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Research Article

The 2-Extra Connectivity of Wheel Networks

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Connectivity is a significant metric for evaluating the error lenience of an interconnected net $G = (V, E)$. In the paper (Fàbrega and Fiol, 1996), the authors addressed the g-extra connectivity which is applied to better measure the consistency and error tolerance of G. As a special Cayley graph, the n-dimensional wheel network CW_n has some desirable features. This paper shows that the g-extra connectivity of $CW_n (n \geq 6)$ is $6n - 12$ when $g = 2$.

1. Introduction

An interconnected net (abbreviated IN) is usually charactered as a graph, whereat each node (vertex) resembles a CPU and an edge resembles the connection between a couple of CPUs, correspondingly. Connectivity of a graph G, written as $\kappa(G)$, is defined as the minimum number of nodes whose deletion from G implies a detached graph or has only 1 node. Connectivity is an important index in evaluating the dependability and error tolerance of an IN. The greater the connectivity is, the more consistent an IN is. Nevertheless, an understandable drawback of the index is that it adopts that each node neighboring the same node of G can flop simultaneously. In fact, it is highly unlikely in applied net uses. Hence, the traditional connectivity is inappropriate for huge computing systems.

To overcome this deficiency, Harary [1] firstly presented the idea of restricted connectivity, which is a more refined index in determining the consistency and error tolerance of INs. Consider G to be a linked directionless basic graph, and p be a given 1-dimensional topology theory feature. The restricted connectivity, written as $\kappa(G; \mathrm{p})$, is obtained as the minimum cardinality of a set of nodes, if any, whose deletion detaches G and each residual factor has feature p. Amid the restricted connectivity, the g-extra connectivity has been initially presented by Fàbrega and Fiol [2]. A subset S of nodes is known as a cutset if $G - S$ is detached. For a nonnegative integer g, a cutset S is named a g-extra cut, if each element of $G - S$ contains t nodes, where $t \geq g + 1$. The g-extra connectivity of G, if there must exist 1 g-extra cut, written as $\tilde{\kappa}^{(g)}(G)$, is now obtained as the minimum cardinality above all g-extra cuts of G, namely, $\tilde{\kappa}^{(g)}(G) = \kappa(G; \mathrm{p}_g)$, whereat p_g is the feature that each residual element takes at least $(g + 1)$ nodes. Obviously, $\tilde{\kappa}^{(0)}(G) = \kappa(G)$ for each linked incomplete graph G. Therefore, the traditional connectivity may be regarded as a simplification of g-extra connectivity, and it is able to serve for more precisely measuring the consistency and error lenience for INs. The g-extra connectivity of numerous INs was widely investigated (see [2–17]).

It is worthwhile to mention that the algebraic connectivity [18] is another important metric in measuring how fine a graph is linked and is very significant in controller theory, communications, etc. Further research relating to algebraic connectivity are described in references [19, 20].

The n-dimensional wheel network, a promising topology arrangement of INs, has some respectable things. Here, we establish that the g-extra connectivity of $CW_n (n \geq 6)$ is $6n - 12$, when $g = 2$.

2. Preliminaries

Definitions and notations used through this paper are provided. The n-dimensional wheel network and its basic properties are examined. We consider [21] for terms and notations not introduced in this paper.

2.1. Definitions and Notations. Consider a basic undirected graph $G = (V, E)$. Given $U \subseteq V, U \neq \emptyset$, the prompted graph determined by U in G, written as $G[U]$, is a graph in which set of nodes is U and the set of edges contains all the edges of G having both endpoints in U. For $x \in V$, the degree x in G is defined as the amount of edges neighboring to x, written as $d_G(x)$. Let the minimum degree $\delta(G) = \min\{d_G(x) : x \in V\}$ in G. For arbitrary node $x \in V$, the neighborhood $N_G(x)$ of x is determined as the set of nodes neighboring to x in a graph G. The node y is named a neighbor node of x, where $y \in N_G(x)$. For $U \subseteq V$, $N_G(U)$ is applied to represent $\cup_{x\in U} N_G(x) \backslash U$. If no confusion arises, then $N_G(x)$, $\delta(G)$, and $d_G(x)$ can be abbreviated into $N(x)$, δ, and $d(x)$, correspondingly. The set F containing all error nodes of G is named as a faulty set in G. Each node in F is named faulty and any node in $V(G)\backslash F$ is named faulty-free.

2.2. The Wheel Networks. The wheel networks have been known as a desirable topology structure of INs. Here, we recall its definition and some important aspects.

Consider D to be a bounded group, Z to be a spanning set of D, where the identity element of D cannot be in Z. The linked Cayley graph $\mathrm{Cay}(Z, D)$ has a set of nodes D and a set of arcs $\{(d, dz) : d \in D, z \in Z\}$. The condition that Z is a spanning set of D ensures that $\mathrm{Cay}(Z, D)$ is linked. The assumption that the identity element of D cannot be in Z guarantees that $\mathrm{Cay}(Z, D)$ is simple. For convenience, let $[n] = \{1, 2, \ldots, n\}$. Here, we concentrate on the Cayley graphs produced by transpositions. We choose the symmetric group S_n on $[n]$ as D, and a set of transpositions of S_n as Z. Notice that Z contains only transposition, and there exists an arc (x, y) iff there exists an arc (y, x), where x and y are two nodes. Thus, the corresponding Cayley graph can be viewed as an undirected Cayley graph.

Consider a basic linked graph H whose set of nodes is $[n]$ $(n \geq 3)$. Each edge of H is viewed as a transposition of the symmetric graph S_n on $[n]$; hence, the set of all edges of H resembles a transposition set Z of S_n. Therefore, H is entitled a transposition basic graph, and the resulting Cayley graph is said to be the corresponding Cayley graph of H, written as $\mathrm{Cay}(H, D)$. Akers et al. [22] proved that $D = S_n$ in $\mathrm{Cay}(H, D)$.

If H mentioned above is a tree (resp. a path, a star), then the resulting Cayley graph is named a transposition tree (resp. a bubble-sort graph, a star graph) [22], written as $C\Gamma_n$ (resp. B_n, S_n). When H is a sector SE_n of n $(n \geq 3)$ nodes, i.e., $V(\mathrm{SE}_n) = [n]$ and $E(\mathrm{SE}_n) = \{(1, k) : 2 \leq k \leq n\} \cup \{(k, k+1) : 2 \leq k \leq n-1\}$, the resulting Cayley graph is named a bubble-sort star graph [23], written as BS_n. If H is a wheel W_n of n $(n \geq 4)$ nodes, i.e., $V(W_n) = [n]$ and $E(W_n) = \{(1, k) : 2 \leq k \leq n\} \cup \{(k, k+1) : 2 \leq k \leq n-1\} \cup \{(2, n)\}$, then the resulting Cayley graph is named a n-dimensional wheel network [24], written as CW_n. In other words, CW_n represents a graph with set of nodes $V(\mathrm{CW}_n) = S_n$, where 2 nodes x and y are neighboring iff $x = y(1, k)$, $2 \leq k \leq n$, or $x = y(k, k+1)$, $2 \leq k \leq n-1$, or $x = y(2, n)$. Figure 1 illustrates the Cayley graph CW_4.

To discuss conveniently, we denote by $a_1 a_2 \cdots a_n$ the permutation $\begin{pmatrix} 1 & 2 & \ldots & n \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}$, where $k \longrightarrow a_k$.

Theorem 1 (see [25]). *Each permutation different from identity within the symmetric group is the only (considering the order of the factors) multiply of cycles that are not joined, where every cycle has length greater or equal to 2.*

Theorem 2 (see [26]). *Suppose that H is a basic linked graph where $n = |V(H)| \geq 3$, and H^1 and H^2 are a pair of diverse graphs gained by labelling H with $[n]$. Then, $Cay(H^1, S_n)$ must be isomorphic to $Cay(H^2, S_n)$.*

By Theorem 1, each permutation must be written as a multiplication of cycles. For instance, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$. In particular, $\begin{pmatrix} 1 & 2 & \ldots & n \\ 1 & 2 & \ldots & n \end{pmatrix} = (1)$. The product $\eta_1 \eta_2$ of 2 permutations η_1 and η_2 is the composition function η_2 trailed by η_1, i.e., $(13)(23) = (132)$. For terms and notations not mentioned here we follow [25].

As a special Cayley graph, CW_n owns many attractive properties.

Proposition 1 (see [27]). *CW_n is $(2n-2)$-regular, node transitive, $\forall n \geq 4$.*

Proposition 2 (see [27]). *CW_n is bipartite, $\forall n \geq 4$.*

Proposition 3 (see [28]). *The girth of CW_n is 4, $\forall n \geq 4$.*

In the next discussion, we often partition CW_ninto n disjoint subgraphs $CW_n^1, CW_n^2, \ldots, CW_n^n$, where each node $x = x_1 x_2 \ldots x_n \in V(CW_n^k)$takes a specified integer k *in the latter place* x_n *for* $k \in [n]$. Evidently, each $CW_n^k$is isomorphic to BS_{n-1}, where $BS_n$is the bubble-sort star graph. For each node, $x \in V(CW_n^k)$, $x(1n)$, $x(n-1, n)$, and $x(2n)$are all named external neighbors of x, written as $x^+ = x(1n)$, $x^- = x(n-1, n)$, *and* $x^* = x(2n)$, respectively. Any edge is named a cross-edge respectful to a fixed factorization if its 2 nodes are in diverse CW_n^k*'s.*

Proposition 4 (see [28]). *Let $x \in V(CW_n^k)$ $(k = 1, 2, \ldots, n)$, where CW_n^k is mentioned as previously. Then, x^+, x^-, and x^* are in three diverse CW_n^l's $(l \neq k)$.*

Proposition 5 (see [29]). *For $x, y \in V(CW_n^k)$, then $\{x^+, x^-, x^*\} \cap \{y^+, y^-, y^*\} = \emptyset$, where $k \in [n]$.*

Proposition 6 (see [28]). *Let CW_n^k be denoted as previously. There exist exactly $3(n-2)!$ autonomous cross-edges among 2 diverse CW_n^k's.*

Theorem 3 (see [28]). *Let CW_n be denoted as previously. If 2 nodes x and y are neighboring, no shared neighboring nodes*

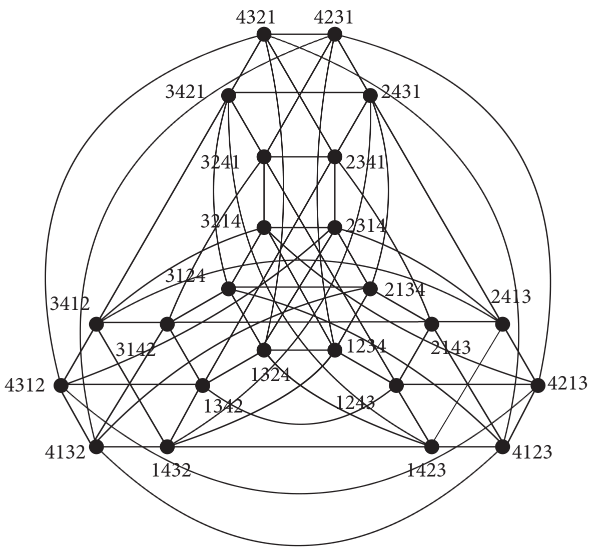

Figure 1: The wheel network CW_4.

exist of those nodes, that is, $|N(x) \cap N(y)| = 0$. If node x is not neighboring to y, maximally 3 mutual neighboring nodes exist of these nodes, namely, $|N(x) \cap N(y)| \leq 3$.

3. $\widetilde{\kappa}^{(2)}(\mathrm{CW}_n)$

Here, we establish $\widetilde{\kappa}^{(2)}(\mathrm{CW}_n) = 6n - 12$.

Proposition 7 (see [30]). *Suppose that BS_n is denoted as previously. Thus, BS_n is $(2n-3)$-regular, node transitive, $\forall n \geq 2$.*

Proposition 8 (see [30]). *Suppose that BS_n is denoted as previously. The connectivity $\kappa(BS_n) = 2n - 3$, $\forall n \geq 4$.*

Lemma 1 (see [30]). *For n greater or equal to 4 and F is a subset of $V(BS_n)$, where $|F| \leq 4n - 9$. When $BS_n - F$ is detached, a single of these situations for $BS_n - F$ is valid:*

(1) $BS_n - F$ has 2 constituents, with 1 isolated node

(2) $BS_n - F$ has 3 constituents, with 2 isolated nodes

Corollary 1 (see [29]). *For n greater or equal to 4 and F is a subset of $V(BS_n)$, where $|F| \leq 4n - 9$. When $BS_n - F$ has 3 components, with 2 isolated nodes, it is valid that $|F| \leq 4n - 9$.*

Lemma 2. *The 2-extra connectivity $\widetilde{\kappa}^{(2)}(CW_n) \leq 6n - 12$ $(n \geq 5)$.*

Proof. Observe $n \geq 5$ and $Z = \{(12), (13), \ldots, (1n), (23), (34), \ldots, (n-1, n)\} \cup \{(2n)\}$. From Theorem 2, assume that $A = \{(1), (12), (123)\}$. It is valid: $N((1)) \cap N((123)) = \{(12), (13), (23)\}$, $|(N((1)) \cap N((123))) \backslash A| = 2$. From Proposition 2, CW_n contains none of 3-cycles. Applying it and Theorem 3, it is valid that $|N_{\mathrm{CW}_n}(A)| = 6n - 12$. Consider $F_1 = N_{\mathrm{CW}_n}(A)$ and $F_2 = N_{\mathrm{CW}_n}(A) \cup A$, and $\mathrm{CW}_n - F_1$ has two parts $\mathrm{CW}_n - F_2$ and $\mathrm{CW}_n[A]$ (see Figure 2). □

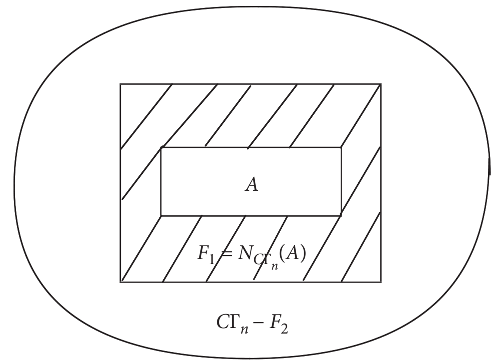

Figure 2: A depiction regarding the demonstrations of Lemma 2 and Lemma 3.

Let us now prove that $\mathrm{CW}_n - F_2$ is linked and $\delta(\mathrm{CW}_n - F_2) \geq 2$.

We factorize CW_n lengthways the latter location, written as $\mathrm{CW}_n^k (k = 1, \ldots, n)$. Recall that each CW_n^k and BS_{n-1} are isomorphic for $k \in [n]$. Hence, CW_n^k is $(2n-5)$-regular using Proposition 7, and $\kappa(\mathrm{CW}_n^k) = 2n - 5$ by Proposition 8 for each $k \in [n]$.

Notice that $A \subseteq V(\mathrm{CW}_n^n)$, and $|N(A) \cap V(\mathrm{CW}_n^1)| = 2$, $|N(A) \cap V(\mathrm{CW}_n^2)| = 3$, $|N(A) \cap V(\mathrm{CW}_n^3)| = 1$, $|N(A) \cap V(\mathrm{CW}_n^4)| = \cdots = |N(A) \cap V(\mathrm{CW}_n^{n-2})| = 0$, and $|N(A) \cap V(\mathrm{CW}_n^{n-1})| = 3$. Using Proposition 6 and $\kappa(\mathrm{CW}_n^k) = 2n-5$, $\mathrm{CW}_n[\cup_{k=1}^{n-1} V(\mathrm{CW}_n^k - F_1)]$ is linked. Using Proposition 5, every node of $\mathrm{CW}_n^n - F_2$ is neighboring to 3 faulty-free nodes of $\mathrm{CW}_n[\cup_{k=1}^{n-1} V(\mathrm{CW}_n^k - F_1)]$, and $\mathrm{CW}_n[\cup_{k=1}^{n-1} V(\mathrm{CW}_n^k - F_1) \cup V(\mathrm{CW}_n^n - F_2)]$ is linked. Combining $|N(A) \cap V(\mathrm{CW}_n^k)| \leq 3$ and $\kappa(\mathrm{CW}_n^k) = 2n-5$, hence $\delta(\mathrm{CW}_n^k - F_2) \geq 2$, where $k \in \{1, 2, \ldots, n-1\}$ and $n \geq 5$. Therefore, $\mathrm{CW}_n - F_2$ is linked and $\delta(\mathrm{CW}_n - F_2) \geq 2$. Consequently, $|V(\mathrm{CW}_n - F_2)| \geq 3$. $\mathrm{CW}_n - F_1$ has 2 constituents: $\mathrm{CW}_n - F_2$ and $\mathrm{CW}_n[A]$. Notice that $|V(\mathrm{CW}_n[A])| = 3$ is valid. Hence, F_1 is really a 2-extra cut and $\widetilde{\kappa}^{(2)}(\mathrm{CW}_n) \leq 6n - 12$.

According to Lemma 2, the next lemma is formulated.

Lemma 3. *Let $A = \{(1), (12), (123)\}$. For n greater or equal to 5, $F_1 = N_{CW_n}(A)$, $F_2 = N_{CW_n}(A) \cup A$, now $|F_1| = 6n - 12$ and $|F_2| = 6n - 9$, F_1 is a 2-extra cut of CW_n, and $CW_n - F_1$ must have 2 constituents: $CW_n - F_2$ and $CW_n[A]$ (see Figure 2).*

Lemma 4. *For any integer $n \geq 6$, $\widetilde{\kappa}^{(2)}(CW_n) \geq 6n - 12$.*

Proof. Consider F to be a minimum 2-extra cut, and suppose $|F| \leq 6n - 13$. We shall obtain a contradiction. Assume $F_k = F \cap \mathrm{CW}_n^k$. Observe 2 situations. □

Situation 1. $|F_k| \leq 2n - 6$ for all k.

Notice that $\kappa(\mathrm{CW}_n^k) = 2n - 5$; hence, $\mathrm{CW}_n^k - F_k$ is linked for each k. Using Proposition 6 and $2(n-2)! - |F_k| - |F_l| \geq 2(n-2)! - 2(2n-6) > 0$ for $n \geq 6$, we have that there must exist at least one edge to connect CW_n^k and CW_n^l, where $k, l \in [n]$ and $k \neq l$. Therefore, $\mathrm{CW}_n - F$ is linked, which contradicts the assumption that F is a minimum 2-extra cut.

Situation 2. $|F_k| \geq 2n-5$ for some k.

Assume $K = \{k \mid |F_k| \geq 2n-5\}$. Recall that $|F| \leq 6n-13$, then we have $|K| \leq 3$. For any $l \notin K$, $|F_l| \leq 2n-6$. From an analogous statement as in Situation 1, it is valid that $\cup_{l \notin K}(\mathrm{CW}_n^l - F_l)$ is linked, denoted it by $\widetilde{B}$.

Situation 2.1. $|K| = 1$.

In general, suppose $K = \{1\}$. Let C is a liked element of $\mathrm{CW}_n^1 - F_1$. When $|V(C)| \leq 2$, C must be linked to $\widetilde{B}$, since F is a 2-extra cut. If $|V(C)| \geq 3$, then C has a path P of three nodes. Suppose $N_1 = N_{\mathrm{CW}_n^1}(P) \cap F_1$ and $N_2 = (N_{\mathrm{CW}_n^1}(P) - N_1) \cup V(P)$. Thus, $N_1 \subseteq F_1$ and $N_2 \subseteq V(C)$ (see Figure 3.). Combining Theorem3 and CW_n^k is $(2n-5)$-regular, $|N_{\mathrm{CW}_n^1}(P)| \geq 3(2n-5) - 4 - 2 = 6n-21$. By Propositions 4 and 5, $|N_{\cup_{l \notin} K\mathrm{CW}_n^l}(N_2)| = 3\ |N_2| = 3(|N_{\mathrm{CW}_n^1}(P)| - |N_1|) + 3|V(P)| \geq |N_{\mathrm{CW}_n^1}(P)| - |N_1| + 3|V(P)| \geq 6n-21 - |F_1| + 3 \times 3 = 6n-12-|F_1| > |F| - |F_1|$. Then, there must exist at least one node $x \in C$ which has an external neighbor $x' \in \widetilde{B}$; thus, C is linked to $\widetilde{B}$. Using the randomness of C, $\mathrm{CW}_n - F$ is linked, which contradicts that F is a minimum 2-extra cut.

Situation 2.2. $|K| = 2$.

In general, suppose that $K = \{1, 2\}$ and C is a linked element of $(\mathrm{CW}_n^1 \cup \mathrm{CW}_n^2) - F_1 - F_2$. When $|V(C)| \leq 2$, C must be linked to $\widetilde{B}$, since F is an 2-extra cut. If $|V(C)| \geq 3$, then C has a path P of 3 nodes.

First, suppose that one of $\mathrm{CW}_n^1 - F_1$ and $\mathrm{CW}_n^2 - F_2$, say $\mathrm{CW}_n^1 - F_1$, has a path P on 3 nodes of C. Let $N_1 = N_{\mathrm{CW}_n^1}(P) \cap F_1$ and $N_2 = (N_{\mathrm{CW}_n^1}(P) - N_1) \cup V(P)$. Then, $N_1 \subseteq F_1$, $N_2 \subseteq V(C)$, and $|F| - |F_1| - |F_2| \leq 6n-13-(2n-5) - |F_1| \leq 4n-8-|F_1|$. When $n \geq 6$, $|N_{\mathrm{CW}_n^1}(P)| \geq 3(2n-5) - 4 - 2 = 6n-21$. By Propositions 4and 5, $|N_{\cup_{l \notin K}\mathrm{CW}_n^l}(N_2)| \geq |N_2| = |N_{\mathrm{CW}_n^1}(P)| - |N_1| + |V(P)| \geq 6n-21-|F_1| + 3 = 6n-18-|F_1| > 4n-8-|F_1| \geq |F| - |F_1| - |F_2|$ for $n \geq 6$. Then, there must exist at least one node $x \in C$ which has an external neighbor $x' \in \widetilde{B}$, and C is linked to $\widetilde{B}$. From the randomness of C, $\mathrm{CW}_n - F$ is linked, which contradicts that F is a minimum 2-extra cut.

If neither $\mathrm{CW}_n^1 - F_1$ nor $\mathrm{CW}_n^2 - F_2$ has a path on three nodes of C, then any path P of $(\mathrm{CW}_n^1 - F_1) \cup (\mathrm{CW}_n^2 - F_2)$ on three nodes has 2 nodes in a single side and 1 node in the different side, say $P = y_1 y_2 y_3$ with $y_1, y_2 \in V(\mathrm{CW}_n^1 - F_1)$ and $y_3 \in V(\mathrm{CW}_n^2 - F_2)$. Thus, $N_{\mathrm{CW}_n^1}(\{y_1, y_2\}) \subseteq F_1$, $|F_1| \geq |N_{\mathrm{CW}_n^1}(\{y_1, y_2\})| = 2(2n-5) - 2 = 4n-12$ from Proposition 2, $\sum_{l=3}^{n}|F_l| = |F| - |F_1| - |F_2| \leq 6n-13-(4n-12)-(2n-5) = 4$. By Propositions 4 and 5, we see that $|N_{\cup_{l \notin K}\mathrm{CW}_n^l}(P)| \geq 6 > \sum_{l=3}^{n}|F_l|$; hence, P is linked to $\widetilde{B}$; thus, C is also linked to $\widetilde{B}$.

In any situations, we have shown that $\mathrm{CW}_n - F$ is linked, which contradicts the assumption that F is a minimum 2-extra cut.

Situation 2.3. $|K| = 3$.

In general, suppose $K = \{1, 2, 3\}$. Since $|F| \leq 6n-13$ and $|F_k| \geq 2n-5$ for each $k \in K$, we have that $2n-5 \leq |F_k| \leq 2n-3$, for $k = 1, 2, 3$, and $\sum_{k=4}^{n}|F_k| \leq 2$. Notice that $2n-3 < 4n-13$ for $n \geq 6$, combining each CW_n^k is isomorphic to BS_{n-1} and Lemma 1 and Corollary 1; we have that, for each $k \in K$, if $\mathrm{CW}_n^k - F_k$ is disconnected, then $\mathrm{CW}_n^k - F_k$ has exactly 2 constituents, one of which is an inaccessible node which is written by y_k. Notice that $\sum_{l=4}^{n}|F_l| \leq 2$ and $n \geq 6$; hence, there must exist some CW_n^l which does not contain any faulty node. By Proposition 6 and $3(n-2)! > 2n-3+1+0 \geq |F_k| + |F_l|$, $\mathrm{CW}_n^k - F_k - \{y_k\}$ and CW_n^l are linked for each $k \in K$. Hence, we have that $(\cup_{k=1}^{3}(\mathrm{CW}_n^k - F_k - \{y_k\})) \cup \widetilde{B}$which is written as $\widehat{B}$ is linked. Next, we merely need to consider the situation that $\mathrm{CW}_n - F$ is disconnected. Let $\mathrm{CW}_n - F$ be disconnected and C be an arbitrary linked component. If $|V(C)| \leq 2$, then C is linked to $\widehat{B}$ since F is a 2-extra cut; thus, y_1, y_2, and y_3 must form a path, and the path is the only other linked component of $\mathrm{CW}_n - F$ different from $\widehat{B}$. Using Propositions 2, 4, and 5, then y_1, y_2, and y_3 does not form a 3-cycle, and each of y_1, y_2, and y_3 must have an external neighbor in $\cup_{l=4}^{n}\mathrm{CW}_n^l$. Notice that $\sum_{l=4}^{n}|F_l| \leq 2$; thus, there must exist at least one node y_k which has an external neighbor $y_k' \in \widetilde{B}$; hence, it is shown that $\mathrm{CW}_n - F$ is linked, which contradicts the assumption that F is a minimum 2-extra cut.

By Situations 1 and 2, F must be not a 2-extra cut of CW_n and therefore $|F| \geq 6n-12$. Hence, $\widetilde{\kappa}^{(2)}(\mathrm{CW}_n) \geq 6n-12$ for $n \geq 6$.

According to obtained Lemmas 2 and 4, the next is valid.

Theorem 4. $\widetilde{\kappa}^{(2)}(CW_n) = 6n-12,\ \forall n \geq 6$.

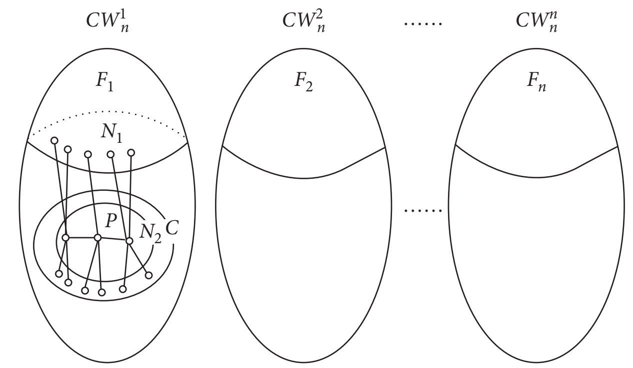

Figure 3: A diagram about the demonstration of Situation 2.1 of Lemma 4.

Table 1: Some situations of g-extra connectivity of CW_n continue.

g/n	4	5	$n \geq 6$
0	4	6	$2n-4$
1	?	14	$4n-6$
2	?	?	$6n-12$
$g \geq 3$	?	?	?

4. Conclusion

The traditional connectivity may be regarded as a simplification of the g-extra connectivity. It is gained that $\widetilde{\kappa}^{(2)}(\mathrm{CW}_n) = 6n-12\ (n \geq 6)$. Notice that $\kappa^{(0)}(\mathrm{CW}_n) = \kappa(\mathrm{CW}_n) = 2n-2$ when $n \geq 4$ and $\widetilde{\kappa}^{(1)}(\mathrm{CW}_n) = 4n-6$, When $n \geq 5$ are obtained in [9]. We conclude the paper by summarizing which situations of g-extra connectivity of CW_n have been not solved in Table 1.

Data Availability

The data used to support the findings of the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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