcsstk19	4.6544E-015	83.6374	1.1299E-015	84.1706	1.6893E-015	83.6942
bp_1600	5.6919E-15	119.6776	2.1991E-015	119.9996	3.3914E-015	123.9625
qh882	2.7482E-010	98.5387	1.8575E-010	93.9466	1.0929E-010	94.3459
Sherman1	2.2421E-014	148.6958	9.1555E-015	147.3571	2.1974E-015	160.7665
tols1090	6.4706E-14	203.0137	7.8958E-14	186.5922	8.0820E-14	175.2136
mdh1280	1.1013E-014	357.0195	3.8742E-015	363.0094	6.2742E-015	357.0245
nnc261	2.8400E-015	391.5121	2.9017E-015	369.4831	3.2134E-015	363.7672
bcsstk12	1.0533E-014	615.0851	8.1737E-015	624.6021	1.8975E-015	582.1993
utm1700	1.1256E-14	985.1695	6.1700E-015	1031.4640	5.5940E-015	968.1327
Plat1919	6.6578E-015	1173.1310	4.1980E-015	1294.0850	5.8991E-015	1296.4698
bcsstk13	7.4671E-015	1366.7534	3.5188E-015	1408.8137	3.2373E-015	1385.8719

Example 5. In this example, we selected seventeen matrices from the UF sparse matrix collection. Two major requirements were used in the selection procedure: matrices with different parameters and matrices arising in different application areas were chosen. We consider the following symptoms parameters.

The order N , the number of nonzero elements NZ , and the condition number CON . The application areas of the selected matrices are listed in Table 5. Matrices from the different areas were selected and thus obtained results by running the matrices will be typical in several scientific fields. We applied the BSSRR method for the calculation of relative residual generalized eigenvalue problem (1) when A is one of the selected matrices in Table 5 and B is identity matrix. As the dimension A , B is equal. We computed relative residual respect to $-_2$ In Table 6 and computed respect to other norms in Table 7, too, the number of eigenpairs for each matrix was sixteen in Table 7.

7. General Conclusions and Plans for Future Work

Several specific conclusions were drawn in connection with the numerical results presented in the previous section. Some general conclusions are given as follows:

- (1) All numerical experiments indicate that CIRR, BSS, and BSSRR methods have higher stability than respect to the SS method
- (2) BSSRR method has less relative residual respect to SS, CIRR, and BSS methods
- (3) If $-_2$ is used for calculation of relative residual in the BSSRR method, then we have higher accuracy and less consuming time

Designing quadrature points with higher performance and a more precise error analysis of the BSSRR method is a part of our future work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The support of Eng. Akbar Shahidzadeh Arabani is gratefully acknowledged.

References

- [1] Z. Bai, "Krylov subspace techniques for reduced-order modeling of large-scale dynamical systems," *Applied Numerical Mathematics*, vol. 43, no. 1-2, pp. 9-44, 2002.
- [2] Y. Inadomi, T. Nakano, K. Kitaura, and U. Nagashima, "Definition of molecular orbitals in fragment molecular orbital method," *Chemical Physics Letters*, vol. 364, no. 1-2, pp. 139-143, 2002.
- [3] M. M. Magolu, "Incomplete factorization-based preconditions for solving the Helmholtz equation," *International Journal for Numerical Methods in Engineering*, vol. 50, pp. 1088-1101, 2001.
- [4] K. Karthikeyan, "Small-signal stability enhancement using STATCOM based eigenvalue analysis," *Indian Journal of Science Technology*, vol. 21, 2015.
- [5] J. Saira Banu, R. Babu, and R. Pandey, "Parallel implementation of singular value decomposition (SVD) in image compression using open Mp and sparse matrix representation," *Indian Journal of Science Technology*, vol. 35, 2015.

- [6] S. Gudarzi, H. H. Wan, M. H. Anisi, and A. Soleymani, "A comparative review of vertical handover decision-making mechanisms in heterogeneous wireless networks," *Indian Journal of Science Technology*, vol. 58, 2015.
- [7] M. Kaviarasan, P. Geetha, and K. P. Soman, "Multivariate statical technique for the assessment of groundwater quality in Coonoor taluk, Nilgiri district, Tamilnadu, India," *Indian Journal of Science Technology*, vol. 112, 2015.
- [8] B. N. Datta, *Numerical Linear Algebra and Applications*, Brooks, New York, NY, USA, 2010.
- [9] H. S. Najafi, A. Refahi, and M. Akbari, "Weighted FOM-inverse vector iteration method for computing a few smallest (largest) eigenvalues of pair (A,B)," *Applied Mathematics and Computation*, vol. 192, no. 1, pp. 239–246, 2007.
- [10] H. Saberi Najafi and A. H. Refahi, "FOM-inverse vector iteration method for computing a few smallest (largest) eigenvalues of pair (A, B)," *Applied mathematics and computation*, vol. 17, pp. 614–647, 2007.
- [11] H. Saberi Najafi and A. Refahi, "A new restarting method in the Lanczos algorithm for generalized eigenvalue problem," *Applied Mathematics and Computation*, vol. 184, no. 2, pp. 421–428, 2007.
- [12] F. Gantmacher, *The Theory of Matrices*, Chelsea, New York, NY, USA, 1959.
- [13] T. Sakurai and H. Sugiura, "A projection method for generalized eigenvalue problems using numerical integration," *Journal of Computational and Applied Mathematics*, vol. 159, no. 1, pp. 119–128, 2003.
- [14] H. S. Najafi, S. A. Edalatpanah, and A. H. Refahi Sheikhan, "Convergence analysis of modified iterative methods to solve linear systems," *Mediterranean Journal of Mathematics*, vol. 11, no. 3, pp. 1019–1032, 2014.
- [15] H. Saberi Najafi, S. A. Edalatpanah, and A. H. Refahisheikhan, "An analytical method as a preconditioning modeling for systems of linear equations," *Computational and Applied Mathematics*, vol. 37, no. 2, pp. 922–931, 2018.
- [16] F. Shariffar, A. H. Refahi Sheikhan, and H. Saberi Najafi, "An efficient Chebyshev semi-iterative method for the solution of large systems," *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, vol. 80, no. 4, pp. 239–252, 2018.
- [17] T. Sakurai and H. Tadano, "CIRR: a Rayleigh-Ritz type method with contour integral for generalized eigenvalue problems," *Hokkaido Mathematical Journal*, vol. 36, no. 4, pp. 745–757, 2007.
- [18] T. Ikegami, T. Sakurai, and U. Nagashima, "A filter diagonalization for generalized eigenvalue problems based on the sakurai – sugiura projection method," *Journal of Computation and Applied Mathematics*, vol. 41, pp. 1927–1936, 2010.
- [19] P. Kravanja, T. Sakurai, and M. Van Barel, "On locating clusters of zeros of analytic functions," *Bit Numerical Mathematics*, vol. 39, no. 4, pp. 646–682, 1999.
- [20] T. Ikegami and T. Sakurai, "Contour integral eigensolver for non-hermitian systems: a Rayleigh-ritz-type Approach," *Taiwanese Journal of Mathematics*, vol. 14, no. 3A, pp. 825–837, 2010.
- [21] A. Imakura, L. Du, and T. Sakurai, "A block Arnoldi-type contour integral spectral projection method for solving generalized eigenvalue problems," *Applied Mathematics Letters*, vol. 32, pp. 22–27, 2014.