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Research Article

Applications of Homogenous Balance Principles Combined with Fractional Calculus Approach and Separate Variable Method on Investigating Exact Solutions to Multidimensional Fractional Nonlinear PDEs

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We investigate the exact solutions of multidimensional time-fractional nonlinear PDEs (fnPDEs) in this paper. In terms of the fractional calculus properties and the separate variable method, we present a new homogenous balance principle (HBP) on the basis of the $(1+1)$-dimensional time fnPDEs. Taking advantage of the new types of HBP together with fractional calculus formulas that subtly avoid the chain rule, the fnPDEs can be reduced to spatial PDEs, and then we solve these PDEs by the fractional calculus method and the separate variable approach. In this way, some new type exact solutions of the certain time-fractional $(2+1)$-dimensional KP equation, $(3+1)$-dimensional Zakharov–Kuznetsov (ZK) equation, and Jimbo–Miwa (JM) equation are explicitly obtained under both Riemann–Liouville derivatives and Caputo derivatives. The dynamical analysis of solutions is shown by numerical simulations of taking property parameters as well.

1. Introduction

The fractional PDEs (fPDEs) have been more and more widely followed with interest up to now since they can be used to accurately describe many nonlinear stochastic phenomena which depend on both time instant and the previous time history in the real-time problem [1–4]. Models set up from the fPDEs play very important roles in a range of scientific fields, such as viscoelastic flow [5, 6], signal processing [7, 8], control systems [9, 10], material diffusion including normal diffusion and anomalous diffusion(superdiffusion, subdiffusion, fast diffusion, and slow diffusion) [11–15], biological mathematics [16, 17], and magnetohydrodynamics (MHD) [18, 19]. Similarly, as some time-fractional PDEs, the space fractional models are also very frequently used in some elastic materials (see [20–24]) in the field of mechanical engineering. Therefore, investigating their solutions also draws much attention from the mathematical and physical points of view, and it can help to concisely characterize and well understand the qualitative features of the concerned phenomena and nonlinear processes in various areas of natural science and engineering, which involved the ubiquitous time memory effects [20, 21] (including some short memories) and space viscoelastic effects (see [22–26]), in particular many complex random walks and material motions in the microscopic space, such as dynamical behaviors of fractional diffusion, particles spread in heat bath, and soft matter interaction with viscoelasticity.

In the past few years, there were excessive studies on the $(1+1)$-dimensional fractional nonlinear PDEs(fnPDEs), and a lot of excellent tools were used for solving them, which include Adomian decomposition method [27, 28], homotopy analysis method (HAM) [29, 30], fractional variational method [31, 32], Lie symmetry method [33–40], invariant subspace method [40–44], and homogenous balance principle (HBP) [45, 46]. Recently, a few of the above methods were also applied to solve several $(2+1)$-dimensional fnPDEs [47–50]. Although approximate analytic solutions or exact solutions of some $(1+1)$-dimensional fnPDEs and $(2+1)$-dimensional fnPDEs can be successfully obtained by using the above methods, this is far from enough, and these methods still have many limitations in solving more complex multidimensional fnPDEs. In this paper, we suggest a new technique to solve the following type of fnPDEs with certain time-fractional derivatives:

$$\partial_t^\alpha u = F\left(x, t, u, \partial_x u, \ldots, \partial_x^m u\right), \tag{1}$$

where $x = (x_1, \ldots, x_N), \partial_x^k u = (\partial^{i_1+\cdots+i_N} u/\partial x_1^{i_1} \ldots \partial x_N^{i_N})$, $(1 \le k \le m, 1 \le i_1 + i_N \le k)$, and the index $0 < \alpha \le 1$. The α-th order time derivative $(\alpha \in (0,1])$ is well defined as an abnormal derivative in the real applications for explaining short memories of evolutional physical systems; there are several kinds of definitions of fractional derivatives, and the two frequently used, classical, and very widely influential definitions are still Riemann–Liouville definition and Caputo definition (see Definitions 1 and 2). Indeed, with the help of Riemann–Liouville derivative and Caputo derivative, the results derived from the time-fractional PDE models are more precise and more general in nature than those of the integer-order ones, while the smaller the α is, the faster effective time memory enjoys [20, 21]. When $\alpha = 1$, expressions (3) and (4) are exactly in accordance with the classical derivatives $(\partial u/\partial t)$; however, compared with the Riemann–Liouville type of derivative, the Caputo type of derivative possesses weaker singularity for handling some fractional initial problems.

Moreover, it is necessary to point out that the two travelling wave transformations $u = U(\xi)(\xi = \sum_{i=1}^N k_i x_i - \omega(t^\alpha/\Gamma(1+\alpha)))$ and $u = U(\xi)(\xi = \sum_{i=1}^N k_i x_i - \omega t)$, which were often used to reduce the integer-order PDE to ODE, are actually not valid for the fractional-order one since it had been successfully verified that the compound fractional derivatives disagree with the following chain rule (refer to [45, 46, 51, 52]):

$$D_x^\alpha f(g(x)) = f'(g(x)) D_x^\alpha g(x) = D_{g(x)}^\alpha f(g(x)) \left(g'(x)\right)^\alpha, \tag{2}$$

thus, we hardly take the fractional derivative of compound function straightly in accordance with the classical chain rule, and the normal chain rule (see [51]) of fractional derivatives was ineffectively applied to solve equation (1) since it contains infinite series. That is to say, the exact solutions of the compound function type of equation (1) were impossible to be obtained by the invalid chain rule (2), and we hardly find the travelling wave solutions (even the exact soliton solutions) of fnPDEs via the two transformations mentioned above or some other complex transformations such as Darboux transformation [53], bilinear method [54], and F-expansion method [55].

To the best of our knowledge, avoiding the invalid fractional chain rule (2), there is no more direct and effective method to obtain the exact solutions for equation (1). The exact solutions of higher-dimensional nonlinear PDEs with time-fractional derivatives were not well obtained. The main difficulty is how to construct solutions to reduce the multidimensional fnPDEs to the classical spatial PDEs. Inspired by the previous homogenous balance principle (HBP) [45–47] including fractional calculus method [40–44, 56] and separate variable approach, we improve the way [45–47] and introduce a new type of HBPs for (1) so that the solutions can be assumed as a general separated variable form and (1) can be reduced to spatial PDEs, and then new type exact solutions of some multidimensional fnPDEs will be successfully obtained by solving these reduced PDEs in the variable separate way under both Riemann–Liouville derivatives and Caputo derivatives.

The main contents of this paper are organized as follows. The definitions and properties of the fractional calculus and fractional Laplace transformation are briefly described in Section 2. In Section 3, based on the fractional derivative formulas and the method of separate variable, the new homogenous balance principle (HBP) is suggested for (1), and in this way, the certain time-fractional $(2+1)$-dimensional KP equation, $(3+1)$-dimension Zakharov–Kuznetsov (ZK) equation, and $(3+1)$-dimension Jimbo–Miwa (JM) equation are reduced to the spatial PDEs and explicitly solved in general separated variable forms; we can see that some of these solutions possess new types including some arbitrary functions which were never attained before by other way, which means more general solutions are obtained. Furthermore, there are real differences between the Riemann–Liouville case and the Caputo case: the singularity occurs in solutions under Riemann–Liouville derivatives but no singularity appears in solutions under Caputo derivatives. The dynamical profiles of these solutions are displayed as can be seen from Figures 1–13 with property parameters, and we also analyze the long time behaviors for some of them as well. The last section is the conclusions of our works.

2. Preliminaries

In this section, we recall some useful definitions, properties, and theorem.

2.1. Definitions and Properties of Two Type Fractional Derivatives

Definition 1. The Riemann–Liouville fractional derivative of order $\alpha > 0$ is defined by the following expression:

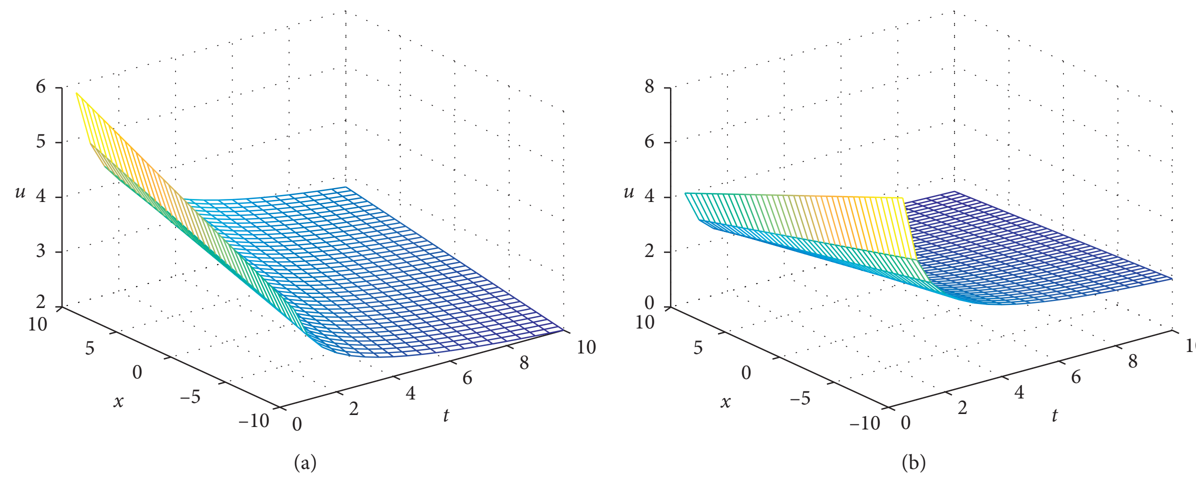


FIGURE 1: Dynamical profiles of solution (28) on the (x, t, u)-plane when (a) $\alpha = (1/3)$ and (b) $\alpha = (2/3)$.

$$ {}^{RL}\partial_t^{\alpha} u(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} u(x,\tau)\mathrm{d}\tau, & (n = [\alpha]+1), \\ \dfrac{\partial^n u(x,t)}{\partial t^n}, & (n = \alpha). \end{cases} \tag{3} $$

Definition 2. The Caputo fractional derivative of order $\alpha > 0$ is defined by the following expression:

$$ {}^{C}\partial_t^{\alpha} u(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \dfrac{\partial^n u(x,\tau)}{\partial \tau^n}\mathrm{d}\tau, & (n = [\alpha]+1), \\ \dfrac{\partial^n u(x,t)}{\partial t^n}, & (n = \alpha). \end{cases} \tag{4} $$

Proposition 1. *Some properties of the fractional derivative:*

(i) ${}^{R}D_t^{\alpha} c = c\left(t^{-\alpha}/\Gamma(1-\alpha)\right)$ *(c is a constant)*

(ii) ${}^{C}D_t^{\alpha} c = 0$ *(c is a constant)*

(iii) $D_t^{\alpha}\left(\sum_{k=1}^{n} c_k f_k(t)\right) = \sum_{k=1}^{n} c_k D_t^{\alpha} f_k(t)$ *(c_k are constants)*

(iv) $D_t^{\alpha} t^{\mu} = \left(\Gamma(1+\mu)/\Gamma(1+\mu-\alpha)\right) t^{\mu-\alpha}$, $(\mu \neq \alpha - 1)$ *holds for* $\mu > -1$ *under the Riemann–Liouville case and* $\mu > 0$ *under the Caputo case (5).*

Proposition 2. *The rules of the partial fractional derivative of a separate variable form:*

(i) *For* $\varphi(t) \neq Const$, $D_t^{\alpha} f(x)\varphi(t) = f(x) D_t^{\alpha}\varphi(t)$.

(ii) *For* $\varphi(t) = 1$,

$$ D_t^{\alpha} f(x) = \begin{cases} f(x)\dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}, & \text{(under Riemann–Liouville derivative)}, \\ 0, & \text{(under Caputo derivative)}. \end{cases} \tag{5} $$

2.2. Definitions of Mittag-Leffler Function and Some Properties

Definition 3. The Mittag-Leffler function is defined by the infinity series expression:

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \left(\frac{z^k}{\Gamma(\alpha k + \beta)} \right), \tag{6} $$

with $\alpha > 0$ and $\beta > 0$.

Proposition 3. *The properties of the Mittag-Leffler function:*

(i) $E_{1,1}(z) = e^z$,

(ii) $D_t^\gamma(t^{(\alpha k+\beta-1)}E_{\alpha,\beta}^{(k)}(\lambda t^\alpha)) = t^{(\alpha k+\beta-\gamma-1)}E_{\alpha,\beta-\gamma}^{(k)}(\lambda t^\alpha)$,

(iii) ${}^C D_t^\alpha E_{\alpha,1}(\lambda t^\alpha) = \lambda E_{\alpha,1}(\lambda t^\alpha)$,

(iv) ${}^R D_t^\alpha E_{\alpha,1}(\lambda t^\alpha) = (t^{-\alpha}/\Gamma(1-\alpha)) + \lambda E_{\alpha,1}(\lambda t^\alpha)$,

(v) *The Mittag-Leffler function* $E_{\alpha,1}(\lambda t^\alpha)$ *is increasing as* $\lambda > 0$ *while decreasing as* $\lambda < 0$ *when* $\alpha > 0$ *(Definition 3).*

The brief proof of (iii) and (iv) is given as follows.
For the Caputo derivatives, we have

$$\begin{aligned} {}^C D_t^\alpha E_{\alpha,1}(\lambda t^\alpha) &= {}^C D_t^\alpha \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty} \frac{\lambda^k\left({}^C D_t^\alpha t^{k\alpha}\right)}{\Gamma(\alpha k+1)} \\ &= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}(\Gamma(\alpha k+1)/\Gamma(\alpha k+1-\alpha))t^{(k-1)\alpha}}{\Gamma(\alpha k+1)} \\ &= \lambda \sum_{n=0}^{\infty} \frac{(\lambda t^\alpha)^n}{\Gamma(\alpha n+1)} \\ &= \lambda E_{\alpha,1}(\lambda t^\alpha). \end{aligned} \tag{7}$$

For the Riemann–Liouville derivatives, we obtain

$$\begin{aligned} {}^R D_t^\alpha E_{\alpha,1}(\lambda t^\alpha) &= {}^R D_t^\alpha \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty} \frac{\lambda^k\left({}^R D_t^\alpha t^{k\alpha}\right)}{\Gamma(\alpha k+1)} \\ &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}(\Gamma(\alpha k+1)/\Gamma(\alpha k+1-\alpha))t^{(k-1)\alpha}}{\Gamma(\alpha k+1)} \\ &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \lambda \sum_{n=0}^{\infty} \frac{(\lambda t^\alpha)^n}{\Gamma(\alpha n+1)} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \lambda E_{\alpha,1}(\lambda t^\alpha). \end{aligned} \tag{8}$$

These achieve (iii) and (iv) in Proposition 3.

3. HBP for Some Multidimensional Time-Fractional PDEs

In the beginning of this section, we improve the previous way to construct exact solutions to some multidimensional time-fractional nonlinear PDEs (1) according to ideas of homogenous balance and the fractional properties (5).

The main procedures are stated as follows:

Step 1. As can be seen from the properties of power function and Mittag-Leffler function in Section 2 and motivated by the HBP of $(1+1)$-dimensional time-fractional nonlinear PDE, for equation (1), we suppose the exact solutions as the following two types:

(1) For the Riemann–Liouville case:

$$u = a_0 t^{\mu_1} + a_1 v(x_1,\ldots,x_N)t^{\mu_2}, \quad \mu_1,\mu_2 \neq \alpha-1. \tag{9}$$

(2) For the Caputo case:

$$u = a_0 + a_1 v(x_1,\ldots,x_N)E_{\alpha,1}(\lambda t^\alpha). \tag{10}$$

Here μ_1, μ_2, and λ are undetermined parameters and $v(x_1,\ldots,x_N)$ is a undetermined function.

Step 2. Substituting (9) or (10) into equation (1) by comparing the powers of t or the coefficients of $E_{\alpha,1}(\lambda t^\alpha)$, we have parameters μ_1, μ_2, or λ and the reduced spatial PDE system for $v(x_1,\ldots,x_N)$:

$$v(x_1,\ldots,x_N) = \Phi(x,t,v,\partial_x v,\ldots,\partial_x^m v), \tag{11}$$

where

$$x = (x_1,\ldots,x_N), \partial_x^k v = \frac{\partial^{i_1+\cdots+i_N}v}{\partial x_1^{i_1}\ldots\partial x_N^{i_N}}, \quad 1\le k\le m, 1\le i_1+i_N\le k. \tag{12}$$

Step 3. Solving the reduced PDE system (10) by using complex transformation $\xi = c_0 + \sum_{i=1}^N c_i x_i$ or a separate variable approach (in product or summary form) leads to the exact solutions.

Remark 1. When $\mu_1, \mu_2 = \alpha - 1$, singular solutions will appear under the Riemann–Liouville derivative and while solutions appear, no singularity will be obtained in the Caputo sense. t should involve minus power, thereby the assumption (9) fits for the Riemann–Liouville derivative, and since the fractional derivative of Mittag-Leffler function includes an additional term $(t^{-\alpha}/\Gamma(1-\alpha))$ under the Riemann–Liouville derivative, this will complicate the calculation of solutions, and the assumption (10) fits for the Caputo derivative.

We will study time-fractional $(2+1)$-dimensional KP equation, $(3+1)$-dimensional ZK equation, and JM equation in the following paragraphs.

3.1. *HBP for the $(2+1)$-Dimensional Time-Fractional KP Equation.*

We first begin with the $(2+1)$-dimensional fractional KP equation:

$$(\partial_t^\alpha u - 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \quad 0<\alpha\le 1, \tag{13}$$

which developed from the classical KdV equation, describes long dispersive wave propagating in two dimension in shallow water.

3.1.1. *Exact Solutions under Riemann–Liouville Derivatives.*

According to (9), we suppose the exact solutions as

$$u = a_0 t^{\mu_1} + a_1 v(x,y)t^{\mu_2}, \tag{14}$$

where a_0, a_1, μ_1, and μ_2 are the undetermined constants.

Balancing the coefficients of t-power yields the following two subcases:

Subcase I

$$\mu_2 - \alpha = \mu_1 + \mu_2 = 2\mu_2. \tag{15}$$

Thus, $\mu_1 = \mu_2 = -\alpha$.

Substituting (15) into (14) by comparing the coefficients of $t^{-\alpha}$ and $t^{-2\alpha}$ yields

$$\Omega_0 v_x = 6(a_0 + a_1 v) v_{xx} + 6a_1 v_x^2, \left(\Omega_0 = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\right), \tag{16a}$$

$$v_{xxxx} + 3v_{yy} = 0. \tag{16b}$$

Then, integrating (16a) corresponding to variable x once by selecting an integrate constant 0 leads to

$$\frac{dv}{dx} = \frac{\Omega_0 v}{6(a_0 + a_1 v)}. \tag{17}$$

This gives rise to the implicit relation:

$$6(a_0 \ln v + a_1 v) = \Omega_0 (x + \phi(y)), \tag{18}$$

where $\phi(y)$ is an undetermined function.

Calculating the derivatives of implicit function (18) reads that

$$v_{xxxx} = \frac{a_0 \Omega_0^4 (6a_1^2 v^2 - 8a_0 a_1 v + a_0^2) v}{1296 (a_1^7 v^7 + 7a_0 a_1^6 v^6 + 21 a_0^2 a_1^5 v^5 + 35 a_0^3 a_1^4 v^4 + 35 a_0^4 a_1^3 v^3 + 21 a_0^5 a_1^2 v^2 + 7 a_0^6 a_1 v + a_0^7)}, \tag{19a}$$

$$v_{yy} = \frac{\Omega_0 \left(\Omega_0 a_0 \phi'(y)^2 + 6(a_0 + a_1 v)^2 \phi''(y) v^2\right) v}{36 (a_1^3 v^3 + 3a_0 a_1^2 v^2 + 3a_0^2 a_1 v + a_0^3)}. \tag{19b}$$

Plugging (19a) and (19b) into (15), we have

$$a_0 = 0, \quad \phi''(y) = 0, \tag{20}$$

which leads to the solution of (15)

$$v(x, y) = \frac{\Omega_0}{6a_1}(x + b_1 y + b_2), \tag{21}$$

with arbitrary constants $a_1, b_1,$ and b_2 and the exact solution of (11)

$$u(t, x, y) = \left[a_0 + \frac{\Gamma(1-\alpha)}{6\Gamma(1-2\alpha)}(x + b_1 y + b_2)\right] t^{-\alpha}, \quad \left(\alpha \neq \frac{1}{2}\right), \tag{22}$$

where c_1 and c_2 are the arbitrary constants.

Subcase II. $\mu_2 - \alpha = 2\mu_2$ and $\mu_1 + \mu_2 = \mu_2$.

Thus, $\mu_1 = 0$ and $\mu_2 = -\alpha$.

Similarly as above, substituting (13) into (11) by comparing the coefficients of $t^{-\alpha}$ and $t^{-2\alpha}$ yields

$$\Omega_0 v_x = 6a_1 v v_{xx} + 6a_1 v_x^2, \left(\Omega_0 = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\right), \tag{23a}$$

$$v_{xxxx} + 6a_0 v_{xx} + 3v_{yy} = 0. \tag{23b}$$

Then, integrating (23a) corresponding to variable x by choosing an integrate constant 0 once leads to

$$\Omega_0 v = 6a_1 v v_x. \tag{24}$$

Since $v \neq 0$ (only nontrivial solutions are considered), we have

$$\frac{dv}{dx} = \frac{\Omega_0}{6a_1}, \tag{25}$$

which give rises to

$$v = \frac{\Omega_0}{6a_1} x + \phi(y). \tag{26}$$

Plugging (26) into (23b) yields

$$\phi(y) = c_1 y + c_2. \tag{27}$$

By rewriting $b_1 = a_1 c_1$ and $b_2 = a_2 c_2$, we have the exact solution

$$u(t, x, y) = a_0 + \left(\frac{\Gamma(1-\alpha)}{6\Gamma(1-2\alpha)} x + b_1 y + b_2\right) t^{-\alpha}, \quad \left(\alpha \neq \frac{1}{2}\right), \tag{28}$$

where $a_0, b_1,$ and b_2 are the arbitrary constants.

Remark 2. We have the same type solutions under the complex transformation $\xi = c_1 x + c_2 y + c_0$, so this case is omitted here.

3.1.2. Exact Solutions under Caputo Derivatives. According to (10), we assume the exact solutions formed as

$$u = a_0 + a_1 v(x, y) E_{\alpha,1}(\lambda t^\alpha). \tag{29}$$

Substituting (29) into (11) by comparing the coefficients of $E_{\alpha,1}(\lambda t^\alpha)$ leads to

$$\lambda a_1 v_x + a_1 v_{xxxx} + a_0 v_{xx} + 3a_1 v_{yy} = 0, \tag{30a}$$

$$v v_{xx} = v_x^2. \tag{30b}$$

We consider the following two cases:

Case I. By using the total differential rule, from (30b), we have

$$\frac{v v_{xx} - v_x^2}{v^2} = \left(\frac{v_x}{v}\right)_x = 0. \tag{31}$$

Integrating (31) leads to

$$v_x = C(y)v. \tag{32}$$

Thus,

$$v = \phi(y)e^{C(y)x}, \tag{33}$$

where $\phi(y)$ and $C(y)$ are the undetermined functions. Plugging (33) into (30a) yields

$$\phi''(y) + \frac{c\lambda + c^4 + a_0 c^2}{3}\phi(y) = 0. \tag{34}$$

Then, we have following three type solutions:

(i) $\delta = c\lambda + c^4 + a_0 c^2 < 0$.
We obtain $\phi(y) = c_1 e^{-\sqrt{-(\delta/3)}y} + c_2 e^{\sqrt{-(\delta/3)}y}$ and the exact solution

$$u(t,x,y) = a_0 + \left(c_1 e^{cx-\sqrt{-(\delta/3)}y} + c_2 e^{cx+\sqrt{-(\delta/3)}y}\right)E_{\alpha,1}(\lambda t^\alpha), \tag{35}$$

where $a_0, c, c_1,$ and c_2 are the arbitrary constants.

(ii) $\delta = c\lambda + c^4 + a_0 c^2 > 0$.
We obtain the exact solution

$$u(t,x,y) = a_0 + \left(c_1 \cos\left(\sqrt{\frac{\delta}{3}}y\right) + c_2 \sin\left(\sqrt{\frac{\delta}{3}}y\right)\right)e^{cx}E_{\alpha,1}(\lambda t^\alpha), \tag{36}$$

where $a_0, c, c_1,$ and c_2 are the arbitrary constants.

(iii) $\delta = c\lambda + c^4 + a_0 c^2 = 0$.
We obtain $\phi(y) = c_1 y + c_2$ and the exact solution

$$u(t,x,y) = a_0 + (c_1 y + c_2)e^{cx}E_{\alpha,1}(\lambda t^\alpha), \tag{37}$$

where $a_0, c, c_1,$ and c_2 are the arbitrary constants.

Case II. Using the complex differential transformation $\xi = c_1 x + c_2 y + c_0$ on (30a) and (30b), we have

$$c_1^4 v^{(4)}(\xi) + \left(a_0 c_1^2 + 3c_2^2\right)v''(\xi) + \lambda c_1 v'(\xi) = 0, \tag{38a}$$

$$v(\xi)v''(\xi) = v'^2(\xi). \tag{38b}$$

Solving (38b) similarly as (25) yields

$$v(\xi) = re^{c\xi}. \tag{39}$$

Taking (39) into (38a), we obtain $c_1^4 c^3 + (a_0 c_1^2 + 3c_2^2)c + \lambda c_1 = c_0$ and the exact solution by rewriting $a = a_1 re^{cc_0}, b_1 = cc_1, b_2 = cc_2$

$$u(t,x,y) = a_0 + ae^{(b_1 x + b_2 y)}E_{\alpha,1}(\lambda t^\alpha), \tag{40}$$

where $a_0, a, b_1,$ and b_2 are the arbitrary constants.

Remark 3. (33) contains product forms of separate variable $v = f(x)g(y)$, and for omitting the trivial solution, we do not consider the summary form of a separate variable $v = f(x)g(y)$.

Remark 4. The singularity only appears when $\alpha = (1/2)$ in solutions (22) and (28) in the Riemann–Liouville case.

3.1.3. Dynamical Analysis of Exact Solution for Fractional KP Model. Under the Riemann–Liouville case:

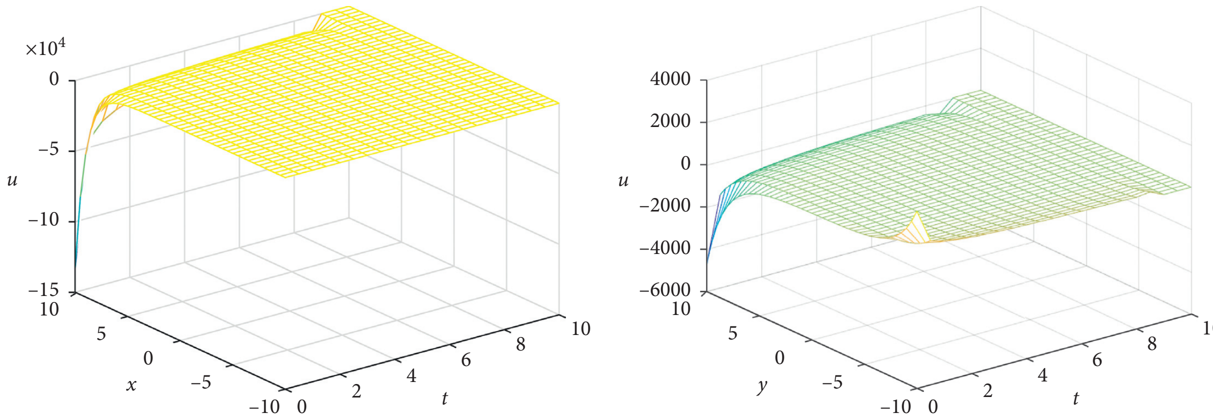
As can be seen from Figure 1, taking parameters $\alpha = (1/3)$ and $\alpha = (2), a_0 = 1, b_1 = 2, b_2 = 1, y = 1$, and interval $x \in [-10, 10], t \in [0, 10]$, it is shown that solution (28) increases with increase in the spatial variable and decreases with increase in time. When x and y are fixed, the solution tends to 0 at an α decay rate as $t \longrightarrow \infty$, the larger the α is, the faster the u decay is.

Under the Caputo case:

(i) As can be seen from Figure 2, choosing $c_1 = 1, c_2 = -2, c = 1, \lambda = -2, \alpha = (1/2), \delta = -1, y = 2, x \in [-10, 10], t \in [0, 10]$, and $x = 2, y \in [-10, 10], t \in [0, 10]$, it is shown that solution (35) decreases with increase in time-space since slope $(u_x < 0)$ or $(u_y < 0)$; however, it is unbounded as x or y tends to ∞.

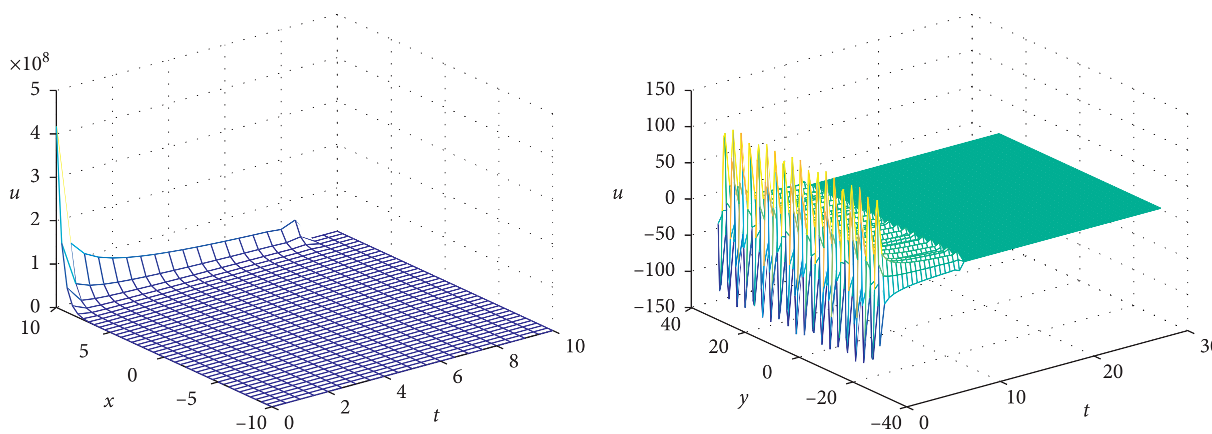
(ii) As we see from Figure 3, selecting $c_1 = 1, c_2 = -2, c = 2, \lambda = 12, \alpha = (1/2), \delta = -1, y = 2, x \in [-10, 10], t \in [0, 10]$, and $x = 2, y \in [-30, 10], t \in [0, 10]$, solution (36) decreases as time increases, periodic bounded as y, t tends to ∞ if x is fixed while unbounded as x, and t tends to ∞ if y is fixed.

(iii) As we see from Figure 4, choosing $c_1 = 1, c_2 = -2, c = 1, \lambda = -1, \alpha = (1/2), \delta = 0, y = 3, x \in [-10, 10], t \in [0, 10]$, and $x = 2, y \in [-10, 10], t \in [0, 10]$, solution (37) decreases as time increases and unbounded as time-space tend to ∞.



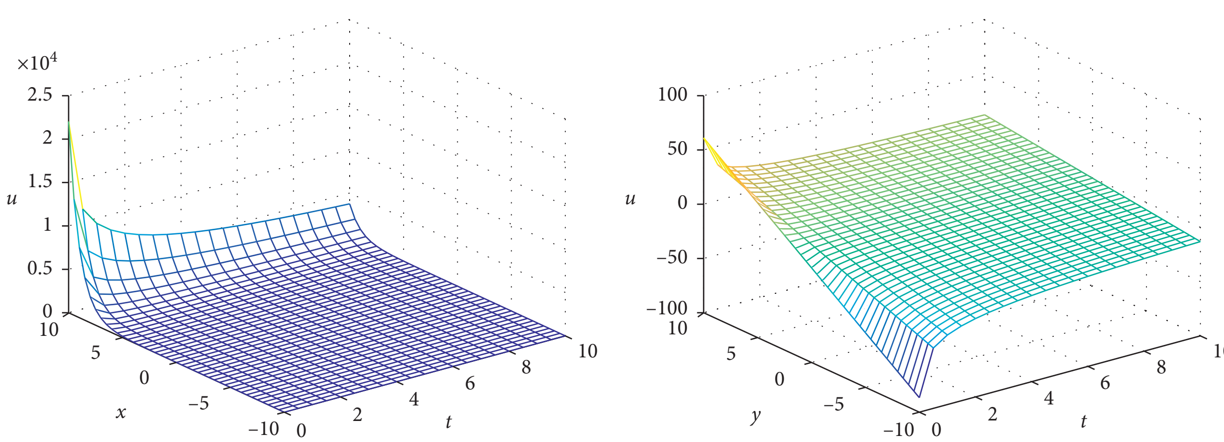
(a) (b)

Figure 2: Dynamical profiles of solution (35) on (a) the (x, t, u)-plane and (b) the (y, t, u)-plane.



(a) (b)

Figure 3: Dynamical profiles of solution (36) on (a) the (x, t, u)-plane and (b) the (y, t, u)-plane.



(a) (b)

Figure 4: Dynamical profiles of solution (37) on (a) the (x, t, u)-plane and (b) the (y, t, u)-plane.

3.2. HBP for the (3 + 1)-Dimensional Fractional ZK Equation

3.2.1. Exact Solutions under Riemann–Liouville Derivatives.

The (3 + 1)-dimensional fractional ZK equation is investigated in this section.

$$\partial_t^\alpha u + puu_x + qu_{zzz} + ru_{xxz} + su_{yyz} = 0, \quad 0 < \alpha \leq 1, \tag{41}$$

which represented the acoustic dynamics in a magnetized plasma in three-dimensional space with a low pressure.

The exact solution can be assumed as

$$u = a_0 t^{\mu_1} + a_1 v(x, y, z) t^{\mu_2}. \tag{42}$$

Balancing the coefficients of t-power by plugging (42) into (41) yields the following two cases:

Cases I. $\mu_1 - \alpha = \mu_1 + \mu_2 = 2\mu_2$

That is,

$$\mu_1 = \mu_2 = -\alpha. \tag{43}$$

By comparing the coefficients of $t^{-\alpha}$ and $t^{-2\alpha}$, we have

$$\Omega_0 + pa_1 v_x = 0, \quad \Omega_0 = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}, \tag{44a}$$

$$qv_{zzz} + rv_{xxz} + sv_{yyz} = 0. \tag{44b}$$

By solving (44a) directly, we obtain

$$v(x, y, z) = -\frac{\Omega_0}{pa_1} x + \phi(y, z). \tag{45}$$

Substituting (45) into (44b) yields

$$q\phi_{zzz} + s\phi_{yyz} = 0. \tag{46}$$

Subcase I. Assume $\eta = b_1 y + b_2 z + b_0$, then we have

$$\left(qb_2^2 + sb_1^2\right)\phi'''(\eta) = 0. \tag{47}$$

Solving (47) leads to the following two solutions:

(i) When $qb_2^2 + sb_1^2 = 0$, $(sq < 0)$, then

$$u(x, y, z, t) = \left[a_0 - \frac{\Gamma(1-\alpha)}{p\Gamma(1-2\alpha)} x + \phi(b_1 y + b_2 z + b_0)\right] t^{-\alpha}, \quad \alpha \neq \frac{1}{2}, \tag{48}$$

where a_0 and b_0 are the arbitrary constants and ϕ is an arbitrary function.

(ii) When $\phi = c_2(\eta^2/2) + c_1\eta + c_0$, then

$$u(x, y, z, t) = \left[a - \frac{\Gamma(1-\alpha)}{p\Gamma(1-2\alpha)} x + c_2 \frac{(b_1 y + b_2 z + b_0)^2}{2} + c_1(b_1 y + b_2 z)\right] t^{-\alpha}, \quad \alpha \neq \frac{1}{2}, \tag{49}$$

where $a = a_0 + c_0 + c_1 b_0, c_1$, and c_2 are the arbitrary constants.

Subcase II. Note that (46) is a linear equation, and if $sq < 0$, by the linear PDE method, we have

$$\phi(y, z) = f_1(y) - \frac{2\sqrt{-sq}}{s} f_1\left(\frac{y}{2} - \frac{sz}{2\sqrt{-sq}}\right) + f_2\left(\frac{\sqrt{-sq}}{s} y + z\right), \tag{50}$$

which gives rise to exact solution when $sq < 0$:

$$u(x, y, z, t) = \left[a_0 - \frac{\Gamma(1-\alpha)}{p\Gamma(1-2\alpha)} x + a_1\left(f_1(y) - \frac{2\sqrt{-sq}}{s} f_1\left(\frac{y}{2} - \frac{sz}{2\sqrt{-sq}}\right) + f_2\left(\frac{\sqrt{-sq}}{s} y + z\right)\right)\right] t^{-\alpha}, \quad \alpha \neq \frac{1}{2}, \tag{51}$$

where a_0 and a_1 are the arbitrary constants and f_1 and f_2 are the two arbitrary functions.

Subcase III. According to the linear PDE method, we assume the solution of (46) as a product form of a separate variable $\phi(y, z) = f(y) g(z)$, thus

$$qg'''(z) = c_0 g'(z), sf''(y) + c_0 f(y) = 0, \tag{52}$$

where c_0 is an arbitrary constant.

By solving them directly, we obtain the following four solutions (67)–(70) where $a_0, a_1, c_1, c_2, c_3, c_4$, and c_5 are the arbitrary constants and $\alpha \neq (1/2)$:

(i) $\phi = (c_1 \sin(\sqrt{(c_0/s)}\, y)) + (c_2 \cos(\sqrt{(c_0/s)}\, y)$ $(c_3 + c_4 \sin(\sqrt{-(c_0/q)}\, z) + c_5 \cos(\sqrt{-(c_0/q)}\, z)))$,

$$u(x,y,z,t) = \left[a_0 - \frac{\Gamma(1-\alpha)}{p\Gamma(1-2\alpha)}x + a_1\left(c_1 \sin\left(\sqrt{\frac{c_0}{s}}y\right) + c_2 \cos\left(\sqrt{\frac{c_0}{s}}y\right)\right)\left(c_3 + c_4 \sin\left(-\sqrt{\frac{c_0}{q}}z\right) + c_5 \cos\left(-\sqrt{\frac{c_0}{q}}z\right)\right)\right]t^{-\alpha}. \tag{53}$$

(ii) $\phi = (c_1 \sin(\sqrt{(c_0/s)}\, y) + c_2 \cos(\sqrt{(c_0/s)}\, y)$ $(c_3 + c_4 e^{\sqrt{(c_0/q)}\, z} + c_5 e^{-\sqrt{(c_0/q)}\, z}))$,

$$u(x,y,z,t) = \left[a_0 - \frac{\Gamma(1-\alpha)}{p\Gamma(1-2\alpha)}x + a_1\left(c_1 \sin\left(\sqrt{\frac{c_0}{s}}y\right) + c_2 \cos\left(\sqrt{\frac{c_0}{s}}y\right)\right)\left(c_3 + c_4 e^{\sqrt{(c_0/q)}\,z} + c_5 e^{-\sqrt{(c_0/q)}\,z}\right)\right]t^{-\alpha}. \tag{54}$$

(iii) $\phi = (c_1 e^{\sqrt{-(c_0/s)}\, y} + c_2 e^{-\sqrt{-(c_0/s)}\, y})$ $(c_3 + c_4 \sin(\sqrt{-(c_0/q)}\, z) + c_5 \cos(\sqrt{-(c_0/q)}\, z))$,

$$u(x,y,z,t) = \left[a_0 - \frac{\Gamma(1-\alpha)}{p\Gamma(1-2\alpha)}x + a_1\left(c_1 e^{\sqrt{-(c_0/s)}\,y} + c_2 e^{-\sqrt{-(c_0/s)}\,y}\right)\left(c_3 + c_4 \sin\left(\sqrt{-\frac{c_0}{q}}z\right)\right) + c_5 \cos\left(\sqrt{-\frac{c_0}{q}}z\right)\right]t^{-\alpha}. \tag{55}$$

(iv) $\phi = (c_1 e^{\sqrt{-(c_0/q)}\, y} + c_2 e^{-\sqrt{-(c_0/q)}\, y})(c_3 + c_4 e^{\sqrt{(c_0/q)}\, z} + c_5 e^{-\sqrt{(c_0/q)}\, z})$,

$$u(x,y,z,t) = \left[a_0 - \frac{\Gamma(1-\alpha)}{p\Gamma(1-2\alpha)}x + a_1\left(c_1 e^{\sqrt{-(c_0/s)}\,y} + c_2 e^{-\sqrt{-(c_0/s)}\,y}\right)\left(c_3 + c_4 e^{\sqrt{(c_0/q)}\,z} + c_5 e^{-\sqrt{(c_0/q)}\,z}\right)\right]t^{-\alpha}. \tag{56}$$

Subcase IV. We can also assume the solution of (47) as a summary form of a separate variable $\phi(y,z) = f(y) + g(z)$, then

$$g'''(z) = 0. \tag{57}$$

Thus,

$$g(z) = \frac{c_2^2 z^2}{2} + c_1 z + c_0, \tag{58}$$

which leads to the exact solution by rewriting $b_2 = a_1 c_2^2, b_1 = a_1 c_1$ and $a = a_0 + a_1 c_0$:

$$u(x,y,z,t) = \left(a - \frac{\Gamma(1-\alpha)}{p\Gamma(1-2\alpha)}x + a_1 f(y) + \frac{b_2 z^2}{2} + b_1 z\right)t^{-\alpha}, \quad \alpha \neq \frac{1}{2}, \tag{59}$$

where a, a_1, b_2, and b_1 are the arbitrary constants and $f(y)$ is an arbitrary function.

Cases II. $\mu_2 - \alpha = 2\mu_2$ and $\mu_1 - \alpha = \mu_1 + \mu_2$.

By balancing equation (41), we have

$$\mu_1 = 0, \quad \mu_2 = -\alpha. \tag{60}$$

Comparing the coefficients of $t^{-\alpha}$ and $t^{-2\alpha}$ yields

$$a_0\omega_0 + pa_0a_1v_x + qa_1v_{zzz} + ra_1v_{xxz} + sa_1v_{yyz} = 0,$$

$$\omega_0 = \frac{1}{\Gamma(1-\alpha)}, \tag{61a}$$

$$\Omega_0 v + pa_1 v v_x = 0, \quad \Omega_0 = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}. \tag{61b}$$

Solving (61b) leads to the following form:

$$v = -\frac{\Omega_0}{pa_1}x + \phi(y,z). \tag{62}$$

Substituting (62) into (61a) leads to

$$qa_1\phi_{zzz} + sa_1\phi_{yyz} = a_0(\Omega_0 - \omega_0). \tag{63}$$

(i) Considering the complex transformation $\eta = b_1 y + b_2 z + b_0$, we obtain

$$a_1 b_2 \left(qb_2^2 + sb_1^2\right)\phi'''(\eta) = a_0(\Omega_0 - \omega_0). \tag{64}$$

Solving (81a) and (81b) leads to

$$\phi(\eta) = \frac{a_0(\Omega_0 - \omega_0)}{6a_1 b_2 \left(qb_2^2 + sb_1^2\right)}\eta^3 + \frac{c_2}{2}\eta^2 + c_1\eta + c_0. \tag{65}$$

Then, we obtain the exact solution

$$u(t,x,y,z) = a_0 + \left[-\frac{\Gamma(1-\alpha)}{p\Gamma(1-2\alpha)}x + \frac{a_0((\Gamma(1-\alpha)/\Gamma(1-2\alpha)) - (1/\Gamma(1-\alpha)))}{6b_2\left(qb_2^2 + sb_1^2\right)}(b_1 y + b_2 z + b_0)^3 + \frac{a_1 c_2}{2}(b_1 y + b_2 z + b_0)^2 + a_1 c_1 (b_1 y + b_2 z + b_0) + a_1 c_0\right]t^{-\alpha}, \quad \alpha \neq \frac{1}{2}, \tag{66}$$

where $a_0, b_1, b_2, a_1, c_0, c_1,$ and c_2 are the arbitrary constants.

(ii) Note that (63) is a linear equation, and by the linear PDE method, we have

$$\phi(y,z) = f_1(y) + \frac{12f_2((\sqrt{-sq}/s)y + z)sqa_1 - 24f_1((y/2) - (sz/2\sqrt{-sq}))\sqrt{-sq}qa_1 + a_0(\Omega_0 - \omega_0)sz^3 + 3qa_0(\Omega_0 - \omega_0)zy^2}{12sqa_1}, \tag{67}$$

which leads to the exact solution

$$u(t,x,y,z) = a_0 + \left(-\frac{\Omega_0}{p}x + a_1 f_1(y) + \frac{12f_2((\sqrt{-sq}/s)y + z)sqa_1 - 24f_1((y/2) - (sz/2\sqrt{-sq}))\sqrt{-sq}qa_1 + a_0(\Omega_0 - \omega_0)sz^3 + 3qa_0(\Omega_0 - \omega_0)zy^2}{12sq}\right)t^{-\alpha}, \quad \alpha \neq \frac{1}{2}, \tag{68}$$

where a_0 and a_1 are constants and f_1 and f_2 are the two arbitrary functions.

(iii) Using summary form of the separate variable $\phi = f(y) + g(z)$ yields

$$qa_1 g'''(z) = a_0(\Omega_0 - \omega_0). \tag{69}$$

Solving (69) directly, we arrive at

$$g(z) = \frac{a_0(\Omega_0 - \omega_0)}{6qa_1}z^3 + \frac{b_2}{2}z^2 + b_1 z + b_0, \tag{70}$$

which gives rise to the exact solution

$$u(t,x,y,z) = a_0 + \left(-\frac{\Gamma(1-\alpha)}{p\Gamma(1-2\alpha)}x + a_1 f(y) + \frac{a_0}{6q}\left(\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} - \frac{1}{\Gamma(1-\alpha)}\right)z^3 + a_1\frac{b_2}{2}z^2 + a_1 b_1 z + a_1 b_0\right)t^{-\alpha}, \quad \alpha \neq \frac{1}{2}, \tag{71}$$

where $a_0, a_1, b_2, b_1,$ and b_0 are the arbitrary constants and $f(y)$ is an arbitrary function.

(iv) When $a_0 = 0$, using the product form of the separate variable $\phi = f(y)g(z)$ yields

$$qf(y)g'''(z) + sf'''(y)g(z) = 0, \tag{72}$$

which give rises to

$$f''(y) = c_0 f(y), qg'''(z) + sc_0 g(z) = 0. \tag{73}$$

From (73), we have

$$g(z) = c_1 e^{\left(\sqrt[3]{-sc_0q^2/q}\right)z} + e^{\left(\sqrt[3]{-sc_0q^2/2q}\right)z}\left(c_2\cos\left(\frac{\sqrt{3}\sqrt[3]{-sc_0q^2}}{2q}z\right) + c_3\sin\left(\frac{\sqrt{3}\sqrt[3]{-sc_0q^2}}{2q}z\right)\right), \tag{74}$$

and following three solutions (75)–(77) where a_1, b_1, b_2, c_1, c_2, and c_3 are constants and $\alpha \neq (1/2)$:

(i) If $c_0 > 0$, then

$$u(t,x,y,z) = \left\{-\frac{\Gamma(1-\alpha)}{pa_1\Gamma(1-2\alpha)}x + \left(b_1 e^{-\sqrt{c_0}y} + b_2 e^{\sqrt{c_0}y}\right)c_1 e^{\left(\sqrt[3]{-sc_0q^2/q}\right)z} + e^{\left(\sqrt[3]{-sc_0q^2/2q}\right)z}\left(c_2\cos\left(\frac{\sqrt{3}\sqrt[3]{-sc_0q^2}}{2q}z\right) + c_3\sin\left(\frac{\sqrt{3}\sqrt[3]{-sc_0q^2}}{2q}z\right)\right)\right\}t^{-\alpha}. \tag{75}$$

(ii) If $c_0 < 0$, then

$$u(t,x,y,z) = \left\{-\frac{\Gamma(1-\alpha)}{pa_1\Gamma(1-2\alpha)}x + \left(b_1\cos\left(\sqrt{-c_0}y\right) + b_2\sin\left(\sqrt{-c_0}y\right)\left[c_1 e^{\left(\sqrt[3]{-sc_0q^2/q}\right)z} + e^{\left(\sqrt[3]{-sc_0q^2/2q}\right)z}\left(c_2\cos\left(\frac{\sqrt{3}\sqrt[3]{-sc_0q^2}}{2q}z\right) + c_3\sin\left(\frac{\sqrt{3}\sqrt[3]{-sc_0q^2}}{2q}z\right)\right)\right]\right\}t^{-\alpha}. \tag{76}$$

(iii) If $c_0 = 0$, then

$$u(t,x,y,z) = \left(-\frac{\Gamma(1-\alpha)}{pa_1\Gamma(1-2\alpha)}x + (b_1y + b_2)\left(c_1z^2 + c_2z + c_3\right)\right)t^{-\alpha}. \tag{77}$$

Remark 5. In case I, we do not concern the complex transform $\xi = ax + by + cz + d$ for (48) contains solutions of this type.

Remark 6. In case II, we do not consider the complex transform $\xi = ax + by + cz + d$ for omitting the trivial solution.

Remark 7. The singularity only appears when $\alpha = (1/2)$ in solutions (53)–(56), (66), (68), (71), and (76)–(78) in the Riemann–Liouville case.

Remark 8. If taking form (10) into the fractional ZK model (41), we see that the nonlinear term uu_x will lead to trivial solutions, thus the Caputo derivative is not considered.

3.2.2. Dynamical Analysis of Exact Solution for Fractional ZK Model

(i) As can be seen from Figure 5, taking $\alpha = (3/5), a = 3, a_0 = c_0 = 1, c_1 = (1/2), b_0 = 2, b_1 = 1, b_2 = -2, c_2 = 2, t \in [0, 10], y = 1.5, z = 2.5, x \in [-10, 10], x = -2.5, z = 2.5, y \in [-10, 10]$, and $x = -2.5, y = 1.5, z \in [-10, 10]$, it is shown that solution (49) increases with increase in spatial variables and decreases as time increases. When x, y, and z are fixed, solution (49) tends to 0 at α decay rate as $t \longrightarrow \infty$, the larger the α is, the faster the u decay is.

(ii) From Figure 6, by taking $\alpha = (2/5), a_0 = (1/2), a_1 = 2, c_1 = 1.5, c_2 = -2, c_3 = -3, c_4 = -2.5, c_5 = 1, c_0 = 2, p = 3, s = q = 1/2, t \in [0, 30], x = -2.5, z = 2.5, y \in [-30, 30]$ and $t \in [0, 10]$, and $x = -2.5, y = 1.5, z \in [-10, 10]$, we see solution (54) is periodic bounded and tends to 0 at α decay rate as y and t tend to ∞ if x and z are fixed. The larger the α is, the faster the u decay is. When x and y are fixed, solution (54) decreases to $-\infty$ as z increases to $+\infty$.

(iii) As can be seen from Figure 7, selecting $\alpha = (2/3), p = 3, a_0 = 2, a_1 = 1, b_1 = 2, b_2 = -1, b_0 = 3, c_0 = 1, c_1 = (1/2), c_2 = 2, q = s = (1/2), t \in [0, 10], y = 1.5, z = 2.5, x \in [-10, 10], x = -2.5, z = 2.5, y \in [-10, 10]$, and $x = -2.5, y = 1.5, z \in [-10, 10]$, while from Figure 8 by choosing $\alpha = (2/3), p = 3, a_1 = 1, b_1 = 1, b_2 = -1.5, c_1 = 1.5,$

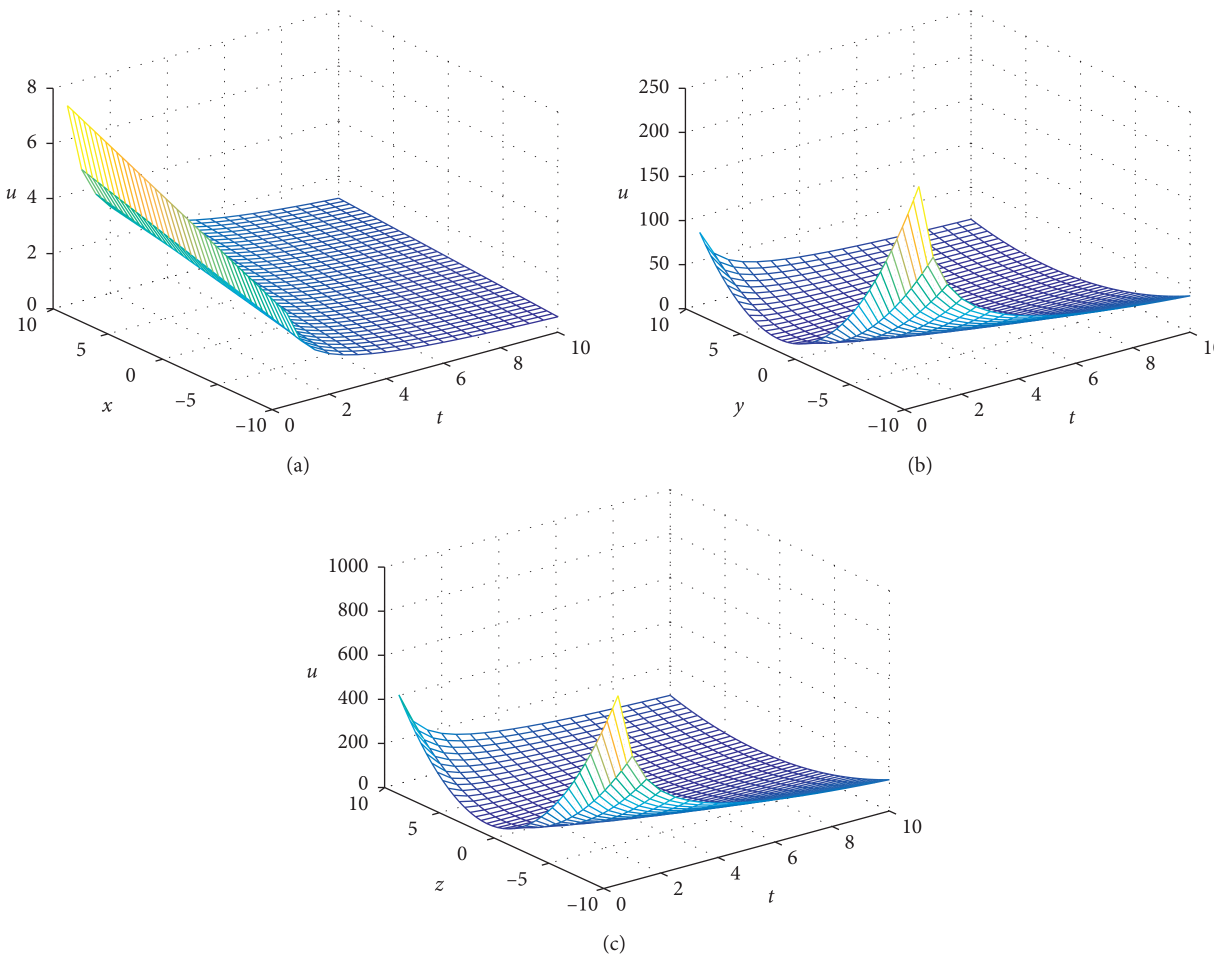


Figure 5: Dynamical profiles of solution (49) on (a) the (x, t, u)-plane, (b) the (y, t, u)-plane, and (c) the (z, t, u)-plane.

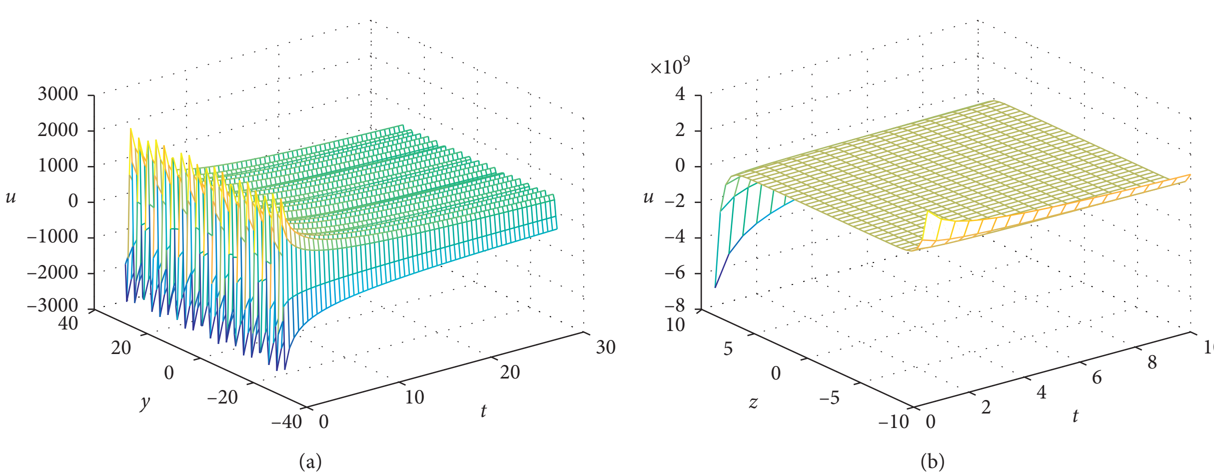


Figure 6: Dynamical profiles of solution (54) on (a) the (y, t, u)-plane and (b) the (z, t, u)-plane.

$c_2 = 2, c_3 = -2.5, c_0 = 2, s = 8, q = 2, t \in [0, 10]$, $y = 1.5, z = 2.5, x \in [-10, 10]$, $x = -2.5, z = 2.5$, $y \in [-10, 10]$, and $x = -2.5, y = 1.5, z \in [-10, 10]$, it is shown that solutions (66) and (75) increase to ∞ or decrease to $-\infty$ as x, y, or z increases to ∞, and ones also tends to 0 at α decay rate as $t \longrightarrow \infty$ if x, y, and z are fixed. The larger the α is, the faster the u decay is.

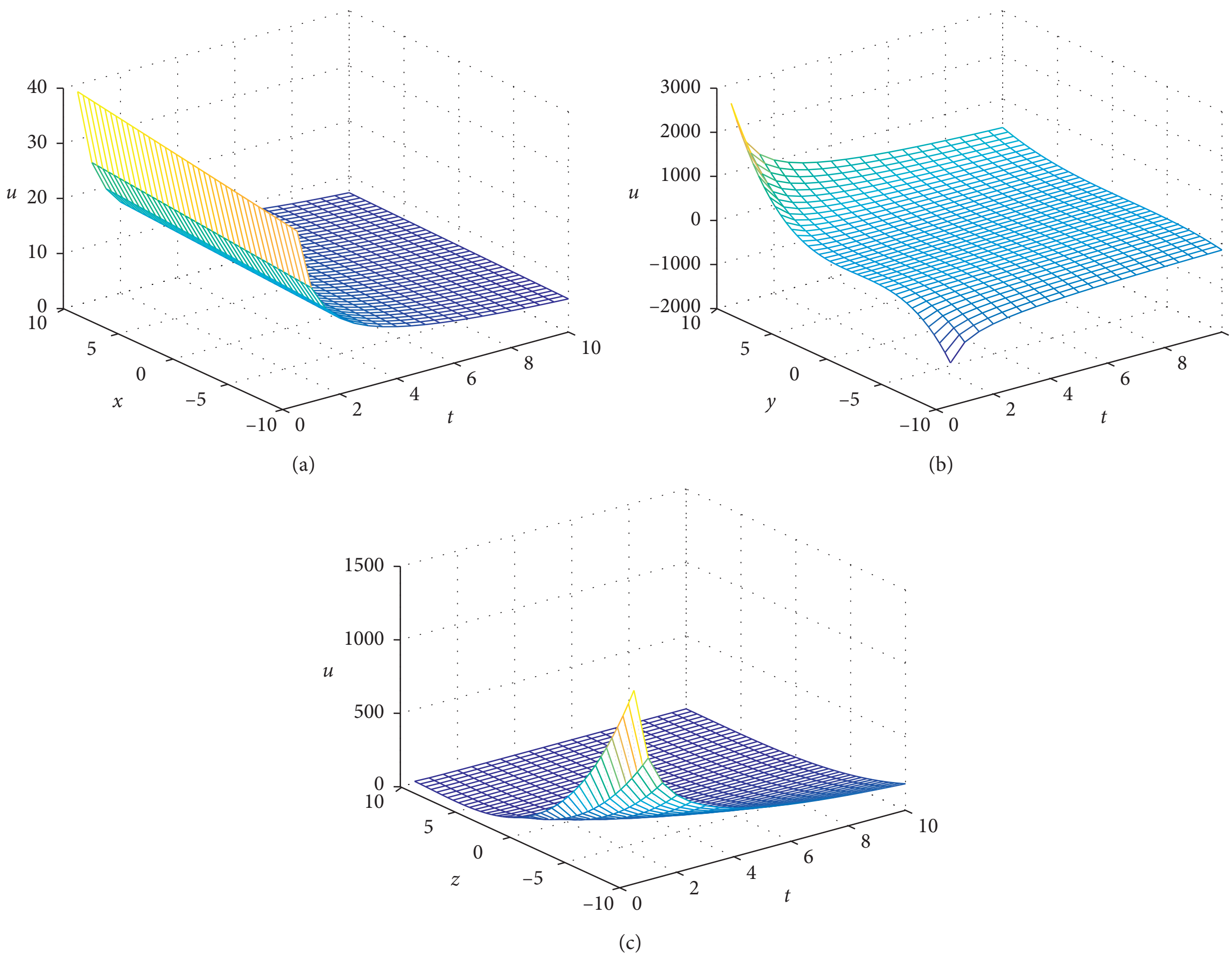


Figure 7: Dynamical profiles of solution (66) on (a) the (x, t, u)-plane, (b) the (y, t, u)-plane, and (c) the (z, t, u)-plane.

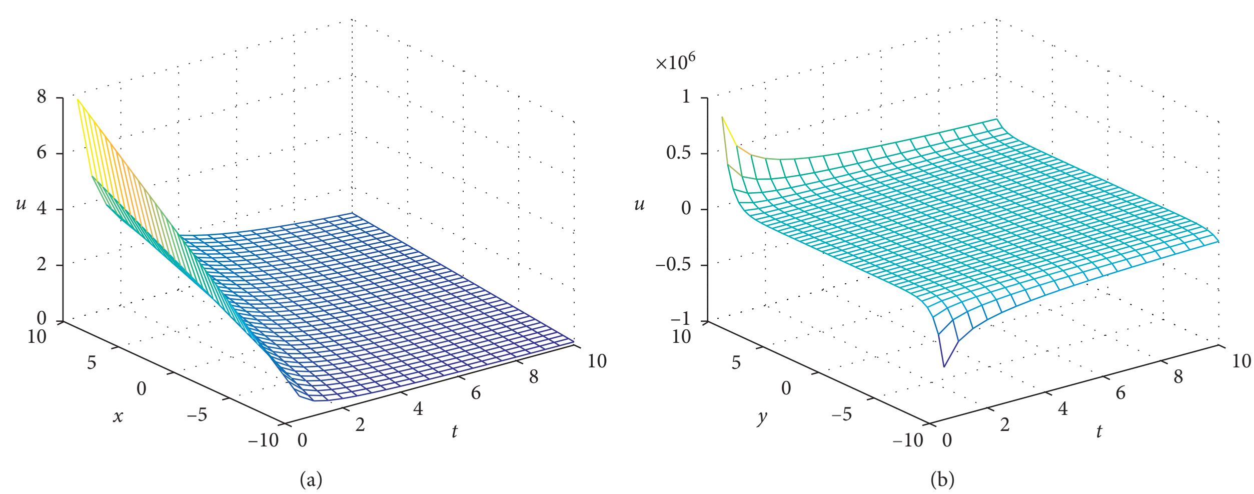
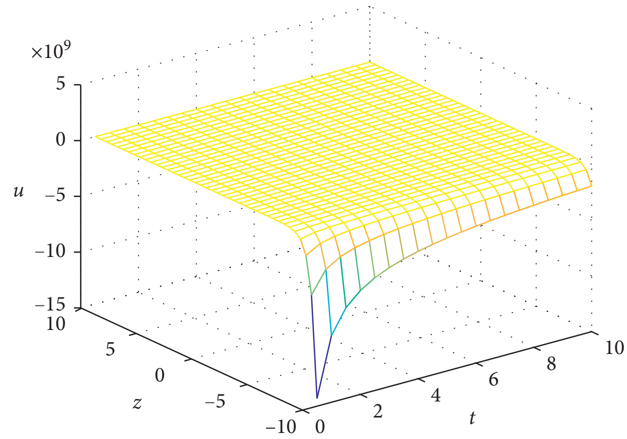


Figure 8: Continued.



(c)

FIGURE 8: Dynamical profiles of solution (75) on (a) the (x, t, u)-plane, (b) the (y, t, u)-plane, and (c) the (z, t, u)-plane.

(iv) From Figure 9, as we choose $\alpha = (2/3), p = 1, a_1 = 3, b_1 = 1, b_2 = -1.5, c_0 = 2, c_1 = 1, c_2 = 2, c_3 = -2.5, s = 8, q = 2, x = z = -2.5, y \in [-30, 30], t \in [0, 30]$, the solution (76) is periodic bounded and has α decay rate as $t \longrightarrow \infty$ if x and z are fixed, and one has the same dynamical properties as (54) if else.

3.3. HBP for the (3 + 1)-Dimensional Fractional JM Equation. The $(3+1)$-dimensional fractional JM equation will be researched in the following contents:

$$u_{xxxy} + pu_y u_{xx} + qu_x u_{xy} + r\partial_t^\alpha u_y - su_{xz} = 0, \quad 0 < \alpha \le 1, \tag{78}$$

which is developed from the second members of integrable systems of the classical KP hierarchy.

3.3.1. Exact Solutions under Caputo Derivatives. Assume

$$u = a_0 + a_1 v(x, y, z) E_{\alpha,1}(\lambda t^\alpha). \tag{79}$$

Injecting (79) into (78), we arrive at

$$v_{xxxy} + r\lambda v_y - sv_{xz} = 0, \tag{80a}$$

$$pv_y v_{xx} + qv_x v_{xy} = 0. \tag{80b}$$

(i) We assume that $v = v(\xi), \xi = ax + by + cz + d$, then (80a) and (80b) becomes the following ODEs:

$$a^3 bv^{(4)}(\xi) + r\lambda bv'(\xi) - sacv''(\xi) = 0, \tag{81a}$$

$$(p + q)v'(\xi)v''(\xi) = 0. \tag{81b}$$

When $q = -p$, (81b) is a free equation, and then solving (81a), we have following solutions by noting $\Delta = \sqrt[3]{(\sqrt{3}(\sqrt{27r^2\lambda^2 b^3 - 4s^3c^3/b}) - 9r\lambda b)b^2}$,

$$v(\xi) = \frac{c_0}{b\lambda r} + c_1 e^{\left(12^{(1/3)}\Delta^2 + 12^{(2/3)}scb/6ab\Delta\right)\xi} + e^{-\left(12^{(1/3)}\Delta^2 + 12^{(2/3)}scb/12ab\Delta\right)\xi}\left[c_2 \cos\left(\frac{\sqrt{3}\left(12^{(1/3)}\Delta^2 - 12^{(2/3)}scb\right)}{12ab\Delta}\xi\right) + c_3 \sin\left(\frac{\sqrt{3}\left(12^{(1/3)}\Delta^2 - 12^{(2/3)}scb\right)}{12ab\Delta}\xi\right)\right], \tag{82}$$

which leads to the exact solution:

$$u(t, x, y, z) = a_0 + a_1\left\{\frac{c_0}{b\lambda r} + c_1 e^{\left(12^{(1/3)}\Delta^2 + 12^{(2/3)}scb/6ab\Delta\right)(ax+by+cz+d)} + e^{-\left(12^{(1/3)}\Delta^2 + 12^{(2/3)}scb/12ab\Delta\right)(ax+by+cz+d)} \cdot \left[c_2 \cos\left(\frac{\sqrt{3}\left(12^{(1/3)}\Delta^2 - 12^{(2/3)}scb\right)}{12ab\Delta} \cdot (ax + by + cz + d)\right) + c_3 \sin\left(\frac{\sqrt{3}\left(12^{(1/3)}\Delta^2 - 12^{(2/3)}scb\right)}{12ab\Delta}(ax + by + cz + d)\right)\right]\right\} E_{\alpha,1}(\lambda t^\alpha), \tag{83}$$

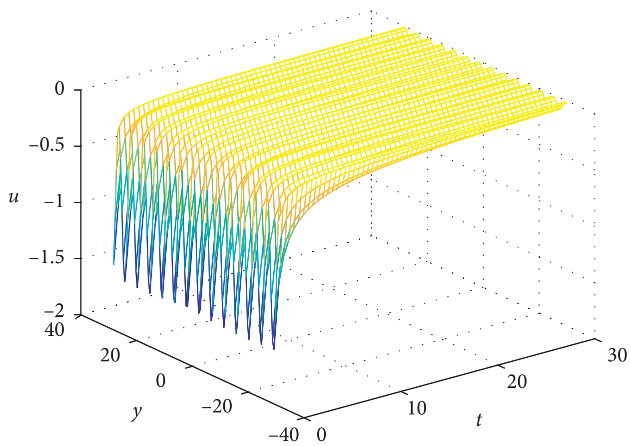


Figure 9: Dynamical profiles of solution (76) on the (y, t, u)-plane.

where a_0, a_1, c_0, a, b, c, and d are the arbitrary constants.

Remark 9. In case of $q \neq -p$, from $v''(\xi) = 0$, we obtain $v'(\xi) = 0$ which only leads to a trivial solution, thus we do not consider this case.

(ii) Assume $v(x, y, z) = v(\eta, z)$, $(\eta = ax + by + c)$, and from (80a) and (80b), we have

$$a^3 b v_{\eta\eta\eta\eta} + r\lambda b v_{\eta} - sav_{\eta z} = 0, \tag{84a}$$

$$(p+q) v_{\eta} v_{\eta\eta} = 0. \tag{84b}$$

Then (84a) and (84b) yields the following two subcases

(1) If $q = -p$, we only consider (84a).

(i) Note that it is a linear equation, and we suppose $v(\eta, z) = f(\eta) g(z)$, which arrives at

$$a^3 b f^{(4)}(\eta) g(z) + r\lambda b f'(\eta) g(z) - sa f'(\eta) g'(z) = 0. \tag{85}$$

Using the separate variable method, we have

$$g'(z) = c g(z), \tag{86a}$$

$$a^3 b f'''(\eta) + (r\lambda b - sac) f(\eta) = c_0. \tag{86b}$$

Solving (86a) reads that

$$g(z) = b_0 e^{cz}, \tag{87}$$

and the solution of (86b) and the exact solution of (78) is described as the following two cases:

If $r\lambda b \neq sac$, then

$$f(\eta) = \frac{c_0}{rb\lambda - sac} + c_1 e^{\left(\sqrt[3]{(sac-rb\lambda)b^2}/ab\right)\eta} + e^{-\left(\sqrt[3]{(sac-rb\lambda)b^2}/2ab\right)\eta}\left[c_2 \cos\left(\frac{\sqrt{3}\sqrt[3]{(sac-rb\lambda)b^2}}{2ab}\eta\right) + c_3 \sin\left(\frac{\sqrt{3}\sqrt[3]{(sac-rb\lambda)b^2}}{2ab}\eta\right)\right], \tag{88}$$

and the exact solution is

$$\begin{aligned} u(t, x, y, z) = {} & a_0 + a_1 b_0 \left\{\frac{c_0}{rb\lambda - sac} + c_1 e^{\left(\sqrt[3]{(sac-rb\lambda)b^2}/ab\right)(ax+by+c)} + e^{-\left(\sqrt[3]{(sac-rb\lambda)b^2}/2ab\right)(ax+by+c)}\right. \\ & \cdot \left[c_2 \left(\cos\left(\frac{\sqrt{3}\sqrt[3]{(sac-rb\lambda)b^2}}{2ab} x + by + c\right)\right)\right. \\ & \left.\cdot \left(a + c_3 \sin\left(\frac{\sqrt{3}\sqrt[3]{(sac-rb\lambda)b^2}}{2ab}(ax + by + c)\right)\right) e^{cz}\right\} E_{\alpha,1}(\lambda t^{\alpha}), \end{aligned} \tag{89}$$

where $a_0, a_1, b_0, c_0, a, b, c, c_1, c_2$, and c_3 are the arbitrary constants.

If $c = (rb\lambda/sa)$, then

$$f(\eta) = \frac{c_0}{6}\eta^3 + \frac{c_2}{2}\eta^2 + c_1 \eta + c_3, \tag{90}$$

and the exact solution is

$$u(t, x, y, z) = a_0 + a\left(\frac{c_0}{6}\left(ax + by + \frac{rb\lambda}{sa}\right)^3 + \frac{c_2}{2}\left(ax + by + \frac{rb\lambda}{sa}\right)^2 + c_1\left(ax + by + \frac{rb\lambda}{sa}\right) + c_3\right) e^{(rb\lambda/sa)z} E_{\alpha,1}(\lambda t^{\alpha}), \tag{91}$$

where a_0, c_0, a, b, c_1, c_2, and c_3 are the arbitrary constants.

(ii) Solving (84a) directly by the linear PDE method reads that

$$v(\eta, z) = c^{-(1/3)}\left[c_1 e^{c^{(1/3)}\eta + ((ba^2c/s)+(br\lambda/sa))z} + 4e^{-(1/2)c^{(1/3)}\eta + ((ba^2c/s)+(br\lambda/sa))z}\left(c_2 \sin\left(\frac{\sqrt{3}\,c^{(1/3)}}{2}\eta\right) + c_3 \cos\left(\frac{\sqrt{3}\,c^{(1/3)}}{2}\eta\right)\right)\right] + f(z),$$

$$u(t,x,y,z) = a_0 + a_1\left\{c^{-(1/3)}\left[c_1 e^{c^{(1/3)}(ax+by+c) + ((ba^2c/s)+(br\lambda/sa))z} + 4e^{-(1/2)c^{(1/3)}(ax+by+c) + ((ba^2c/s)+(br\lambda/sa))z}\right.\right.$$
$$\left.\left.\cdot\left(c_2 \sin\left(\frac{\sqrt{3}\,c^{(1/3)}}{2}(ax+by+c)\right) + c_3 \cos\left(\frac{\sqrt{3}\,c^{(1/3)}}{2}(ax+by+c)\right)\right)\right] + f(z)\right\}E_{\alpha,1}(\lambda t^{\alpha}), \tag{92}$$

where $a_0, a_1, a, b, c, c_1, c_2$, and c_3 are the arbitrary constants and $f(z)$ is an arbitrary function.

(iii) Assume $v(\eta, z) = f(\eta) + g(z)$, and from (84a), we have

$$a^3 f^{(4)}(\eta) + r\lambda f'(\eta) = 0, \tag{93}$$

and then

$$a^3 f'''(\eta) + r\lambda f(\eta) = c_0. \tag{94}$$

By solving (94), we have

$$f(\eta) = \frac{c_0}{r\lambda} + c_1 e^{-(\sqrt[3]{r\lambda}/a)\eta} + e^{(\sqrt[3]{r\lambda}/2a)\eta}\left(c_2 \cos\left(\frac{\sqrt{3}\sqrt[3]{r\lambda}}{2a}\eta\right) + c_3 \sin\left(\frac{\sqrt{3}\sqrt[3]{r\lambda}}{2a}\eta\right)\right), \tag{95}$$

and the exact solution

$$u(t,x,y,z) = a_0 + a_1\left[\frac{c_0}{r\lambda} + c_1 e^{-(\sqrt[3]{r\lambda}/a)(ax+by+c)} + e^{(\sqrt[3]{r\lambda}/2a)(ax+by+c)}\left(c_2 \cos\left(\frac{\sqrt{3}\sqrt[3]{r\lambda}}{2a}(ax+by+c)\right) + c_3 \sin\left(\frac{\sqrt{3}\sqrt[3]{r\lambda}}{2a}(ax+by+c)\right) + g(z)\right)\right]E_{\alpha,1}(\lambda t^{\alpha}), \tag{96}$$

where $a_0, a_1, a, b, c, c_1, c_2$, and c_3 are the arbitrary constants and $g(z)$ is an arbitrary function.

(2) If $q \neq -p$, then for the nontrivial case, we have $v_{\eta\eta} = 0$, and this leads to

$$v = f_1(z)\eta + f_2(z). \tag{97}$$

Then, substituting (97) into (84a), we have

$$r\lambda b f_1(z) - sa f_1'(z) = 0, \tag{98}$$

which yields

$$f_1(z) = c_0 e^{(r\lambda b/sa)z}. \tag{99}$$

This gives rise to the exact solution

$$u(t,x,y,z) = a_0 + a_1\left(c_0 e^{(r\lambda b/sa)z}(ax+by+c) + f_2(z)\right)E_{\alpha,1}(\lambda t^{\alpha}), \tag{100}$$

where a_0, a_1, c_0, a, b, and c are the arbitrary constants and $f_2(z)$ is an arbitrary function.

(III) Assume $v = f(x,z) + g(y,z)$.

Then, from (84a) and (84b), we arrive at

$$g_y f_{xx} = 0, \tag{101a}$$

$$r\lambda g_y - s f_{xz} = 0. \tag{101b}$$

For the nontrivial case, we obtain

$$f_{xx} = 0 \Rightarrow f = f_1(z)x + f_2(z). \tag{102}$$

Injecting (102) into (101b) yields

$$g = \frac{s}{r\lambda} f_1'(z)y + g_0(z), \tag{103}$$

which leads to the exact solution

$$u(t,x,y,z) = a_0 + a_1\left(f_1(z)x + \frac{s}{r\lambda}f_1'(z)y + \psi(z)\right)E_{\alpha,1}(\lambda t^{\alpha}), \tag{104}$$

where a_0 and a_1 are the arbitrary constants and $f_1(z)$ and $\psi(z)$ are the two arbitrary functions.

3.3.2. Exact Solutions under Riemann–Liouville Derivatives. We assume

$$u = a_0 t^{\mu_1} + a_1 v(x, y, z) t^{\mu_2}, \tag{105}$$

leads to $\mu_2 - \alpha = 2\mu_2$ and $\mu_2 = -\alpha$.

Substituting (105) into (78) by comparing the coefficients of $t^{-\alpha}$ and $t^{-2\alpha}$, we have

$$a_1\left(p v_y v_{xx} + q v_x v_{xy}\right) + r\Omega_0 v_y = 0, \quad \Omega_0 = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}, \tag{106a}$$

$$v_{xxxy} - s v_{xz} = 0. \tag{106b}$$

(I) Note that (106b) is a linear equation, and we assume the summary type of a separate variable $v = f(x, z) + g(y, z)$ which yields

$$p a_1 f_{xx} + r\Omega_0 = 0, \tag{107a}$$

$$f_{xz} = 0. \tag{107b}$$

Then, from (107b), we have

$$f(x, z) = f_1(x) + f_2(z). \tag{108}$$

Plugging (108) into (107a), we arrive at

$$f_1(x) = -\frac{r\Omega_0}{2pa_1} x^2 + c_1 x + c_0. \tag{109}$$

This gives rise to the exact solution

$$u(t, x, y, z) = a_0 t^{\mu_1} + \left[-\frac{r\Gamma(1-\alpha)}{2p\Gamma(1-2\alpha)} x^2 + b_1 x + a_1 f_2(z) + a_1 g(y, z) + b_0\right] t^{-\alpha}, \quad \alpha \neq \frac{1}{2}, \mu_1 \neq \alpha - 1, \tag{110}$$

where $\mu_1, a_0, a_1, b_1 = a_1 c_1$ and $b_0 = a_1 c_0$ are the arbitrary constants and $f_2(z)$ and $g(y, z)$ are the two arbitrary functions.

(II) Noticing equation (106a), we assume that $v = v(\xi, z)$, $(\xi = ax + by + c)$, then (106a) and (106b) become

$$a_1 a^2 (p+q) v_{\xi\xi} + r\Omega_0 = 0, \quad \Omega_0 = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}, \tag{111a}$$

$$v_z = -\frac{\psi(z)}{sa}. \tag{111b}$$

Solving (111a) leads to

$$v = -\frac{r\Omega_0}{2a_1 a^2 (p+q)} \xi^2 + \phi_1(z)\xi + \phi_0(z). \tag{112}$$

Substituting (112) into (111b) yields

$$\phi_1'(z)\xi + \phi_0'(z) = -\frac{\psi(z)}{sa}. \tag{113}$$

By comparing the coefficients, we have

$$\begin{aligned} \phi_1(z) &= c_1, \\ \phi_0(z) &= -\frac{1}{sa}\int_z \psi(\overline{z}) d\overline{z} = -\frac{1}{sa}\phi(z), \end{aligned} \tag{114}$$

which gives rise to the exact solution

$$u(t, x, y, z) = a_0 t^{\mu_1} + \left[-\frac{r\Gamma(1-\alpha)}{2a^2(p+q)\Gamma(1-2\alpha)}(ax+by+c)^2 + a_1 c_1 (ax+by+c) - \frac{a_1}{sa}\phi(z)\right] t^{-\alpha}, \quad \alpha \neq \frac{1}{2}, \mu_1 \neq \alpha - 1, \tag{115}$$

where $a_0, a_1, c_1, a, b,$ and c are the arbitrary constants and $\phi(z)$ is an arbitrary function.

Remark 10. We do not consider the transformation $\xi = ax + by + cz + d$ for omitting the trivial solution.

Remark 11. The singularity only appears when $\alpha = (1/2)$ and $\mu_1 = \alpha - 1$ in solutions (110) and (115) in the Riemann–Liouville case.

3.3.3. Dynamical Analysis of Exact Solutions for Fractional JM Equation. Under the Caputo case:

(i) As can be seen from Figure 10, taking $\alpha = (1/2), \mu_1 = \alpha - 1, a_0 = 1, a_1 = 1, c_0 = 1, c_1 = 2, c_2 = 1, c_3 = -3, a = 2, r = b = s = c = 1, \lambda = -1, d = -3 y = 1.5, z = 2, t \in [0, 10], x \in [-10, 10]$, it is shown that solution (83) increases with increase in the spatial variable and decreases as time increases on all (u, t, x), (u, t, y), and (u, t, z)-planes.

(ii) As Figures 11 and 12 show, we see that by choosing $a = b = 1, s = (7/2), c = 2, r = 2, \alpha = (1/2), \lambda = -(1/2), a_0 =$

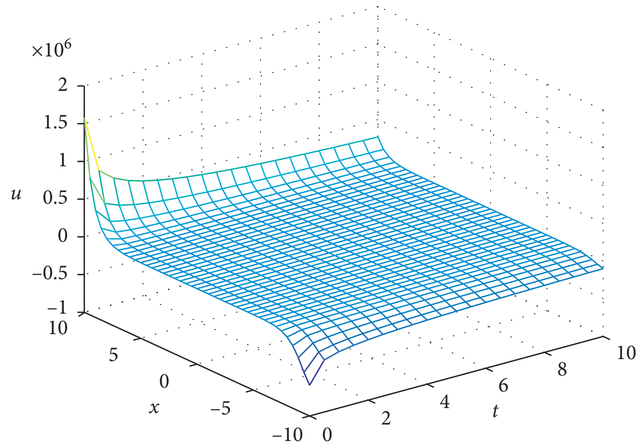
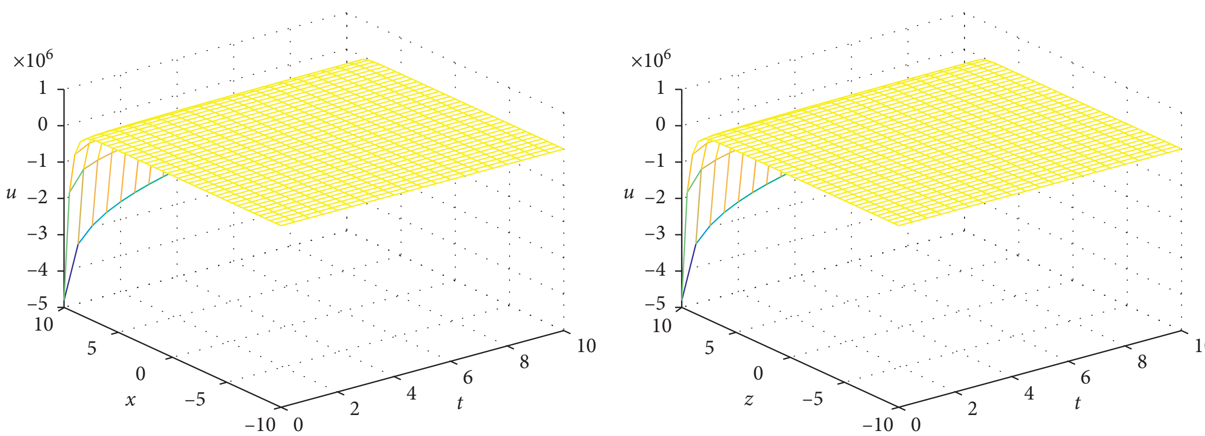
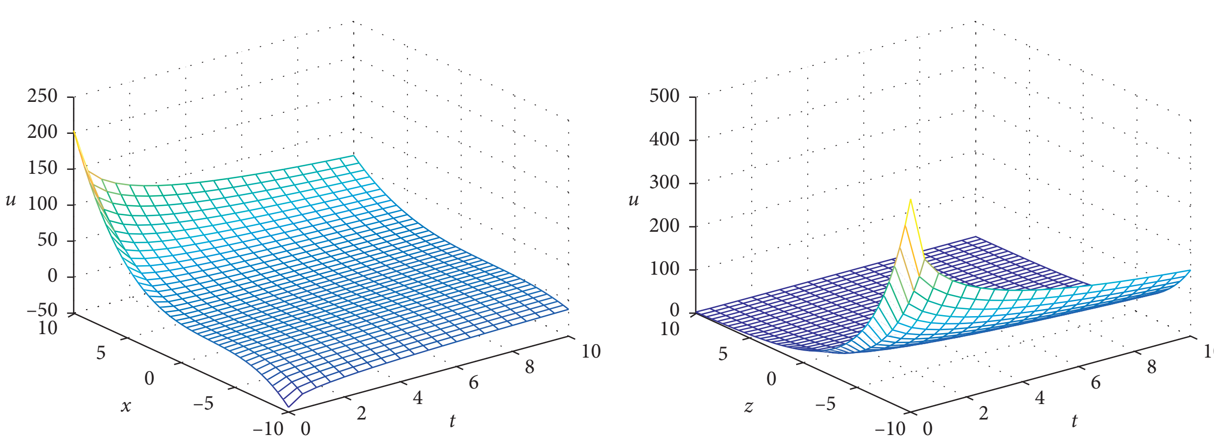


Figure 10: Dynamical profiles of solution (83) on the (x, t, u)-plane.



(a) (b)

Figure 11: Dynamical profiles of solution (89) on (a) the (x, t, u)-plane and (b) the (z, t, u)-plane.



(a) (b)

Figure 12: Dynamical profiles of solution (91) on (a) the (x, t, u)-plane and (b) the (z, t, u)-plane.

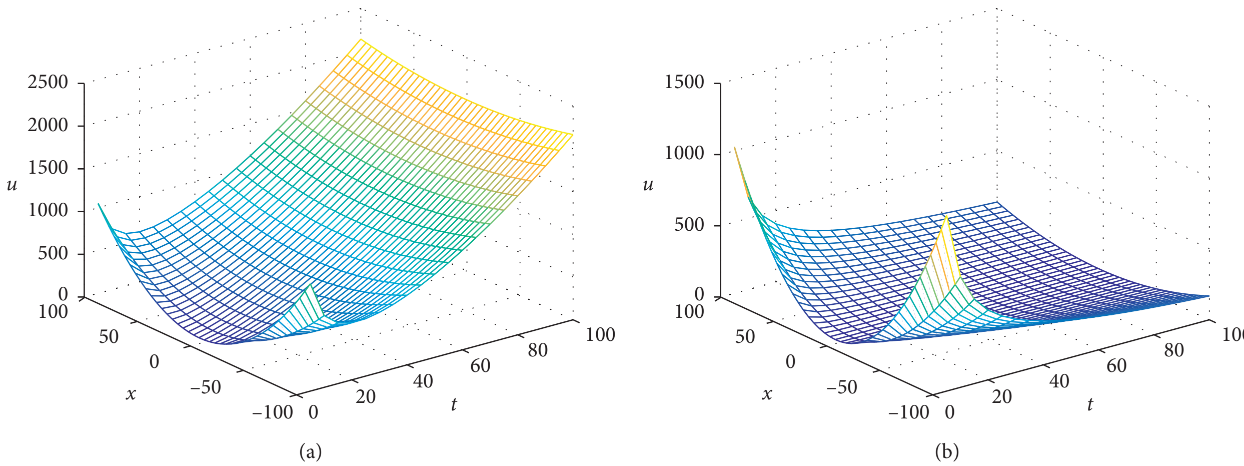


FIGURE 13: Dynamical profiles of solution (110) on the (x, t, u)-plane when (a) $\alpha = (1/3)$ and (b) $\alpha = (2/3)$.

$c_0 = 1, a_1 = -2, b_0 = 1, c_1 = 2, c_2 = 1, c_3 = -1.5, y = 2, z = 2, t \in [0,10], x \in [-10,10]$, and $x = 2.5, y = 2, z \in [-10,10]$, the solutions (89) and (91) increase as x and y increases and decrease as time increases on both (u, t, x) and (u, t, y)-planes; however, ones have exponential decay as $z \longrightarrow \infty$ when x and y are fixed.

Under the Riemann–Liouville case:

As Figure 13 shows, taking $\alpha = (2/3), \mu_1 = \pm(3/2), a_0 = 2, p = r = 1, b_1 = -3, a_1 f(z) + a_1 g(y,z) + b_0 = 10, x \in [-100,100], t \in [0,100]$, we see that

(i) If $\mu_1 > 0$, solutions (110) and (115) will blow up as time-space tends to infinity.

(ii) If $\mu_1 < = 0$, solutions (110) and (115) increase as x increases if y and z are fixed and decay at $\max\{\mu_1, \alpha\}$ rate as $t \longrightarrow \infty$ if the spatial variables are fixed.

4. Conclusions

In this paper, by taking advantage of the fractional calculus methods, we avoid the invalid chain rule and suggest the new types of HBPs to solve some multidimensional time-fractional PDEs whose exact solutions were hardly obtained before. We get some explicit solutions of the $(2+1)$-dimensional KP equation, $(3+1)$-dimensional ZK equation, and JM equation by solving the reduced PDE system in both Riemann–Liouville and Caputo cases. These solutions are all in the general separated variable forms of new type which even include arbitrary functions, and the singularity solutions only appear in the Riemann–Liouville case. Furthermore, the dynamical analysis and long-time behaviors of these solutions are also performed.

However, to the best of our knowledge, that is far from enough, the HBP is only applicable to solve some special fractional nonlinear PDEs which satisfy the balance conditions (see Section 3.1) and get a few special exact solutions but not more general solutions. For even more multidimensional fractional nonlinear partial differential systems, there is still no better way to acquire their exact solutions generally at present. Nevertheless, finding more effective methods for constructing more exact solutions of more general $(N+1)$-dimensional fnPDEs will be quite a meaningful and challenging task in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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